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SOME ECONOMETRIC ANALYSIS OF CONSTRUCTED BINARY TIME SERIES

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Some Econometric Analysis of Constructed Binary Time Series*

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Contents

*Australian National University. An earlier version was an invited paper to the Canadian Econometric Study Group Meeting, Toronto, September 2004. This paper is largely a summary of work done in a number of papers with a variety of co-authors - Larry Taylor, Jonathan Ohn, David Haugh, James Engel and, especially, Don Harding. The new research was supported by ARC Grant DP0342949.

[1]Material in this section comes from Harding and Pagan (2001)

[2]Material in this section comes from Ohn, Taylor and Pagan (2004).

1 Introduction

Macroeconometric and financial econometric research often feature binary random variables. We will designate such a random variable as S_t, and assume that it takes the values of unity and zero. Such binary random variables arise in a number of ways, although they differ in their origin. Because of this it is useful to distinguish between binary random variables that are *primary* and those that are *secondary* or *constructed*. In the first set one would include most of those that arise in micro-econometrics. If a time series is involved there will generally be a panel of data on whether an individual makes a particular decision. In these cases the binary variable is often thought of as deriving from an underlying *continuous latent variable* (as in the Probit model). Also in this set would be cases where a continuous random variable

[3] Some of the material in this section comes from Haugh, Engel and Pagan (2003).

[4] Material in this section comes from Harding and Pagan (2003b)

- on which there are realizations- depends upon a latent binary random variable. The clearest example of the latter would be Markov Switching (MS) models - Hamilton (1989). In contrast to those cases, this paper is concerned with secondary binary random variables which are constructed from the realizations of a continuous random variable (or variables) y_t. This case does not seem to have been studied much, a notable exception being Kedem (1980). However, as we will try to illustrate, quite a few interesting econometric issues arise when such variables are used in empirical work.

Are there many examples of constructed binary time series? There seem to be quite a few, among which we can mention the following.

1. Cycles in economic activity. Here a series y_t is chosen to represent economic activity and a cycle in it involves expansions, $S_t = 1$, and contractions, $S_t = 0$. In the event that the series y_t represents the level of economic activity then it is the *business cycle* that is being isolated. If a permanent component is taken away from y_t we are investigating the *growth cycle*. In the case of the NBER's dating of the business cycle the variable used for y_t is the equivalent of the log of GDP - see *The NBER's Recession Dating Procedure at http://www.nber.org/cycles/recessions.html.*.

2. Bull and bear markets. The underlying variable here will be some asset price e.g. the Dow-Jones or the S&P500 and similar sets of rules as in dating business cycles can be used to perform the segmentation of history into periods of bull and bear markets.

3 Financial crises. Here a unity indicates that a crisis is occurring while a zero indicates that this is not a crisis period. In this case y_t is generally formed from a set of variables such as exchange rates and international reserve changes -see Eichengreen et al (1985) and Kaminsky and Reinhart (1999).

4. IPO markets are often classified as hot ($S_t = 1$) and cold ($S_t = 0$) depending upon the volume of new offers - see Ibbotson et al. (1994).

5. Commodity and real estate markets are often classified as booms and slumps depending upon movements in the underlying prices.

One could continue on in this vein but as the examples above indicate there are many situations in which binary random variables are constructed from some observed continuous random variable. The prevalence of them raises the issue of why this is such a popular strategy. We might put forward a number of reasons:

1. The S_t may be chosen to emphasize some feature in y_t that is not immediately obvious. An example we will use in this paper is the feature

of U.S. business cycles that expansions are not smooth but generally feature a period of very fast growth. This has also been observed in bull markets- see Pagan and Sussonov (2003). This phenomenon will be documented in a later section and we will examine some statistical models that have been proposed to account for it. These are non-linear models in a sense to be explained later. By segmenting time into periods of expansions, and looking at the behaviour of y_t in these periods,we can see a feature that may not be apparent at first glance.

2. Meaningful to decision makers. Because of the well documented phenomenon of loss aversion it is probably not surprising that decision makers are very sensitive to whether there has been a decline (turning point) in series such as GDP and the S&P. Reactions to such an event from the electorate or clients are often very strong and this has led to great interest in being able to predict these events and to examine their causes. This motivates why one might wish to determine the DGP of the S_t given a known DGP for y_t.

3. Often the S_t are objects of interest. An example would be if one wanted to ask whether cycles are synchronized across sectors or countries. It is often the case that the S_t are summarizing many series and so they represent a succinct way of examining such questions. Section 6 looks at a number of questions relating to synchronization between different stock markets, as well as that between the U.S. stock market and the U.S. business cycle (as identified by the NBER). Because the NBER utilize many series in determining the month that a turning point occurred, it is more convenient to examine the coherence of the two cycles, as measured by their representative S_t, then to try to find correlations with the underlying series that they might have been derived from.

4. Sometimes there may be large short lived movements in Δy_t that can affect statistics based upon Δy_t but which have little effect upon the constructed S_t e.g. the stock market crash of October 1987 and the decline in output during the Great Depression. In this instance one might wish to obtain a more robust measure of some feature using the S_t rather than the Δy_t.

5. There are also many situations in which binary variables are used as inputs into a measure of fit. By far the most common examples

are those assessing predictive success. Thus Pesaran and Timmerman (1992) have a sign test of predictive accuracy which has been used to compare output gap estimates by Camba-Mendez and Rodriguez-Palenzuela (2003).

In the next section we will discuss ways of constructing the S_t from the y_t. We will distinguish two classes of methods for doing this that are referred to as *turning point* and *termination* rules and give some illustrations of these in the contexts distinguished above. The nature of these rules turns out to be very important in the determination of the DGP of S_t, and the latter always needs to be carefully derived from that of y_t, since it is unlikely that the DGPs of y_t and S_t will be the same. In particular, it is rare for a constructed S_t to be *i.i.d.*, as is typically assumed in micro-econometrics, even when the underlying variable y_t is *i.i.d.* Indeed, the failure to make an allowance for this feature is a potential problem with many existing studies using these variables.

Because our focus is upon the S_t it is natural to pay attention to a number of questions arising over its DGP.

- How does S_t relate to S_{t-j} (serial correlation in states)?
- How does $\Pr(S_t|S_{t-1})$ relate to $S_{t-1}, S_{t-2}, ...$(duration dependence in states)?
- How does $\Pr(S_t|S_{t-j}, z_t)$ relate to z_t (continuation of the current state in some outcomes in a variable z_t. This might be Δy_{t-1} but could also be some other variables that are not directly related to y_t.)
- How does z_t relate to S_{t-j} i.e. can S_{t-j} be used in a model for z_t?

As well there are many instances in which we will have a number of S_t e.g. we may have series y_t and x_t from which we derive S_{yt} and S_{xt}. In such circumstances we will be interested in the question of

- How does S_{xt} related to S_{xt} (synchronization of states)?

After detailing how the S_t are constructed, and providing some examples of their use, the following sections then take up the questions above and the econometric methods that can be used to answer them.

