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Discovering Stars: Problems in Recovering Latent Variables from Models

CAMA Working Paper 52/2022
September 2022

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Abstract

There exist many latent variables in macroeconometrics that are commonly referred to as "stars". Examples of such "stars" are the NAIRU, potential GDP, and the neutral real rate of interest. Because these "stars" are defined as latent variables, they are estimated using the Kalman filter and/or smoother from models that can be expressed in State Space Form. When there are more shocks than observables in the State Space Form representation of such models, issues arise related to the recoverability of these "stars" from the data. Recoverability is problematic in this setting even if the assumed model for the data is correct and all model parameters are known. In this paper, we examine recoverability in a range of popular models and show that many of these "stars" cannot be recovered.

Keywords

Recoverability, excess shocks, latent variables, neutral rates, Kalman Filter

JEL Classification

E37, C51, C52

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ISSN 2206-0332

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Discovering Stars: Problems in Recovering Latent Variables from Models*

Daniel Buncic[#] Adrian Pagan[⋆]

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Abstract

There exist many latent variables in macroeconometrics that are commonly referred to as "stars". Examples of such "stars" are the NAIRU, potential GDP, and the neutral real rate of interest. Because these "stars" are defined as latent variables, they are estimated using the Kalman filter and/or smoother from models that can be expressed in State Space Form. When there are more shocks than observables in the State Space Form representation of such models, issues arise related to the recoverability of these "stars" from the data. Recoverability is problematic in this setting even if the assumed model for the data is correct and all model parameters are known. In this paper, we examine recoverability in a range of popular models and show that many of these "stars" cannot be recovered.

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*We would like to thank Dan Rees, Tim Robinson and Thomas Lubik for comments on earlier drafts of this paper. Daniel Buncic is grateful to the Jan Wallander and Tom Hedelius Foundation and the Tore Browaldh Foundation for research support.

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1 Introduction

There has been an increase in interest over the past 50 years to recover variables often referred to as "stars" in the macroeconomics literature. These "stars" are supposed to characterize long-run outcomes or steady-state values of some variables of policy relevance. There are many such "stars". Zaman (2022) lists them as arising in blocks describing output, unemployment, productivity, price inflation, wage inflation and interest rates. Because these "stars" are latent variables, they need to be recovered from observable data. Almost always this has meant the use of Unobserved Component (UC) models in conjunction with the Kalman filter. Estimating these models can pose problems due to identification issues, and sometimes there are errors in the estimation of such models. Buncic (2021) shows this for the structural model of Holston, Laubach and Williams (HLW) (2017) of the neutral real rate of interest.[1] Despite the empirical importance of estimation problems in UC models, for the purpose of this paper we will assume that they are absent, and that an acceptable set of parameter values is available for the model of interest that will deliver estimates of the "stars".

The focus of this paper is upon *recovery* of the latent variables from the data. More precisely, the paper examines whether we can recover "stars" from models that contain more shocks than observables, ie., in the terminology of Pagan and Robinson (2022), when there are *'excess shocks'*. As shown in Pagan and Robinson (2022), Chahrour and Jurado (2022) and Canova and Ferroni (2021), one cannot recover all the shocks when excess shocks are present. Excess shocks typically arise when latent variables are estimated using UC models. As an example, in Zaman (2022), there are 10 observable variables and 18 shocks. In this situation, not all 18 shocks can be recovered. Whilst it is the case that not all the shocks can be recovered, perhaps those that are important to the estimation of

[1] This is sometimes also called the natural real rate.

the "stars" can be. This situation should put the onus on those who present estimates of the "stars" from UC models to show that one *can* recover key shocks of relevance. In our experience this is rarely the case.

The remainder of the paper is structured as follows. Section 2 discusses general problems of recovering latent variables when excess shocks are present. Whether particular shocks can be recovered is easily checked by examining quantities that are computed from the Kalman filtering and smoothing equations.

Section 3 then looks at applications. Firstly, recovery of potential output with the widely used Hodrick-Prescott filter is examined. Secondly, recovery of the neutral real rate based on models proposed by Laubach and Williams (2003) and Schmitt-Grohe and Uribe (2022) is analyzed. Thirdly, recovery of potential output, the NAIRU and the neutral real rate is examined with th UC model given in McCririck and Rees (2017), which is essentially a part of the model used for forecasting and policy at the Reserve Bank of Australia (see Ballantyne et al. (2020) for more details on the model known as 'MARTIN').

Section 4 then looks at the question of whether the treatment of the short term nominal interest rate as being exogenous in existing attempts to recover the neutral real rate can be improved upon by making the short rate endogenous. This is done by formulating a monetary policy rule. Pagan and Wickens (2022) suggested that, without such a rule, the output gap may be an $I(1)$ process. Zaman (2022) notes that the estimates of the parameters in his UC model are improved when such a policy rule is included. However, this does not mean that recovery will be any better, as a policy adds another shock to the model, leading to the number of excess shocks remaining unchanged.

Section 5 turns to studies that do not use UC models to capture "stars". Instead, these models define the long-run measures of quantities such as output, unemployment, and other variables as either those derived from a Beveridge Nelson decomposition or the long-horizon forecast from a model with time varying parameters. The first was used in Morley et al. (2022), and the second by Lubik and Matthes (2015). There are issues with both approaches which we will discuss in that section.

2 Recovering Latent Variables from Models

To begin, we ask: what is the key issue when excess shocks are present in a model? We will discuss this in the context of models that can be written in State Space Form (SSF) as:

$$\zeta_t = As_t + B\varepsilon_t \tag{1}$$

$$s_t = Cs_{t-1} + D\varepsilon_t. \tag{2}$$

There may be identification issues in estimating A, B, C and D when there are more shocks than observables. To avoid this, we assume that all the parameters are known, i.e., we will simply use the numerical values provided in the papers of the models. There are two sets of equations (1) and (2) describing the relationship between variables, realizations and shocks. These equations involve the *assumed* shocks ε_t.[2] Given A, B, C, D, characteristics such as variances and covariances of the observed variables ζ_t can be determined. Equation (1) works from right to left. Assumptions are made about the shocks (on the RHS) and these then describe the random variables on the LHS. There are auxiliary assumptions involved in the model about the assumed shocks ε_t, for instance, that they are uncorrelated, while those in equation (1) tell the investigator about the assumed model properties.

The other equations in (2) involve the data, and these can be used to define the *estimated shocks*, which will be either filtered or smoothed. We will largely work with *smoothed shocks* and denote these by $E_T\varepsilon_t$. Smoothed shocks at time t are defined as the expectation of the

[2] All ε_t have zero means and unit variances.

shock ε_t using *all the* T observations in the sample. Filtered shocks are denoted by $E_t\varepsilon_t$ and are estimated using data up to time period t. Designating the data as ζ_t^D the system becomes:

$$\zeta_t^D = AE_T s_t + BE_T\varepsilon_t \tag{3}$$

$$E_T s_t = CE_T s_{t-1} + DE_T\varepsilon_t. \tag{4}$$

Here equation (3) works from the left to the right. Given a set of data (and A, B, C, D), one obtains smoothed shocks from the Kalman smoother. The key question that we ask is under what conditions we can recover ε_t from the data using $E_T\varepsilon_t$? It is important to highlight here that this is not an estimation issue, as we have assumed all parameters to be known. It is a *recovery* issue, and it asks whether we can separate the estimated shocks.

To look at this question, it is useful to think about this in the simplest possible scenario, where we have one observable and two shocks. Then we obtain the following two equations corresponding to (1) and (3):

$$\zeta_t = \varepsilon_{1t} + \varepsilon_{2t} \tag{5}$$

$$\begin{aligned} \zeta_t^D &= E_T\varepsilon_{1t} + E_T\varepsilon_{2t} && (6) \\ &= \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{G} \underbrace{\begin{bmatrix} E_T\varepsilon_{1t} \\ E_T\varepsilon_{2t} \end{bmatrix}}_{E_T\varepsilon_t} \\ &= GE_T\varepsilon_t. && (7) \end{aligned}$$

From this, it is apparent that $E_T\varepsilon_t$ cannot be recovered uniquely from ζ_t^D, because G is not a square matrix and hence singular. One could use a generalized inverse to solve for $E_T\varepsilon_t$.

This would find an estimated shock $\tilde{\varepsilon}_t = E_T \varepsilon_t$ that is closest to ε_t using a quadratic norm. For the case in (7) above, the generalized inverse of G (denoted by G^+), is given by:

$$\begin{aligned} G^+ &= G'(GG')^{-1} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^{-1} \\ &= \begin{bmatrix} .5 \\ .5 \end{bmatrix}, \end{aligned}$$

so that solving (7) for $E_T \varepsilon_t$ yields

$$\begin{gathered} G^+ \zeta_t^D = G^+ G E_T \varepsilon_t \\ \begin{bmatrix} .5 \\ .5 \end{bmatrix} \zeta_t^D = \begin{bmatrix} E_T \varepsilon_{1t} \\ E_T \varepsilon_{2t} \end{bmatrix}, \end{gathered} \tag{8}$$

which implies that $E_T \varepsilon_{1t} = .5\zeta_t^D = E_T \varepsilon_{2t}$. That is, the smoothed shocks $E_T \varepsilon_{1t}$ and $E_T \varepsilon_{2t}$ are identical, and thus cannot be separately identified from the data.

Now suppose that we think of a realization of the random variable ζ_t that equals the data. This would be associated with some realization of the shocks ε_{1t} and ε_{2t}, say we call these ε_{1t}^* and ε_{2t}^*, and so that:

$$\varepsilon_{1t}^* + \varepsilon_{2t}^* = G E_T \varepsilon_t,$$

meaning that $\tilde{\varepsilon}_{1t} = .5\zeta_t^D = .5(\varepsilon_{1t}^* + \varepsilon_{2t}^*)$. Clearly, $\tilde{\varepsilon}_{1t}$ does not equal ε_{1t}^*, and so we cannot recover the realized shock ε_{1t}.

To get some idea of the difference between the shock estimated from the data and the assumed shock, consider looking at the index $\phi = \text{Var}(\tilde{\varepsilon}_{1t} - \varepsilon_{1t}) = \text{Var}(\frac{1}{2}(\varepsilon_{1t} + \varepsilon_{2t})) = .5$.

If $\phi = 0$, we can recover ε_{1t}. However, for other values, more analysis is needed. Consider the case $\phi = \text{Var}(\tilde{\varepsilon}_t - \varepsilon_t)$. This yields:

$$\phi = \text{Var}(\tilde{\varepsilon}_t) - 2\text{Cov}(\tilde{\varepsilon}_t, \varepsilon_t) + \text{Var}(\varepsilon_t).$$

Since the last term is unity, we get:

$$\phi = \text{Var}(\tilde{\varepsilon}_t) - 2\rho \times \sigma(\tilde{\varepsilon}_t) + 1,$$

where ρ is the correlation between $\tilde{\varepsilon}_t$ and ε_t, and $\sigma(\tilde{\varepsilon}_t) = \sqrt{\text{Var}(\tilde{\varepsilon}_t)}$ the standard deviation of $\tilde{\varepsilon}_t$. Consequently,

$$\begin{aligned} 1 - \phi &= 2\rho \times \sigma(\tilde{\varepsilon}_t) - \text{Var}(\tilde{\varepsilon}_t) \\ &= \sigma(\tilde{\varepsilon}_t)(2\rho - \sigma(\tilde{\varepsilon}_t)). \end{aligned}$$

When $\phi = 0$, we have $\sigma(\tilde{\varepsilon}_t) = 1$ as $\tilde{\varepsilon}_t = \varepsilon_t$, and so $\rho = 0$, meaning that it is possible to recover the shock ε_t. The situation is more complex when ϕ is different from zero. For instance, when $\phi = 1$, then $\rho = \frac{\sigma(\tilde{\varepsilon}_t)}{2}$, and so the correlation between $\tilde{\varepsilon}_t$ and ε_t depends on $\sigma(\tilde{\varepsilon}_t)$, the standard deviation of $\tilde{\varepsilon}_t$.