2 Constructing the States

The variable S_t is found by *segmenting* a period of time $t = 1, ..., T$ into a history of binary outcomes based upon some underlying continuous random variable. This segmentation requires a *rule* and, depending on what one is studying, this could be classified as either a *turning point* or a *termination* rule. A turning point rule performs the segmentation based upon the location of local maxima and minima in the series y_t. A termination rule is one which prescribes an event which would cause a change in the value of the state S_t. In turn termination rules could either be *non-parametric* or derive from a *parametric* model of y_t.

To give some examples suppose we consider the S_t that define a cycle. Perhaps the simplest definition is what might be termed the *calculus rule*. This says that a peak occurs at time t if $\Delta y_t > 0$ and $\Delta y_{t+1} < 0$. The reason for the name is the result in calculus that identifies a maximum with a change in sign of the first derivative from being positive to negative. A trough (or local minimum) can be found using the outcomes $\Delta y_t < 0$ and $\Delta y_{t+1} > 0$. The states S_t are simply defined in this case as $S_t = 1(\Delta y_t > 0)$. This rule has been popular when y_t is yearly data, see Cashin and McDermott(2002) and Neftci (1984).

When data occurs at (say) the quarterly or monthly frequency one needs to recognize that common usage of a word like "recession" would identify it with a *sustained* decline in the *level* of economic activity i.e. something that lasts for several periods. If one applied the calculus rule there would be too many turning points since the growth rate might switch sign between one period and the next. Visualizing a peak in a series leads one to the idea that a local peak in y_t occurs at time t if y_t exceeds values y_s for $t-k < s < t$ and $t + k > s > t$, where k delineates some symmetric window in time around t. One can define a trough in a similar way. By making k large enough we also capture the idea that the level of activity has declined (or increased) in a sustained way. Of course we need to limit the window in time over which this test is applied when performing the test. It is this simple idea that is the basis of the NBER procedures summarized in the Bry and Boschan (1971) dating algorithm. In that program, designed for the analysis of monthly data, $k = 5$. However, because much analysis is conducted with quarterly data, we will take y_t to be a quarterly series and set $k = 2$ as an analogue. One can make the appropriate substitutions if monthly data are being examined for turning points. We will refer to this rule as the BBQ rule.

The calculus rule can also be formulated as a termination rule by expressing it as

$$\begin{aligned}\Pr(S_t &= 1|S_{t-1}) = \Pr(\Delta y_t > 0) \\ \Pr(S_t &= 0|S_{t-1}) = \Pr(\Delta y_t < 0)\end{aligned}$$

but it is clear that it involves no dependence on the past states. A termination rule that does have such dependence is the extended Okun rule that often appears in the financial press and which can be summarized as

$$\begin{aligned}\Pr(S_t &= 0|S_{t-1} = 1) = \Pr(\Delta y_{t+1} < 0, \Delta y_{t+2} < 0|S_{t-1} = 1) \\ \Pr(S_t &= 1|S_{t-1} = 0) = \Pr(\Delta y_{t+1} > 0, \Delta y_{t+2} > 0|S_{t-1} = 0).\end{aligned}$$

All of the rules above are non-parametric in the sense that they simply look for patterns in the data without making any assumptions about the DGP of y_t. Parametric (model-based) termination rules proceed by working with a parametric model of Δy_t. Perhaps the best known of these arises by assuming that Δy_t is a function of a latent binary variable ξ_t that follows a Markov chain and to then construct a series of binary states using the *MS rule* $\zeta_t = 1[\Pr(\xi_t = 1|F_t) - .5]$, where F_t is a set containing either the past history of the observed random variable Δy_t or perhaps the complete sample of observations - see Hamilton (1989). Of course one could use other parametric models of Δy_t to produce ζ_t e.g. a SETAR model. In all these cases a classification into binary outcomes is produced which is based on whether movements in some function of the Δy_t (and its lags) exceeds a threshold, and the magnitude of the movements involves the parameters of the model.

Each type of rule generates binary random variables but they will not be the same. For this reason we will use the symbol S_t to designate those that come from either a turning point or non-parametric termination rule and reserve ζ_t for those that come from a parametric termination rule. Applied to the same data series y_t the states ζ_t and the states S_t are conceptually distinct but, in practice, they are often quite close. Thus the ζ_t states estimated in Hamilton (1989) with his MS-based termination rule were close to the S_t coming from using the NBER type rules. Harding and Pagan (2003a) looked at the way in which they differed by using some approximations for getting the ζ_t from the history of Δy_t. From that analysis it was clear that the MS rule used a broader information set then the NBER-type turning point rule (in the sense that the latter uses $\{\Delta y_{t\pm j}\}_{j\leq 2}$ whereas the former uses $\{\Delta y_{t\pm j}\}_{j\leq T}$, with downweighting as j rises). Notice that the latent states in the MS model ξ_t are *not the same* as the ζ_t and so $\Pr(\xi_t = 1) \neq \Pr(\zeta_t = 1)$ − failure to recognize this is a common error in many studies that use parametric dating rules. We will focus upon S_t type measures in the lecture but everything said about these holds for the ζ_t type measures. The states S_t are then how one summarizes information on the cycle in y_t and it is possible to use these to investigate questions such as synchronization of business cycles across countries, regions and sectors.

Pagan and Sussonov (2003) provide a set of rules for locating turning points in the equivalent of the S&P500, while Lunde and Timmermann(2000) use a non-parametric termination rule. There have been quite a few adaptions of this approach e.g. to study booms and slumps in commodity markets - Cashin, McDermott and Scott (2002). There is also a literature which uses parametric termination rules e.g. the MS model in Maheu and McCurdy(2000).

Non-parametric termination rules to construct the indicators of financial crises most often involve a consideration of the size of the movements in a combination of a number of series. Thus Eichengren et al. (1995) define a crisis as occurring whether a weighted average of changes in exchange rates, reserves and interest rates exceeds some threshold value. Parametric termination rules have also been applied, mainly based on a MS model e.g. Abiad (2003).

3 Looking at Some Features of Business Cycles and Bull and Bear Markets

We have applied the turning point rule set out in the BBQ program described in Harding and Pagan (2002) to US quarterly GDP data over 1947/1-2002/2 to date the business cycle. Bull and bear market markets are itemized by applying the turning point rule in Pagan and Sossonov (2003), with y_t being the log of monthly data on stock prices, where the latter were found by cumulating the total returns series used by Maheu and McCurdy (2000). We then use this information on the S_t and y_t for each series to highlight some of the features of the cycles.