3 Some Applications aimed at recovering Stars

We start here by looking at recovering potential output using a UC model that delivers the Hodrick Prescott (1997) filter solution, and then move on to the models of Laubach and Williams (2003) and Schmitt-Grohe and Uribe (2022) to recover the neutral real rate. Finally, we look at a model with Australian data used by McCririck and Rees (2017) to recover the NAIRU, the neutral real rate and potential output. In all cases we find issues in recovering the "stars".

3.1 Recovering Potential Output from the HP Filter

One "star" is potential output. It is used primarily to measure the output gap. In the typical setting, potential output is driven by permanent shocks, while the output gap is assumed to be stationary, and is thus exposed to transitory shocks only. There are various different specifications of these two components.

A popular one, due to Hodrick and Prescott (HP) (1997), was motivated on the basis of a UC model. As Pagan and Robinson (2022) observe, this involved excess shocks. In the appendix to Pagan and Robinson (2022) it is shown that the output gap computed from a one-sided HP filter which uses only current and past data leads to the filtered permanent shocks $E_t \varepsilon_t^{perm}$ and the filtered output gap $E_t \varepsilon_t^{gap}$ being perfectly correlated. That is, $E_t \varepsilon_t^{perm}$ is a multiple of $E_t \varepsilon_t^{gap}$. If one uses a two-sided filter, as the original HP filter, then the smoothed permanent $E_T \varepsilon_t^{perm}$ and the smoothed output gap shocks $E_T \varepsilon_t^{gap}$ are *dynamically correlated* via the identity:

$$(1 - 2L + L^2) E_T \varepsilon_t^{perm} = (1/\lambda) E_T \varepsilon_{t-2}^{gap}, \tag{9}$$

where λ is the smoothing parameter (commonly set to 1600 for monthly macroeconomic data) in the HP filter.

In a broader setting, the same type of dynamic correlation comes up with many multivariate filters that are derived from UC models. One example is the filter constructed from MAPMOD by Alichi et al. (2017).

3.2 Recovering the Neutral Real Rate - Laubach and Williams (2003)

Laubach and Williams (LW) (2003) proposed the following "baseline" model to recover the neutral real rate of interest (r_t^*). Numerous variants of the LW model exist in the literature, and are widely used at central banks. The baseline version of the model consists of the following equations:

$$\tilde{y}_t = \alpha_1 \tilde{y}_{t-1} + \alpha_2 \tilde{y}_{t-2} + \frac{a_r}{2}\sum_{i=1}^{2}(r_{t-i} - r_{t-i}^*) + \sigma_1 \varepsilon_{1t} \tag{10}$$

$$\pi_t = B(L)\pi_{t-1} + b_I(\pi_t^I - \pi_t) + b_o(\pi_{t-1}^o - \pi_{t-1}) + b_y \tilde{y}_{t-1} + \sigma_2 \varepsilon_{2t} \tag{11}$$

$$\Delta z_t = \sigma_3 \varepsilon_{3t} \tag{12}$$

$$\Delta y_t^* = g_{t-1} + \sigma_4 \varepsilon_{4t} \tag{13}$$

$$\Delta g_t = \sigma_5 \varepsilon_{5t} \tag{14}$$

$$r_t^* = c g_t + z_t \tag{15}$$

$$\tilde{y}_t = y_t - y_t^* \tag{16}$$

where $\tilde{y}_t$ is the output gap, y_t^* is potential GDP, r_t is a real interest rate, r_t^* is the neutral real rate, and π_t, π_t^I and π_t^o are various measures of inflation. There are evolving processes for the trend growth of GDP g_t, and “other determinants” z_t which affect r_t^*. In LW, there are five shocks $\{\sigma_j \varepsilon_{jt}\}_{j=1}^5$ with standard deviations $\{\sigma_j\}_{j=1}^5$, where the $\{\varepsilon_{jt}\}_{j=1}^5$ have unit variances.

In order to assess recoverability in this model, we follow the framework set out in Pagan and Robinson (2022) and Chahrour and Jurado (2022), and write the LW model in SSF which expresses all observables in ζ_t on the LHS of (1) and collects all shocks as well as remaining latent states in the state vector s_t. That is, we use (16) to re-write (10) as:

$$y_t = y_t^* + \alpha_1(y_{t-1} - y_{t-1}^*) + \alpha_2(y_{t-2} - y_{t-2}^*) - \frac{a_r}{2}\sum_{i=1}^{2}(r_{t-i} - r_{t-i}^*) + \sigma_1\varepsilon_{1t}.$$

Then the first measurement equation of the SSF is:

$$\zeta_{1t} = y_t^* - \alpha_1 y_{t-1}^* - \alpha_2 y_{t-2}^* + \frac{a_r}{2}\sum_{i=1}^{2} r_{t-i}^* + \sigma_1\varepsilon_{1t}, \tag{17}$$

where $\zeta_{1t} = y_t - \alpha_1 y_{t-1} - \alpha_2 y_{t-2} + \frac{a_r}{2}\sum_{i=1}^{2} r_{t-i}$ contains the relevant observable variables. Similarly, the second measurement equation is:

$$\zeta_{2t} = b_y y_{t-1}^* + \sigma_2\varepsilon_{2t}, \tag{18}$$

where $\zeta_{2t} = \pi_t - B(L)\pi_{t-1} - b_I(\pi_t^I - \pi_t) - b_o(\pi_{t-1}^o - \pi_{t-1}) - b_y y_{t-1}$ consists of the observable variables.