Our discussion of these features will be assisted by Figure 1 which shows

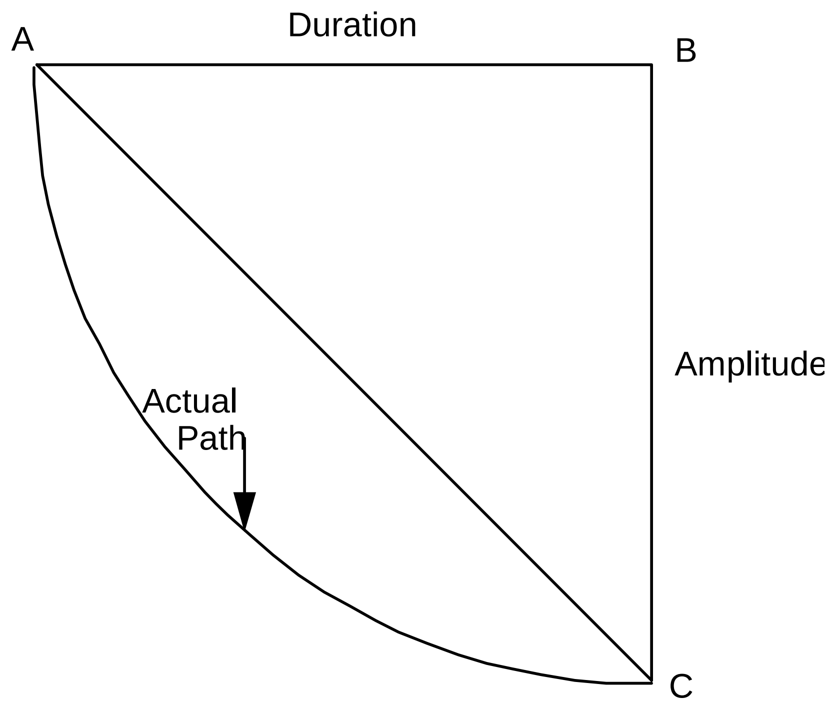

Figure 1:

a stylized contraction. There is an equivalent one for expansions. Here the y axis is y_t (the log of whatever the underlying levels variable Y_t is) and the x axis is time.[5] Thus the graph shows turning points at A (a peak) and C (a trough). The length of AB is the duration of the contraction, and the vertical distance between A and C is the amplitude of it, effectively expressed as a fraction of the peak, since it is the difference $logY_C - \log Y_A$.

Let the turning point dates produce $\hat{K}$ estimated expansions and contractions with the duration of the $i'th$ $(i = 1, ..., \hat{K})$ expansion being D_i^E and that of the contractions being D_i^C. Then, dropping the i subscript, the quantities summarizing durations of the average phases will be

$$\overline{D}^E = \frac{1}{\hat{K}} \sum_{i=1}^{\hat{K}} D_i^E, \overline{D}^C = \frac{1}{\hat{K}} \sum_{i=1}^{\hat{K}} D_i^C.$$

[5] We will refer to y_t as output for convenience, even though that is only the case for the business cycle. Note that the turning points in Y_t are the same as in $\log Y_t$.

Generally these refer to completed durations i.e. if the expansion (contraction) is still on-going at the end (beginning) of the sample period it will not be counted when computing the average. It is also possible to produce the same measures $\overline{A}^E$ and $\overline{A}^C$ for the amplitudes of the phases of the average cycle. Notice that, since the y-axis is the log of output the amplitude A_i^C will be the difference between the log of output at the trough and at the peak. Provided these changes are not too large this difference is the percentage change in output over the contraction. However, as is well known, this approximation breaks down for larger changes, and so it is the case that it will only be approximately correct for bull and bear markets since the amplitudes of these tends to be large. Since we will largely be doing comparative analysis we will just report the changes in the logs and readers should bear in mind that an adjustment would be needed to get the exact change.

It is useful to write the expressions above in terms of the states S_t. Then the number of peaks is $\hat{K} = \sum_{t=1}^{T-1}(1 - S_{t+1})S_t$, since the series $(1 - S_{t+1})S_t$ equals unity only when a peak occurs at time t i.e. when $S_t = 1, S_{t+1} = 0$.[6] Because the total time spent in expansions is $\sum_{t=1}^{T} S_t$ the average duration of an expansion will therefore be

$$\overline{D}^E = \hat{K}^{-1} \sum_{t=1}^{T} S_t.$$

Similarly the average amplitude of expansions will be

$$\hat{A} = \hat{K}^{-1} \sum_{t=1}^{T} S_t \Delta y_t.$$

These expressions show that the random variable $\overline{D}^E$ is the ratio of two random variables. We can write it as

$$\overline{D}^E = (\frac{T}{\hat{K}})\frac{1}{T} \sum_{t=1}^{T} S_t.$$

Now the term $\frac{1}{T}\sum_{t=1}^{T} S_t$ is likely to converge quite quickly to $E(S_t)$, but the random variable $(\frac{T}{\hat{K}})$ may be much less well behaved. In Pagan and Sussonov

[6]Some difficulties can arise with these formulae due to the possibility of incomplete phases at both ends of the sample. If one used *completed* phases the summation should run from the beginning of the first completed phase until the end of the last one rather than over $1, ..., T$.

(2003) simulations showed that the distribution of $\overline{D}^E$ could be quite highly skewed and may have bias. The situation is reminiscent of that which occurs with weak instruments since $\overline{D}^E$ is the ratio of two random variables. But it was also found that the conclusions reached by treating the t ratio as normal tended to indicate the right outcomes.

The graph also shows another feature that we might like to measure. This relates to the cumulated gain or loss during a phase i.e. the area under the curve that describes the actual path for the log of GDP. Basically we might think of approximating the area by summing a series of rectangles of unit length and height $y_j - y_0$. But we know that this overstates the area since the length on one side of the rectangle is $y_j - y_0$ and on the other is $y_{j+1} - y_0$. Hence we correct for this by taking away the triangle that has length unity and height $(y_{j+1} - y_j)$ i.e. $\frac{(y_{j+1}-y_j)}{2}$. Summing over all cycles this latter term is $\frac{A}{2}$. Thus the cumulated loss is measured by (see Harding and Pagan (2002))

$$F = \sum_{j=1}^{D}(y_j - y_0) - \frac{A}{2}.$$

A useful benchmark for the size of F is the area of the triangle ABC ($AR = \frac{D\times A}{2}$) and we term the difference between the two of them, divided by the triangular area, the *excess area*

$$E = \frac{F - AR}{AR}.$$

This shows how much extra output is gained or lost during an expansion or contraction as compared to the situation if the economy had expanded or contracted at a constant growth rate. It is computed for both phases, although it is unlikely to be very reliable for contractions, as these are very short, so that later we sometimes only present it for expansions. It can be regarded as a measure of the *shape* of the cycle. It isn't entirely satisfactory for that as it would be possible for E to be zero if the curve initially went below the hypotenuse of the triangle, but then went above the hypotenuse for the remainder of the contraction. Although the absolute value might be a better measure of "shape", there are advantages in using the version described above.

Table 1 below contrasts the characteristics of expansions and contractions in economic activity with bull and bear markets. There are some obvious differences in the duration and amplitudes of expansions but the value of the "excess" statistics seem quite similar.