The relevant state dynamics are driven by the equations:

$$\Delta y_t^* = g_{t-1} + \sigma_4\varepsilon_{4t} \tag{19}$$

$$\Delta g_t = \sigma_5\varepsilon_{5t} \tag{20}$$

$$\Delta r_t^* = c\sigma_5\varepsilon_{5t} + \sigma_3\varepsilon_{3t}. \tag{21}$$

The full SSM corresponding to the relations in (17) to (21) is given by:

$$\underbrace{\begin{bmatrix} \zeta_{1t} \\ \zeta_{2t} \end{bmatrix}}_{\zeta_t} = \underbrace{\begin{bmatrix} 1 & -\alpha_1 & -\alpha_2 & 0 & \frac{a_r}{2} & \frac{a_r}{2} & 0 & \cdots & 0 \\ 0 & b_y & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}}_{A_{2\times 12}} \underbrace{\begin{bmatrix} y^*_t \\ y^*_{t-1} \\ y^*_{t-2} \\ r^*_t \\ r^*_{t-1} \\ r^*_{t-2} \\ g_t \\ \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix}}_{s_t} + \underbrace{\begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix}}_{\varepsilon_t} \tag{22}$$

$$\underbrace{\begin{bmatrix} y^*_t \\ y^*_{t-1} \\ y^*_{t-2} \\ r^*_t \\ r^*_{t-1} \\ r^*_{t-2} \\ g_t \\ \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix}}_{s_t} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} y^*_{t-1} \\ y^*_{t-2} \\ y^*_{t-3} \\ r^*_{t-1} \\ r^*_{t-2} \\ r^*_{t-3} \\ g_{t-1} \\ \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{4t-1} \\ \varepsilon_{5t-1} \end{bmatrix}}_{s_{t-1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & \sigma_4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 & c\sigma_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_5 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix}}_{\varepsilon_t} \tag{23}$$

All numerical values of the relevant parameters are taken from LW and are reproduced below for convenience.[3]

$$\sigma_1 = .387,\ \sigma_2 = .731,\ \sigma_3 = .323,\ \sigma_4 = .605,\ \sigma_5 = .102$$
$$c = 1.068,\ \alpha_1 = 1.51,\ \alpha_2 = -.57,\ b_y = .043,\ c = 1.068,\ \alpha_r = -.098$$

The SSF corresponding to LW's structural model has two observables and five shocks. This means that it will not be possible to recover more than two unique shocks from this model.

From the SSF defined in (22) and (23), we can determine which shocks are likely to be recoverable, and which ones are not. Following the recoverability criterion of Chahrour and Jurado (2022) as used by Pagan and Robinson (2022), all that needs to be computed is the mean squared error (MSE) of the Kalman filtered or smoothed estimates of the states s_t, ie., either the $E_t s_t$ or $E_T s_t$. These two MSEs are commonly denoted by $P_{t|t}$ and $P_{t|T}$. We will label those using the steady-state Kalman gain matrix in their computation as $P^*_{t|t}$ and $P^*_{t|T}$, respectively (see section 3.2 in Pagan and Robinson (2022) for more details). Now, a shock will be recoverable from the model if the diagonal element of $P^*_{t|T}$ corresponding to the five shocks of interest (the last five entries in state vector s_t in (23)) has a zero entry and will be unrecoverable when these equal unity.

For LW's model, we find for the smoothed shocks contained in the last five entries of $E_T s_t$ the following values for $\text{diag}(P^*_{t|T})$:

$$\text{diag}(P^*_{t|T}) = \begin{bmatrix} .71 & .02 & .98 & .36 & .94 \end{bmatrix}. \tag{24}$$

It is clear from (24) that the "other determinants" z_t shock (ε_{3t}) and the trend growth g_t shock (ε_{5t}) cannot be recovered. Note from (21) that these two shocks define the neutral

[3] LW give the sum of $\alpha_1 + \alpha_2$. The values here come from Table 3 of Buncic (2022).

rate shock. Moreover, the smoothed versions of these shocks are dynamically correlated. Defining $\eta_{jt} = \sigma_j \varepsilon_{jt}$, we have with the trend output y_t^* shock (η_{4t}):

$$\Delta E_T \eta_{5t} = \Delta E_T \eta_{3t} - .028 E_T \eta_{4t}.$$

An updated version of the LW model is given in the recent paper by Holston et al. (2017), which uses a somewhat different formulation of the Phillips curve equation (11), and also estimates the model over a longer sample period. The results HLW present regarding the estimate of the neutral rate r_t^* are, nonetheless, much the same as in the original LW paper. Taking the parameter estimates of HLW's model from Table 3 in Buncic (2022) we find that:

$$\text{diag}(P^*_{t|T}) = \begin{bmatrix} .72 & .02 & .99 & .30 & .97 \end{bmatrix},$$

indicating that the lack of recoverability of the neutral rate in this model is unchanged.

The following identity holds further using the smoothed quantities

$$\begin{aligned} E_T \Delta^2 r_t^* = &-.034 E_T \eta_{1t} - .00023 E_T \eta_{2t} + .00247 E_T \eta_{4t} \\ &+ .0159 E_T \eta_{1t-1} - .00538 E_T \eta_{4t-1}. \end{aligned}$$

That is, whatever $E_T \Delta^2 r_t^*$ is measuring, can be equally explained by the smoothed demand, technology and Phillips curve shocks $\{\eta_{it}\}_{i=1,2,4}$, respectively. Of course, it is also true that

$$E_T \Delta^2 r_t^* = E_T \eta_{5t} + E_T \eta_{3t}.$$

The presence of excess shocks therefore creates interpretation difficulties.

3.3 Recovering the Neutral Real Rate - Schmitt-Grohe and Uribe (2022)

Schmitt-Grohe and Uribe (SGU) (2022) give the following definitions relating to gaps:[4]

$$\tilde{y}_t = y_t - x_t - \delta x_t^r$$
$$\tilde{\pi}_t = \pi_t - x_t^m$$
$$\tilde{\imath}_t = i_t - (1+\alpha)x_t^m - x_t^r,$$

where x_t, x_t^m and x_t^r are permanent output, monetary and real interest rate components. The neutral real rate is then set to

$$r_t^* = x_t^r + \alpha x_t^m.$$

The observation equations are

$$\Delta y_t = \Delta \tilde{y}_t + \Delta x_t + \delta \Delta x_t^r + \sigma_y \varepsilon_t^y \tag{25}$$

$$\Delta \pi_t = \Delta \tilde{\pi}_t + \Delta x_t^m + \sigma_\pi \varepsilon_t^\pi \tag{26}$$

$$\Delta i_t = \Delta \tilde{\imath}_t + (1+\alpha)\Delta x_t^m + \Delta x_t^r + \sigma_i \varepsilon_t^i, \tag{27}$$

where the three shocks ε_t^y, ε_t^π and ε_t^i here are measurement errors. These will be permanent shocks.