Table 1 US Business and Stock Market Characteristics
1947/1-2002/2 (Bus Cycle), 1854/6-1997/5 (stocks)

	B.C. Data	Stocks
Dur Con (quarters)	3	4.4
Dur Expan (quarters)	20	10.1
Amp Con	-2.1	-27.5
Amp Expan	22.0	61.8
Excess Con	.06	-.15
Excess Expan	.09	.10
CV Dur Con	.343	.69
CV Dur Expan	.601	.51
CV Amp Con	-.52	-.90
CV Amp Expan	.56	.53

In both cases expansions don't seem to exhibit a steady rate of growth, implying that, at some stage, there will be a period of rapid growth. Figures 2-5 show some of the typical expansions in the U.S. business cycle and bull markets. It is obviously a challenge to explain this phenomenon. As observed in Harding and Pagan (2002) linear models of the business cycle with symmetrically distributed errors would produce an excess test statistic of around zero, so that one needs to find a model that will produce some asymmetry in growth rates. Non-linear models are one way to do this and quite a few have been proposed for this task in recent years. We look at a typical representative of this literature later.

4 The Second Order Properties of the Binary States

Treating the S_t as a covariance stationary series we will consider its moments. At this point the focus is upon univariate properties and we leave multivariate

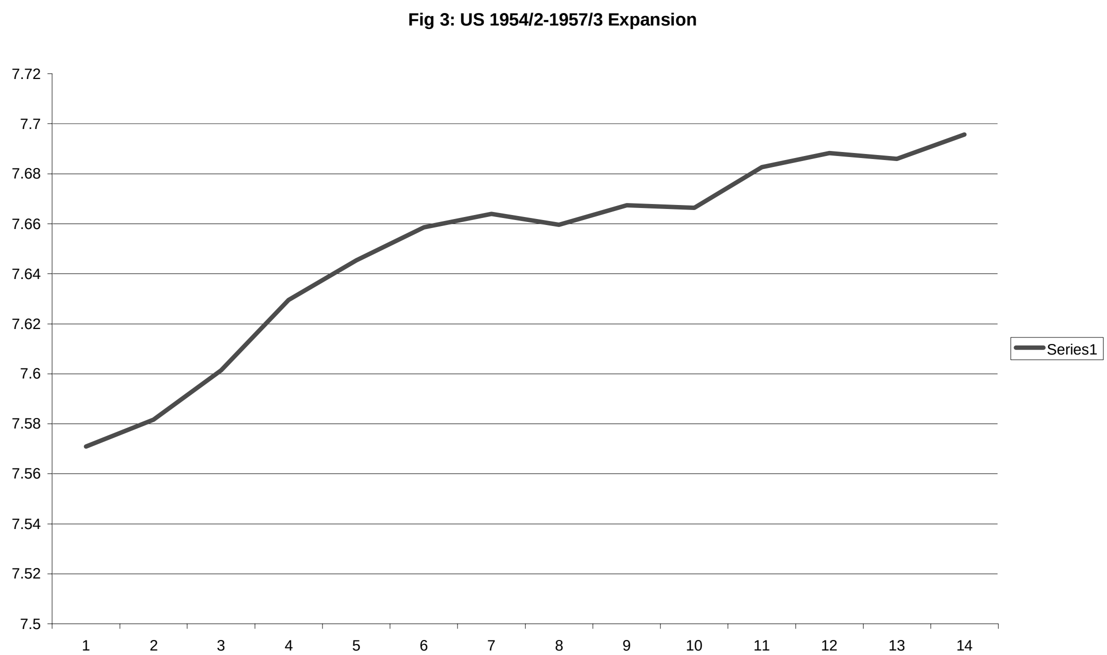
Fig 3: US 1954/2-1957/3 Expansion
7.72
7.7
7.68
7.66
7.64
7.62
7.6
7.58
7.56
7.54
7.52
7.5
1
2
3
4
5
6
7
8
9
10
11
12
13
14
Series1

Figure 2:

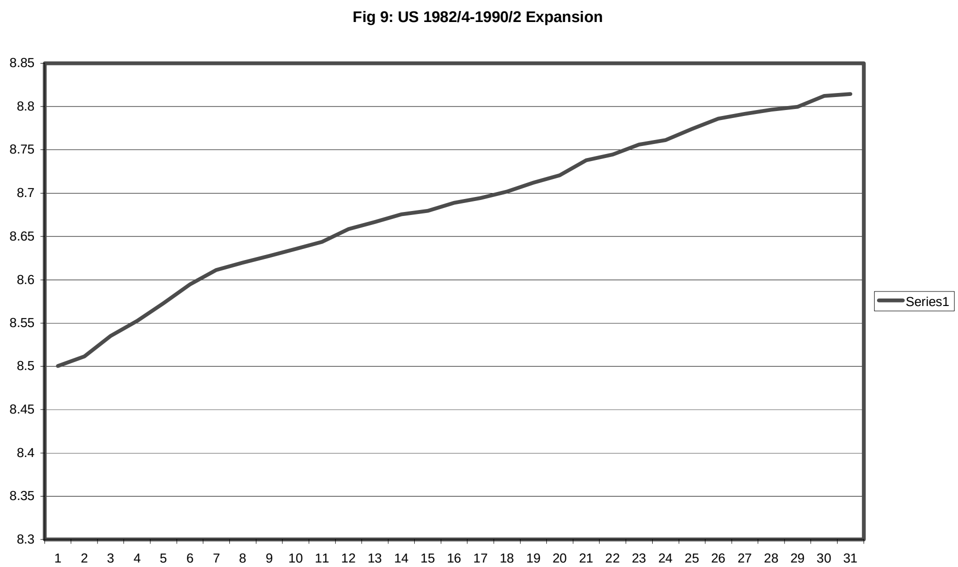

Figure 3:

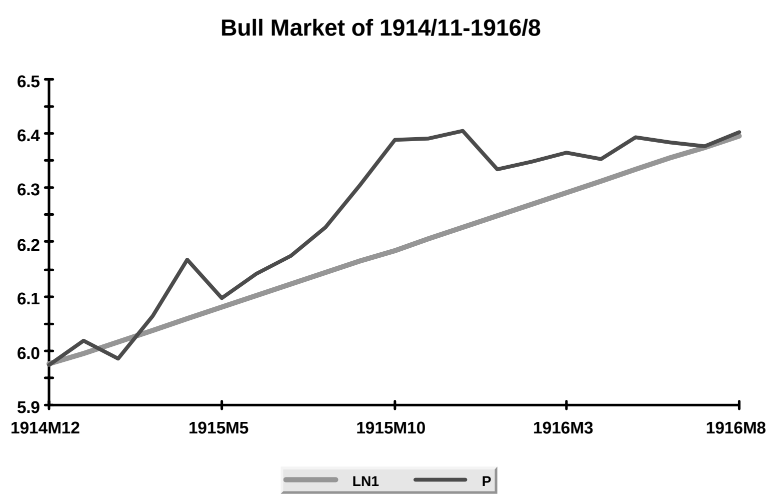

Figure 4:

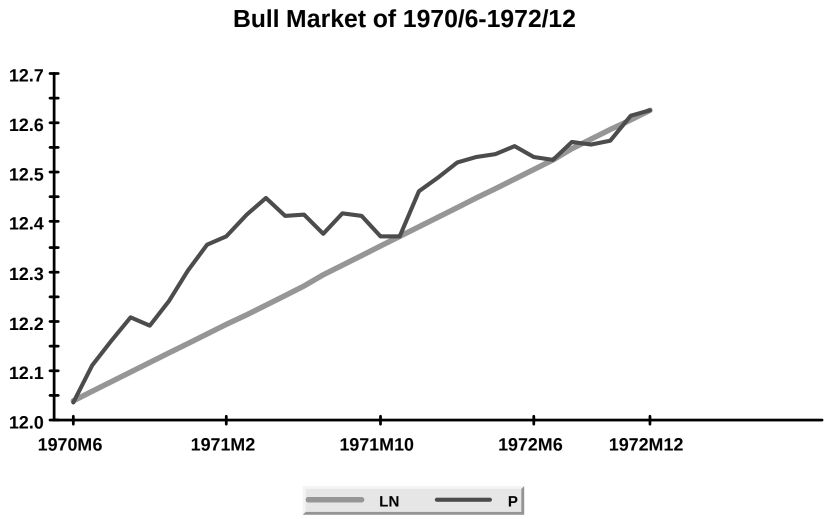

Figure 5:

issues until later.

4.1 Moments

In the univariate case we would be interested in $E(S_t), var(S_t)$ and the a.c.f. of S_t. Of course since

$$\begin{aligned} var(S_t) &= E(S_t^2) - (E(S_t)^2) \\ &= E(S_t) - (E(S_t)^2) \end{aligned}$$

it is clear that the variance has no new information that is not in $E(S_t)$. Hence when there are univariate binary indicators the basic measures that summarize them will be their expected value- which is the fraction of time spent in the state $S_t = 1$ - and the autocorrelation function of the S_t. When there are multivariate series we would clearly be interested in the cross correlations and we will return to this later.

4.2 First Order Serial Correlation Properties[7]

To look at the autocovariances we start with a result in Hamilton (1994 p684) that the evolution of the states can be described through the identity

$$S_t = p_{01} + (1 - p_{01} - p_{10})S_{t-1} + \eta_t, \tag{1}$$

where η_t is discrete and conditionally heteroskedastic since it depends upon S_{t-1} and

$$p_{jk} = \Pr(S_{t+1} = k|S_t = j)$$

The determinants of p_{jk} will depend upon the nature of the DGP for y_t and the type of rule employed to construct S_t. To illustrate this we suppose that

$$\Delta y_t = \mu + \sigma e_t \tag{2}$$

where e_t is $i.i.d(0, \sigma^2)$. Now if the calculus rule is employed i.e. $S_t = 1(\Delta y_t > 0)$,

$$\begin{aligned} p_{10} &= \Pr(S_{t+1} = 0|S_t = 1) \\ &= \Pr(\Delta y_{t+1} < 0|\Delta y_t > 0) \\ &= \Pr(\Delta y_{t+1} < 0) = \psi \end{aligned}$$

due to independence of Δy_t. Hence $p_{01} = 1 - \psi$ and, from (1),

$$S_t = 1 - \psi + (0 \times S_{t-1}) + \eta_t, \tag{3}$$

showing that there is no serial correlation in the states S_t.

Now, what happens if one relaxes the assumption that y_t follows a random walk with drift? Using the calculus rule, combined with Δy_t being a mean-zero stationary Gaussian process, Kedem(1980, p34) sets out the relation between the autocorrelations of the Δy_t and $S(t)$ processes. Letting $\rho_{\Delta y}(k) = corr\left(\Delta y_t, \Delta y_{t-k}\right)$, and $\rho_S(k) = corr\left(S_t, S_{t-k}\right)$, he determines that

$$\rho_S(k) = \frac{2}{\pi} \arcsin\left(\rho_{\Delta y}(k)\right).$$

Thus, given an estimate of $\rho_{\Delta y}(k)$, we can immediately find an estimate of

[7] Material in this section comes from Harding and Pagan (2001)

$\rho_S(k)$ and *vice versa*, making it clear that an AR process for Δy_t will result in a much more complex DGP for S_t than an $AR(1)$.

So the autocovariances in S_t depend upon whether there is serial correlation in Δy_t. But, even if there is no serial correlation in Δy_t, the dating rule itself can induce it into S_t. We can illustrate this with the non-parametric termination rule for business cycles that we earlier called the extended Okun rule, assuming that y_t follows (2). The complications in working with this rule come from the fact that the conditioning event $S_{t-1} = 1$ will place some restrictions upon the signs of past sample paths for $\{\Delta y_t\}$ that are associated with an expansion terminating sequence defining the move from $S_{t-1} = 1$ to $S_t = 0$. For example the sequence

$$\{\Delta y_{t+1}, \Delta y_t, \Delta y_{t-1}, \Delta y_{t-2}, ...\} = \{-, -, -., +, ...\} \tag{4}$$

would be incompatible with $S_{t-1} = 1$, since the negative growth at $t-1$ would match with the negative growth at t, and so the expansion would have been terminated at $t-1$. After enumerating the possible paths one finds that

$$p_{10} = \frac{\psi^2}{(1+\psi)}, p_{01} = \frac{(1-\psi)^2}{2-\psi} \tag{5}$$

$$p_{11} = \frac{1+\psi-\psi^2}{(1+\psi)}, p_{00} = \frac{1+\psi-\psi^2}{2-\psi}. \tag{6}$$

Hence, using (1), we will have

$$S_t = \frac{(1-\psi)^2}{2-\psi} + [1 - \frac{(1-\psi)^2}{2-\psi} - \frac{\psi^2(1-\psi)^2}{1-\psi(1-\psi)}]S_{t-1} + \eta_t \tag{7}$$

To get some feel for the magnitude of the coefficients in this relation assume that Δy_t is $N(\mu, \sigma^2)$, so that $\psi = \Phi(-\frac{\mu}{\sigma})$, where $\Phi(u)$ is the cumulative standard normal distribution function. Using sample estimates of μ and σ^2 for US GDP over the period 1959/1-1997/2 (the same sample as used in the application by Estrella and Mishkin (1998)) gives $\psi = .21$. Inserting this into (7) produces

$$S_t = .35 + .62 S_{t-1} + \eta_t, \tag{8}$$

showing that there is substantial serial correlation in the states. Fitting an AR(1) to the "NBER states" (found from their web page) over the same period yields

$$S_t = .29 + .67S_{t-1} + \eta_t, \tag{9}$$

which shows that the predictions about the nature of the business cycle states identified by the NBER, and those using the extended Okun rule, are quite good.