Define $\phi_t = \begin{bmatrix} y_t & \pi_t & \tilde{\imath}_t \end{bmatrix}'$ and $\xi_t = \begin{bmatrix} \Delta x_t^m & z_t^m & \Delta x_t & z_t & \Delta x_t^r \end{bmatrix}'$. Then the dynamics for the gaps ϕ_t are captured by the following VAR equation:

$$\phi_t = B\phi_{t-1} + C\xi_t.$$

[4]We thank Tim Robinson for drawing our attention to this paper.

These are augmented by univariate processes for the elements of ξ_t:

$$\begin{aligned}
\Delta x_t^m &= \rho_1 \Delta x_{t-1}^m + \sigma_1 \varepsilon_{1t} \\
z_t^m &= \rho_2 z_{t-1}^m + \sigma_2 \varepsilon_{2t} \\
\Delta x_t &= \rho_3 \Delta x_{t-1} + \sigma_3 \varepsilon_{3t} \\
z_t &= \rho_4 z_{t-1} + \sigma_4 \varepsilon_{4t} \\
\Delta x_t^r &= \rho_5 \Delta x_{t-1}^r + \sigma_5 \varepsilon_{5t}.
\end{aligned} \tag{28}$$

SGU estimate the system parameters by Bayesian methods. Some of the entries of C are fixed at values needed for identification of the parameters. The posterior means of the parameters are provided below.[5]

$$B = \begin{bmatrix} .2627 & .0187 & -.5031 \\ .3129 & .3292 & -.1170 \\ .2268 & -.0977 & .5048 \end{bmatrix}, \ C = \begin{bmatrix} -.0956 & 0 & -.2603 & 1 & -.0051 \\ -.4892 & 0 & .5632 & .8727 & .3651 \\ 1.3964 & 1.0 & -.0309 & .2579 & -.2184 \end{bmatrix}$$

$$\begin{aligned}
&\rho_1 = .2426,\ \rho_2 = 3298,\ \rho_3 = .2619,\ \rho_4 = .4254,\ \rho_5 = .3110 \\
&\sigma_1 = .4824,\ \sigma_2 = .6250,\ \sigma_3 = 1.3624,\ \sigma_4 = 1.0913,\ \sigma_5 = .4723 \\
&\delta = 8.3292,\ \sigma_y = \sqrt{1.2304},\ \sigma_\pi = \sqrt{.4862},\ \sigma_i = \sqrt{.3208}.
\end{aligned}$$

It is should be clear that, to recover r_t^*, one needs to be able to recover ε_{5t}. There are eight shocks and three observed variables, which means that we can only recover three shocks. There are 16 states in total consisting of Δx_t^m, z_t^m, Δx_t, z_t, Δx_t^r, $\tilde{y}_t$, $\tilde{\pi}_t$, $\tilde{\imath}_t$ plus the eight shocks $\left\{ \{\varepsilon_t^y, \varepsilon_t^\pi, \varepsilon_t^i, \{\varepsilon_{jt}\}_{j=1}^5 \} \right\}$.

[5] We thank Martin Uribe for providing these. We put $\alpha = 0$ as we didn't get a mean posterior for that and the median in the paper seems very close to zero.

With the posterior means given above, we can check which shocks can be recovered by following again the set up in Pagan and Robinson (2022) by looking at the last eight entries of the diagonal of the $P^*_{t|T}$ matrix as was done earlier.[6]

$$\mathrm{diag}(P^*_{t|T}) = \begin{bmatrix} .81 & .56 & .59 & .56 & .16 & .91 & .74 & .67 \end{bmatrix}. \tag{29}$$

From (29), it seems that the real rate shock ε_{5t} can be recovered, as the value is close to zero. What happens if $\delta = 0$? Then we find a $\mathrm{diag}(P^*_{t|T})$ for ε_{5t} of .64, which indicates that the shock cannot be recovered. This points to a fundamental role of δ in the recovery of shocks in this model, and we need to more closely examine its impact. Now

$$\begin{aligned} \tilde{y}_t &= b_{11}\tilde{y}_{t-1} + b_{12}\tilde{\pi}_{t-1} + b_{13}\tilde{\imath}_{t-1} + c_{11}\Delta x^m_t \\ &\quad + c_{13}\Delta x_t + c_{14}z_t + c_{15}\Delta x^r_t, \end{aligned}$$

so that

$$\begin{aligned} \Delta\tilde{y}_t &= (b_{11}-1)\tilde{y}_{t-1} + b_{12}\tilde{\pi}_{t-1} + b_{13}\tilde{\imath}_{t-1} + c_{11}\Delta x^m_t \\ &\quad + c_{13}\Delta x_t + c_{14}z_t + c_{15}\Delta x^r_t, \end{aligned}$$

which means that the observable output growth is

$$\begin{aligned} \Delta y_t &= (b_{11}-1)\tilde{y}_{t-1} + b_{12}\tilde{\pi}_{t-1} + b_{13}\tilde{\imath}_{t-1} + c_{11}\Delta x^m_t \\ &\quad + c_{13}\Delta x_t + c_{14}z_t + (c_{15}+\delta)\Delta x^r_t + \sigma_y\varepsilon^y_t \\ &= \eta_t + (c_{15}+\delta)\Delta x^r_t. \end{aligned}$$

[6] In SGU's paper, they report the posterior median of δ to be 8.6 (see page 13), so there is little difference between that value, and the posterior mean value which we use.

Now suppose that $\rho_5 = 0$ in equation (28). Then η_t is uncorrelated with Δx_t^r. Moreover, δ does not affect the variance of this variable. Hence the variance of Δy_t would vary directly with δ, once we set all the other parameters to some values (modes, medians, or means). This gives rise to two interesting observations. First, c_{15} is very small. If it was zero, then the model variance of Δy_t will depend on δ^2. This explains why SGU found that there was some evidence of negative values for δ. Indeed, setting $\delta = 8.3292$ produces standard deviations of Δy_t, Δp_t and Δi_t of $4.67, 1.63$ and 1.32. Putting $\delta = -8.3292$ we get $4.65, 1.63$ and 1.32.

Secondly, the fraction of the variance of Δy_t explained by the fifth shock will rise as δ rises. Thus, when $\delta = 8.3292$ we find that 80% of the variation in GDP growth is due to neutral real rate shocks. This appears to be rather high, since these are shocks that, as Schmitt-Grohe and Uribe (2022, p. 4) write: "*could stem from, for example, secular variations in demographic variables, exogenous changes in subjective discount rates, or in other factors determining the domestic or external willingness to save*". To reduce this influence, we need to reduce the magnitude of δ. Now if $\delta = 2$, then the real neutral interest rate shocks explain 18% of growth. Nonetheless, with that value, the $\text{diag}(P^*_{t|T})$ entry for the fifth shock is .7, indicating that it cannot be recovered. Clearly, there are issue here about whether we have strong opinions about the likelihood of these shocks driving so much of growth, while technology shocks determine so little, which is what a value of $\delta = 8.3292$ implies.

Why does one get such a high value of δ from estimation? Fundamentally, δ is a free parameter that enables the model to better match the data on output growth. To see this, note that the standard deviation of GDP growth is 4.89 in the empirical data. Setting $\delta = 8.3292$ in the model, leads to a model based value of the standard deviation of GDP of 4.67, and this evidently matches the data rather well. Setting $\delta = 2$ instead, gives a standard deviation of GDP growth of 2.37, a poor match. Thus, as δ rises, a larger proportion of output growth is accounted for by the real neutral rate shock, making recovery of shocks easier.

3.4 Recovering the Neutral Real Rate and the NAIRU - McCririck and Rees (2017)

McCririck and Rees (MR) (2017) attempt to determine a number of macroeconomic "stars"; namely, growth in potential GDP, the NAIRU, and the neutral real interest rate denoted by Δy_t^*, u_t^* and r_t^*, respectively. Their model is effectively an extension of LW that augments the model with an equation for Okun's law. The model takes the following format:[7]

$$\tilde{y}_t = \alpha_1 \tilde{y}_{t-1} + \alpha_2 \tilde{y}_{t-2} - \frac{a_r}{2} \sum_{i=1}^{2} (r_{t-i} - r_{t-i}^*) + \sigma_1 \varepsilon_{1t} \tag{30}$$

$$\pi_t = (1 - \beta_1)\pi_t^e + \frac{\beta_1}{3} \sum_{i=1}^{3} \pi_{t-i} + \beta_2 (u_{t-1} - u_{t-1}^*) + \sigma_2 \varepsilon_{2t} \tag{31}$$

$$\Delta z_t = \sigma_3 \varepsilon_t, \tag{32}$$

$$\Delta y_t^* = g_t + \sigma_4 \varepsilon_{4t} \tag{33}$$

$$\Delta g_t = \sigma_5 \varepsilon_{5t} \tag{34}$$

$$\Delta u_t^* = \sigma_6 \varepsilon_{6t} \tag{35}$$

$$u_t = u_t^* + \beta(.4\tilde{y}_t + .3\tilde{y}_{t-1} + .2\tilde{y}_{t-2} + .1\tilde{y}_{t-3}) + \sigma_7 \varepsilon_{7t} \tag{36}$$

$$r_t^* = 4g_t + z_t \tag{37}$$

[7] Note that in MR, g_t rather than g_{t-1} is in the potential GDP growth equation, and the sign of the interest rate variables in the IS equation has changed. Also, for ease of comparability, we use the shock numbering $\{\sigma_j \varepsilon_{jt}\}_{j=1}^{7}$ as in LW, rather than the labelling used in MR.

where $\tilde{y}_t = y_t - y_t^*$ is an output gap, y_t^* is potential GDP, r_t is the real interest rate, r_t^* the neutral real rate, u_t is the unemployment rate and u_t^* the NAIRU, π_t is inflation and π_t^e is measured expected inflation.

Following the format of the analysis we used for LW in section 3.2, we get the first observation equation

$$\zeta_{1t} = y_t^* - \alpha_1 y_{t-1}^* - \alpha_2 y_{t-2}^* + \frac{a_r}{2} \sum_{i=1}^{2} r_{t-i}^* + \sigma_1 \varepsilon_{1t}, \tag{38}$$

where $\zeta_{1t} = y_t - \alpha_1 y_{t-1} - \alpha_2 y_{t-2} + \frac{a_r}{2} \sum_{i=1}^{2} r_{t-i}$ is observable. The second observation equation is:

$$\zeta_{2t} = -\beta_2 u_{t-1}^* + \sigma_2 \varepsilon_{2t},$$

where $\zeta_{2t} = \pi_t - \frac{\beta_1}{3} \sum_{i=1}^{3} \pi_{t-i} - \beta_2 u_{t-1} - (1 - \beta_1)\pi_t^e$ is observable. There is now a third observation equation which is:

$$\zeta_{3t} = u_t^* - \beta(.4y_t^* + .3y_{t-1}^* + .2y_{t-2}^* + .1y_{t-3}^*) + \sigma_7 \varepsilon_{7t},$$

with $\zeta_{3t} = u_t - \beta(.4y_t + .3y_{t-1} + .2y_{t-2} + .1y_{t-3})$.