In fact serial correlation can arise from other constraints imposed when deriving the constructed states. It is not uncommon to impose minimum or maximum lengths to cycle phases or crises when constructing the S_t. Harding and Pagan (2001) show that this itself induces serial correlation in S_t. Thus, if one uses the constraint that recessions are at least two quarters long, the process for S_t must be at least second-order Markov. One can see this in the following regression that uses Euro-area business cycle states that have been constructed with BBQ (which imposes the NBER constraint that phases are at least two quarters long).

$$S_t = .5 + .5S_{t-1} - .5S_{t-2} + .46S_{t-1}S_{t-2} + \eta_t,$$

which, if just an AR(2) was fitted, would be

$$\begin{aligned} S_t &= .41 + \underset{(7.7)}{.77}S_{t-1} - \underset{(2.3)}{.23}S_{t-2} + \eta_t. \end{aligned}$$

These results also continue to hold for the S_t found from the monthly S&P500 using the turning point definitions in Pagan and Sussonov (2003). The regression over 1854/6-1997/12 gives

$$\begin{aligned} S_t &= \underset{(8.1)}{.07} + \underset{(80.5)}{.89}S_{t-1} + \eta_t \end{aligned}$$

So it is very unlikely that there will not be serial correlation in the states S_t. This is important since it means that secondary (constructed) states cannot be treated as if they were primary states. In particular, it will not be correct to assume that they are realizations from an *i.d.* process, as is done in the micro-economics literature and in the derivation of many tests based on these states. An example of the latter comes from Pesaran and Timmerman (1992) who implicitly assume that the S_t have no serial correlation in the derivation of their test statistic i.e. it is used to find the variance of the test statistic. In their context this may be a valid assumption since the y_t are forecast errors and the $S_t = 1(y_t > 0)$. But others have applied it to y_t that are possibly serially correlated e.g. in Camba-Mendez and Rodriguez-Palenzuela (2003) the y_t are the revision errors in output gaps and there is no reason to think that these would be serially uncorrelated. In these instances the results above show that the S_t are serially correlated, so that a literal application of the Pesaran-Timmermann statistic may lead to incorrect conclusions. An adjustment needs to be made for the serial correlation in the S_t. In section 6 we will see an illustration of this which shows that the requisite adjustment can be very large indeed.

4.3 Higher Order Serial Correlation Properties[8]

The discussion above suggests that the S_t may follow much higher order processes than a first order one. Indeed there is a substantial literature about this feature that goes under the heading of duration dependence, in which one seeks to determine if

$$\Pr(S_t = j|S_{t-1} = k, \tilde{S}_{t-2}) = \Pr(S_t = j|S_{t-1} = k),$$

where $\tilde{S}_{t-2} = \{S_{t-2-j}\}_{j=0}^{\infty}$. Rather than leaving the formulation as general as this the $\tilde{S}_{t-2}$ are generally represented by a single variable, the duration of time that has been spent in the same state as S_{t-1}. We will designate this as d_t and, for convenience, analyze expansions, so that $S_{t-1} = 1$.

One of the main themes has been to examine the question of whether the probability of exiting the state of interest depends upon how long one has spent in it. If it does, we say that there is duration dependence. There is quite a bit of empirical work on duration dependence in the business cycle, largely motivated by the question of whether it is possible to predict the termination of a boom or a recession. Fisher (1925) was one of the first investigators to consider this question, raising the issue of whether the probability of exiting any phase of the cycle is just a constant. Thus findings, such as those by Diebold and Rudebusch (1990, 1991), that there is evidence of duration dependence in U.S. business cycles, have attracted quite a bit of attention and are often cited.

Often the questions of duration dependence are analyzed not with the S_t but rather with series that measure the duration of time spent in each phase.

[8] Material in this section comes from Ohn, Taylor and Pagan (2004).

Thus, if we divide the T observations into n phases, the duration of time spent in the i'th phase will be designated as X_i. We can then formally define any duration dependence within a given phase in one of two ways. In the first we focus upon the continuation probability $\Pr(S_t = j|\ S_{t-1} = j, \tilde{S}_{t-2})$ and ask if this probability depends upon the time spent in the phase up to (and including) $t-1$. If it does not then the process will be first order Markov and we will have duration independence. When data are discrete the time spent in the j'th phase up to $t-1$ is just the sum of past S_t, so that it is clear that duration dependence is a statement that $\Pr(S_t = j|S_{t-1} = j, \tilde{S}_{t-2})$ $= f(S_{t-1}, S_{t-2}, ..)$ and does not just depend on S_{t-1}. In the second approach the implications of duration dependence for the density of the X_i are derived and then a comparison would be made of the density of X_i with that expected under duration independence.

Consider a random sample of n observations $(X_1, X_2, ...X_n)$ from a continuous distribution F, such that $F(a) = 0$ for $a < 0$. We assume that the X_i are duration data and they represent the time spent in one of two phases. An example of the latter would be an expansion or contraction of the business cycle. Let the density function of the random variable underlying the duration data be $f(x)$. Then the hazard rate function is defined as

$$h(x) = f(x)/G(x)$$

where $h(x)$ is the hazard (or failure) rate and $G(x) = P(X \geq x)$ is known as the survival function. If there is to be no duration dependence then the hazard rate must be constant i.e. it does not depend on x With discretely measured duration data the density of X_i is a geometric density when there is duration independence. For a geometric density the relation between the mean and variance of the random variables is

$$V(X) - [E(X)]^2 - E(X) = 0. \tag{10}$$

A basic test of (10) comes from testing if $\tau = 0$ in the moment condition.

$$V(X) - E(X)^2 - E(X) - \tau = 0.$$

One can test if $\tau = 0$ using the GMM estimator of τ from this moment condition. This was done by Mudambi and Taylor (1995) and is called the MT test here.

It is also possible to implement a simple regression-based test that is closely related to MT but which emphasizes the S_t rather than the durations X_i. We write (1) as

$$S_t = c_0 + c_1 S_{t-1} + \eta_t \tag{11}$$

and test for duration dependence using the following regression equation

$$S_t = c_0 + c_1 S_{t-1} + c_2 S_{t-1} d_{t-1} + e_t, \tag{12}$$

where e_t is some disturbance, S_t is the state variable which is assigned unity if the observed index is a month of expansion (zero for contraction), and d_t is the number of consecutive months spent in an expansion up and through time t i.e. the duration of the current expansion. The d_t for checking duration dependence in expansions can be constructed using

$$d_t = (1 + d_{t-1}) S_t,$$

while, to find the equivalent for contractions, we replace S_t with $(1 - S_t)$. Testing whether c_2 is zero in (12) is then the basis of our test for duration dependence. We term this the SB test to indicate that it is based upon the states S_t rather than durations. It can be shown that the SB test effectively checks if $V(X) - [E(X)]^2 + E(X)$ is zero. At first sight this seems to be inconsistent with the moment implications of a geometric density given earlier, but the resolution of the difference comes from noting that, by the definition of the binary indicator, the duration of any expansion or contraction must be at least one period. Hence it is not possible to get a duration of zero periods when using the S_t. This suggests that we should check the relationship between the moments when the geometric density is left-censored at unity. Now the censored probability function will be $P(X = x) = (1-p)^{x-1}p$ $(1 \leq x < \infty)$ and therefore $E(X) = \frac{1}{p}, V(X) = \frac{1-p}{p^2}$ so that, in the censored geometric case, the relation between moments is $V(X) - E(X)^2 + E(X) = 0$, agreeing with what SB tests.