The state dynamics are given by:

$$\Delta y_t^* = g_{t-1} + \sigma_5 \varepsilon_{5t} + \sigma_4 \varepsilon_{4t} \tag{39}$$

$$g_t = g_{t-1} + \sigma_5 \varepsilon_{5t} \tag{40}$$

$$\Delta r_t^* = 4\sigma_5 \varepsilon_{5t} + \sigma_3 \varepsilon_{3t} \tag{41}$$

$$\Delta u_t^* = \sigma_6 \varepsilon_{6t} \tag{42}$$

This gives a state vector consisting of y_t^*, r_t^*, y_{t-1}^*, r_{t-1}^*, y_{t-2}^*, u_t^*, g_t and the seven shocks

$\{\varepsilon_{jt}\}_{j=1}^{7}$. In MR's model, there are three observables and seven shocks, so that again the full set of seven shocks cannot be recovered. At best, one can recover three shocks. We may, nonetheless, be able to recover three linear combinations of the shocks.

Values of the parameters from MR are as follows:

$$\sigma_1 = .38,\ \sigma_2 = .79,\ \sigma_3 = .22,\ \sigma_4 = .54,\ \sigma_5 = .05,\ \sigma_6 = .15,\ \sigma_7 = .07$$

$$\alpha_1 = 1.53,\ \alpha_2 = -.54,\ \alpha_r = .05,\ \beta_2 = -.32,\ \beta_1 = .39,\ \beta = .62$$

Using these we find that the diagonal elements of $P^*_{t|T}$ are given by:

$$\mathrm{diag}(P^*_{t|T}) = \begin{bmatrix} .54 & .02 & .99 & .24 & .96 & .49 & .76 \end{bmatrix}. \tag{43}$$

So, while there are issues in recovering the NAIRU shock ε_{6t}, the biggest concern is recovering the neutral rate, since the $\mathrm{diag}(P^*_{t|T})$ values corresponding to the two shocks that define r^*_t (ε_{3t} and ε_{5t}) are very close to unity, indicating a lack of recoverability.

There exist again corresponding dynamic correlations between the smoothed shocks in this model, as is visible from the following identities:

$$E_T\eta_{2t} = 86.7E_T\Delta\eta_{6t} + 398E_T\Delta\eta_{7t} \tag{44}$$

$$E_T\Delta\eta_{3t} = -.136E_T\eta_{5t} + .3E_T\eta_{5t-1}. \tag{45}$$

From the relation in (44) it is evident that the two shocks that determine the neutral rate are related via an identity, as are the shocks in the Phillips curve, NAIRU and Okun's law equation.

4 Endogenous Interest Rates: Can this change the outcome?

In the above applications, the interest rate is assumed to be exogenous, ie., there is no equation to explain it's evolution. As Pagan and Wickens (2020) observed, this means that the LW model has some undesirable features. By definition, r_t^* is an $I(1)$ process, but since there is no equation for r_t in LW, there is no mechanism in place to ensure that r_t^* and r_t cointegrate. If they do not cointegrate, then both $\tilde{y}_t$ and inflation will be $I(1)$. Of course the model implies that y_t^* is $I(2)$, so that y_t will be $I(2)$ as well. Evidently, with a monetary rule we may be able to stabilize inflation. To look at this possibility, we add a monetary rule into the LW model. It takes the form of a standard Taylor rule with (interest rate dynamics):

$$i_t - i_t^* = \alpha(i_{t-1} - i_{t-1}^*) + (1-\alpha)(.5\tilde{y}_t + 1.5\pi_t) + \varepsilon_{6t}.$$

Letting $i_t^* = r_t^* + \pi_t$ (with i_t^* being the nominal neutral rate), this yields an equation for r_t of the form:

$$r_t = \alpha r_{t-1} + r_t^* - (1-\alpha)r_{t-1}^* + (1-\alpha)(.5\tilde{y}_t + 1.5\pi_t) + \varepsilon_{6t}.$$

Setting $\alpha = .7$, we find that Δy_t and π_t are $I(0)$. It is then instructive to perform a variance decomposition on the model variables with the model shocks.[8] One finds that, for Δy_t, 77% of the variance is accounted for by the ε_{1t} shock, 18% by ε_{4t}, and only 3% by ε_{5t}. It is also the case that neither Δy_t nor π_t are driven by ε_{3t}, as 94.6% comes from ε_{2t}. So this explains why we found earlier that one could not recover these shocks.

[8] As Pagan and Robinson (2022) show, this is not a variance decomposition of the data in terms of the smoothed shocks, since the latter are dynamically related to each other when there are excess shocks. We have seen the dynamic relations between these shocks already for the LW model.

The above experiment can also be performed on the MR model. Because it was incorporated into the MARTIN model, it is instructive to use the nominal interest rate rule provided therein. The overall format of the model remains the same, with the only exception being that we now have an additional equation to define r_t endogenously through:

$$i_t = .7i_{t-1} + .3(r_t^* + \pi_t - \bar{\pi} - 2(u_t - u_t^*)) - \Delta_2 u_t + 1.19\varepsilon_{8t},$$

so that

$$r_t = .7r_{t-1} - .7\Delta\pi_t + .3r_t^* + .3\bar{\pi} - .6(u_t - u_t^*) - \Delta_2 u_t + 1.19\varepsilon_{8t}.$$

This system then leads to a fourth observable equation:

$$\zeta_{4t} = .3r_t^* + .6u_t^* + 1.19\varepsilon_{8t},$$

where observable $\zeta_{4t} = r_t - .7r_{t-1} + .7\Delta\pi_{t-1} - .3\bar{\pi} + .6u_t + \Delta_2 u_t$ implies that ζ_{4t} is cointegrated with r_t^* and u_t^*. Now there is also an extra shock ε_{8t}.

With these definitions the model can be put into SSF, with ζ_t consisting of the four observables, $\Delta^2 y_t$, Δu_t, $\left\{\pi_t - (1-\beta_1)\pi_t^e - \frac{\beta_1}{3}\sum_{i=1}^3 \pi_{t-i}\right\}$ and Δr_t. Because there are only four observables and eight shocks, we will still not be able to recover all shocks. Taking the parameter values from MR we get:

$$\text{diag}(P_{t|T}^*) = \begin{bmatrix} .54 & .02 & .98 & .24 & .95 & .48 & .76 & .04 \end{bmatrix}. \tag{46}$$

Comparing the relevant entries of $\text{diag}(P_{t|T}^*)$ in (46) to the earlier one in (43), we can see that adding an interest rate rule to the model does not alter the lack of recoverability of the shocks found for the basic MR model. The relevant entries in $\text{diag}(P_{t|T}^*)$ corresponding to ε_{3t} and ε_{5t} do not change materially, and are still very close to unity.