The SB test has a number of attractions. First, it focuses directly upon the conditional probabilities and what influences them. Second, it involves a regression and so it is generally easy to explain the outcome to non-specialists. Thirdly, it can be used to examine prediction issues. Finally, since the parameters can be recursively estimated we can study how duration dependence might have changed over time.

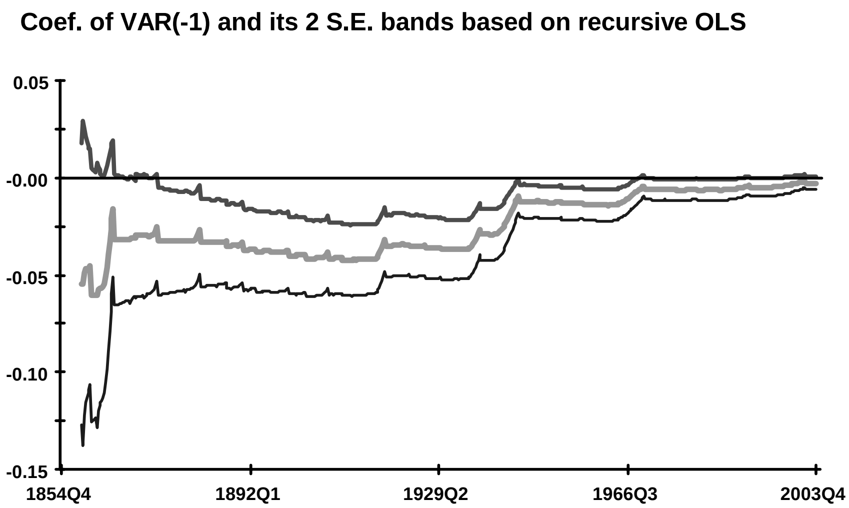

Figure 6:

4.4 Testing for Duration Dependence in Business Cycles and the Stock Market

Data on the US business cycle from the NBER was used to assess whether there was duration dependence in the U.S. cycle. The two test statistics SB and MT yield similar conclusions, viz. that there is evidence for duration dependence in pre-WWII expansions and post-WWII contractions.[9] The t ratio that c_2 is zero is -4.04 in the pre-WWII data and -1.86 in the post war. Recursive estimation of the coefficient associated with the variable $VAR(-1) = S_{t-1}d_{t-1}$ when we attempt to add it on to (11) is given in fig 6, from which it seems clear that duration dependence in expansions has declined a good deal in the post-war world. Notice that the effect is of reasonable size. If we take an estimate of c_2 to be $-.025$ then the probability of staying in an expansion declines according to $-.025 * d_t$ i.e. after one has been in an expansion for 100 months this probability will have been reduced by .25.

We see the same with the S&P500 data. The regression gives an estimated

[9] The results are given in Tables 2-5 of Ohn et al.(2004).

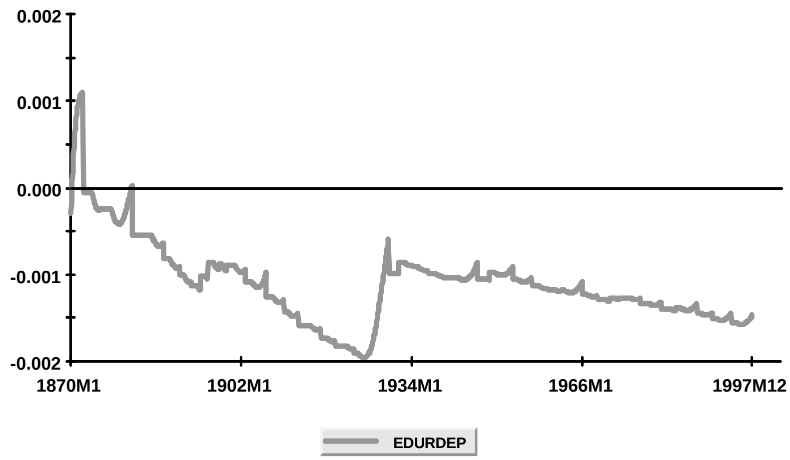

Figure 7:

coefficient of -.001 with a t ratio of -1.65 before WW2 and then a coefficient of -.002 with t ratio of 2.85 after it i.e. positive duration dependence in bull markets seems to have increased over time. In fact the situation seems more complex than this as can be seen from the recursive estimation of the coefficient. The figure suggests that duration dependence was a feature from last century which was interrupted in the first part of this century. It seems useful then to examine regressions involving the states S_t as well as the density of duration statistics, since these recursive results suggest that one cannot assume that the realizations come from a constant density function, which is the standard assumption in micro- econometrics.

5 Relations Between the Constructed States and Other Variables[10]

The transition probabilities found above were unconditional ones associated with the Markov chain nature of the binary states. However, since the states

[10]Some of the material in this section comes from Haugh, Engel and Pagan (2003).

are constructed from y_t, it is generally instructive to examine the transition probabilities conditional upon some function of this variable. In particular, when interest is in turning points, it is natural to condition upon past growth rates Δy_{t-j}. This is a useful diagnostic tool, particularly when it comes to analyzing non-linear models. For this reason we will look at the probability of transiting between states conditional upon such a variable.

5.1 The Transition Between Phases

As seen earlier it is hard to analytically derive the transition probabilities for most dating rules when Δy_t has serial correlation. For this reason we will use the calculus rule in the theoretical analysis below in order to get some idea of how the transition probability would vary with Δy_{t-1} when y_t follows the process

$$\Delta y_t = \mu + \rho \Delta y_{t-1} + \sigma e_t,$$

where e_t is $n.i.d.(0,1)$. Then we can write the conditional transition probability $P(S_t = 0 | S_{t-1} = 1, \Delta y_{t-1})$ as

$$\begin{aligned} \Pr(\Delta y_t &< 0|\Delta y_{t-1}) = \Pr(\mu + \rho\Delta y_{t-1} + \sigma e_t < 0|\Delta y_{t-1}) \\ &= \Pr(e_t < -\alpha - \phi\Delta y_{t-1}|\Delta y_{t-1}) \\ &= \Phi(-\alpha - \phi\Delta y_{t-1}) \end{aligned}$$

where $\Phi(\cdot)$ is the cumulative standard normal.

From the analysis above we would expect that the probability of switching from an expansion to a contraction state depends upon the magnitude of Δy_{t-1} for any dating rule. When there is a small positive growth rate observed during $t-1$ there is a bigger chance of a switch to a contraction phase than if the Δy_{t-1} had been larger. This is sensible due to the positive serial correlation, and it points to the fact that it will be useful to look at how the conditional transition probability will vary with Δy_{t-1} in the data and with different models.