5 Alternative Approaches to Recover r^*

Morley et al. (2022) assume that the real interest rate is $I(1)$. For that reason, they employ a VAR in changes in GDP, the real interest rate and inflation. They use the Beveridge Nelson (BN) formula to compute the long-run equilibrium when there is a measurement error in the data. To simplify the analysis, and to illustrate the lack of recoverability in their model, suppose that there is just one variable, the real interest rate r_t, and that the univariate system corresponding to the VAR they utilize consists of:

$$
\begin{aligned}
r_t &= r_t^* + \varepsilon_{1t} \\
\Delta r_t^* &= \varepsilon_{2t}.
\end{aligned}
$$

From these we can compute:

$$\Delta r_t = \varepsilon_{2t} + \Delta\varepsilon_{1t}. \tag{47}$$

Applying the BN formula defining the permanent component as:

$$
\begin{aligned}
r_t^* &= r_t + \sum_{j=1}^{\infty} E_t \Delta r_{t+j} \\
&= r_t - E_t \varepsilon_{1t},
\end{aligned}
$$

so that when using the BN formula one needs to recover the measurement error from the filtered shocks. Pagan and Robinson (2022), Appendix A.1, show that $E_t\varepsilon_{1t}$ and $E_t\varepsilon_{2t}$ are perfectly correlated in this model, so that the measurement error cannot be recovered.

Morley et al. (2022) proceed by noting that (47) can be written as:

$$\Delta r_t = \eta_t + \alpha\eta_{t-1}$$

and then the BN decomposition gives an estimate of

$$r_t^* = r_t - (1+\alpha)\eta_{t-1}.$$

Now the problem with this analysis is that η_t is a linear combination of all lags of the permanent shock ε_{2t} and the measurement error ε_{1t} (see McDonald and Darroch (1983) and Nelson (1975)), making it impossible to recover two filtered shocks from $E_t\eta_t$.

As a final example, consider the study by Lubik and Matthes (LM) (2015) who estimate a simple TVP-VAR for three variables: the growth rate of real GDP, the PCE inflation rate, and the same real interest rate as in Laubach and Williams (2003). Their proposal is to measure the natural real rate of interest as the (conditional) long-horizon forecast of the observed real rate. In their paper, the chosen time horizon is five years.

To illustrate the issues with their approach here, consider a simpler TVP model for a single equation only, the real interest rate, consisting of:

$$r_t = \rho_t r_{t-1} + \sigma_1\varepsilon_{1t} \tag{48}$$

$$\Delta\rho_t = \sigma_2\varepsilon_{2t}, \tag{49}$$

where ε_{1t} and ε_{2t} are mutually and serially uncorrelated. Suppose we define r_t^* as the prediction of r_t two periods ahead (instead of the five used in LM for simplicity of exposition), that is, $r_t^* = E_t r_{t+2}$. Then, to compute $E_t r_{t+2}$, we construct the following from the relations in (48) and (49):

$$\begin{aligned} r_t^* &= E_t(\rho_{t+2} r_{t+1} + \sigma_1\varepsilon_{1t+2}) \\ &= E_t[(\rho_t + \sigma_2\varepsilon_{2t+2} + \sigma_2\varepsilon_{2t+1})r_{t+1} + \sigma_1\varepsilon_{1t+2}] \end{aligned}$$

$$= E_t[(\rho_t + \sigma_2\varepsilon_{2t+2} + \sigma_2\varepsilon_{2t+1})(\rho_{t+1}r_t + \sigma_1\varepsilon_{1t+1})$$
$$= E_t[(\rho_t + \sigma_2\varepsilon_{2t+2} + \sigma_2\varepsilon_{2t+1})(\rho_t + \sigma_2\varepsilon_{2t+1})r_t)]$$
$$= E_t(\rho_t^2 + \sigma_2^2)r_t. \tag{50}$$

Now in the above, all random variables observed at time t are taken as known, but future ones are unknown and are replaced by their unconditional means of zero, ie., $E_t(\varepsilon_{1t+i}) = 0, \forall i > 0$. It then needs to be recognized that, while r_t is known, ρ_t is not, and the expectation must be conditional on the data. The relation in (50) then leads to a "star" type of estimate of r_t^* having the form:

$$r_t^* = E_t(\rho_t^2)r_t + \sigma_2^2 r_t.$$

The problem then is that $E_t(\rho_t^2)$ is not computed by the Kalman filter and so Lubik and Matthes (2015) did something different. For this case they would define r_t^* as $r_t E_t(\rho_{t+2.})$ not $E_t(\rho_t^2)r_t + \sigma_2^2 r_t$. Now in any TVP VAR we have shocks that would drive the structural equations and shocks that are for the TVPs. There are excess shocks and so there will be linear relations between these filtered quantities, making it again hard to know how to interpret the estimated r^*.

6 Conclusion

The principal concern of this paper is the use of UC models to tackle the estimation of quantities such as the NAIRU, neutral interest rates and long term changes in output. Often, the attitude seems to be that this technology makes fewer assumptions than the older ways of dealing with measuring them, and we have shown that this is not necessarily true. Free lunches are rarely available in econometrics. There is a cost to using these models in that latent variables may not be recoverable, even if the assumptions made about the model and shocks describing them are accurate, and the parameters of the model are known.

Finding recoverable shocks that are consistent with the data and estimating model parameters are two very different questions. Of course at the end of the day we must ask whether shocks need to be recoverable? Because often these are presented in media and policy briefings, and sometimes used as regressors where the assumption that the estimated quantities have the same properties as the assumed shocks, it is an important issue to be aware of. In the examples considered in this paper, it was shown that the estimated quantities do not recover the assumed shocks from the narrative of the model. Instead the estimated shocks are linear combinations of the core recoverable shocks in the model.

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