One also has to give some thought to what conditional probability should be analyzed. It is natural to think of $\Pr(S_t|S_{t-1}, \Delta y_{t-1})$, but this may be less useful than it would appear to be, at least when the dating rule is the BBQ one. To see why, suppose we try to find how $\Pr(S_t|S_{t-1} = 1, \Delta y_{t-1})$ varies with Δy_{t-1} when the BBQ rule is used. The fact that $S_{t-1} = 1$ means that the expansion did not terminate in $t-2$. This puts constraints on the relation between y_t, y_{t-1} and y_{t-2}. In particular, suppose that $y_{t-1} < y_{t-2}$ i.e. $\Delta y_{t-1} < 0$. Then it will be necessary that $y_t > y_{t-2}$ to ensure that $S_{t-2} = 1$ i.e. we must have

$$\{\Delta y_{t-1} \quad < \quad 0, \Delta y_t > |\Delta y_{t-1}|\}$$

Now consider what happens if we condition upon a negative value for Δy_{t-1} and $S_{t-1} = 1$. As just seen this means that a positive value for Δy_t is necessary. But, if that happens, it is impossible for $S_t = 0$, since the positive value for Δy_t means that $y_t > y_{t-1}$ and so one cannot satisfy the criterion for a peak at $t-1$ i.e. $S_t \neq 0$. Such an outcome would not arise with the calculus rule. It occurs with the BBQ rule since knowledge about S_{t-1} and Δy_{t-1} implies some knowledge of future growth outcomes. In a prediction environment the simplest way to avoid the constraints which arise from the use of the BBQ determined dates is to work with $\Pr(S_t|S_{t-2}, \Delta y_{t-1})$, since then we are not using any information about Δy_t. Consequently, in later work we use $\Pr(S_t|S_{t-2} = 1, \Delta y_{t-1})$ to compare models of the cycle.

In passing, it should be noted that it has often been said that a turning point rule does not enable one to predict the change in state (unlike when a parametric model based termination rule is used), and the preceding analysis shows that to be a fallacy. Of course it may not be easy to determine the mapping between the conditional probability and Δy_{t-1} when there are more realistic DGP's and dating rules.

We now need to estimate quantities like $\Pr(S_t|S_{t-2} = 1, \Delta y_{t-1})$. Because S_t is a binary random variable it follows that

$$\Pr(S_t|S_{t-2} = 1, \Delta y_{t-1} = y^*) = E(S_t|S_{t-2} = 1, \Delta y_{t-1} = y^*)$$

Consequently, given observations upon S_t, S_{t-2} and Δy_{t-1} we can estimate this conditional expectation using non-parametric methods of analysis. In what follows we will use a simple kernel based approach in which the kernel has the form $K(\frac{S_{t-2}-1}{h_1}, \frac{\Delta y_{t-1}-y^*}{h_2})$ and $K(a,b)=\phi(a)\phi(b)$ where ϕ is the standard normal density and the h_j are window-widths. The window-width h_1 is chosen so that only values of S_t with $S_{t-2} = 1$ are used in computing the conditional expectation while h_2 is proportional to the standard deviation of Δy_{t-1}. Generally we set y^* to all values of Δy_{t-1} in the sample i.e. if there are $T-1$ observations on Δy_{t-1} we use all these values as values for y^*. This means that for the smallest and largest values of Δy_{t-1} in the sample there are very few observations to compute the conditional mean so it is not likely to be very reliably estimated at these boundaries.

5.2 Some Examples

5.2.1 A Business Cycle Application

The models we choose are a linear AR(1) model and a non-linear SETAR model. With y_t being the log of US GDP, an AR(1) in growth rates Δy_t was fitted to data over 1947/1-2002/2 to produce

$$AR(1) : \Delta y_t = .0055 + .3438\Delta y_{t-1} + .0096e_t.$$

Data was simulated from this model by assuming that e_t is $n.i.d.(0, 1)$, and it is then passed through the BBQ program to produce the range of measures summarizing the business cycle discussed earlier. One thousand simulated series were generated.

Various types of SETAR models exist in the literature e.g. Pesaran and Potter (1997), but the version we use here is that set out in van Dijk and Franses (2003). It has a number of extra features compared to other SETAR models in the literature, notably an ARCH effect in the errors. The model then consists of the following set of equations (defining $\Delta y_t = \frac{1}{4}\psi_t$).

$$
\begin{aligned}
\psi_t &= \phi_0 + \phi_1 \psi_{t-1} + \phi_2 \psi_{t-2} + \theta_1 CDR_{t-1} + \theta_2 OH_{t-1} + v_t \\
& \quad v_t \tilde{} N(0, H_t) \\
H_t &= \sigma_F^2 F_{t-1} + \sigma_{COR}^2 COR_{t-1} + \sigma_C^2 C_{t-1} \\
\\
F_t &= 1(\psi_t < r_F) \qquad\qquad\qquad if \quad F_{t-1} = 0 \\
&= 1(CDR_{t-1} + \psi_t < 0) \qquad\quad if \quad F_{t-1} = 1 \\
C_t &= 1(F_t = 0) I(\psi_t > r_C) I(\psi_{t-1} > r_C) \\
\\
CDR_{t-1} &= (\psi_t - r_F) F_t \qquad\qquad\quad if \quad F_{t-1} = 0 \\
&= (CDR_{t-1} + \psi_t) F_t \qquad\quad if \quad F_{t-1} = 1 \\
OH_t &= C_t (OH_{t-1} + \psi_t - r_C) \\
COR &= 1(F_t + C_t = 0) \\
\phi_0 &= 1.52, \phi_1 = .35, \phi_2 = .21, \theta_1 = -.45, \theta_2 = -.041 \\
\sigma_F &= 5.03, \sigma_{COR} = 3.64, \sigma_C = 2.81, r_F = -3.51, r_C = 2.04
\end{aligned}
$$

Simulations of this model were also performed and Table 2 presents business cycle characteristics of the SETAR model as well as other non-linear models. It is clear that the SETAR model is of limited value in generating business cycle features. It makes expansions too strong and produces much the same variability of durations and amplitudes as the linear AR(1) model. The only way in which it improves on the linear model is in terms of the fact that it predicts a cumulated growth from expansions higher than what would happen if growth was constant i.e. it is closer to the shape of actual expansions, although falling well short of that registered by the data.

Table 2 US Business Cycle Characteristics,
Data and Non-Linear Models: 1947/1-2002/2

	Data	SETAR	AR(1)
Dur Con	3	3	3
Dur Expan	20	23	20
Amp Con	-2.1	-1.9	-1.8
Amp Expan	22.0	23.2	20.8
Excess Expan	.09	.04	-.00
CV Dur Expan	.60	.78	.80
CV Amp Expan	.56	.81	.84

Figure 8 looks at the transition probability computation. For comparison we include the AR(1) mode. Although there is little difference between the estimated probabilities from the SETAR and AR(1) models for most values of Δy_{t-1}, when growth is negative the SETAR model has a probability of an expansion continuing that is higher than the linear model i.e. it has a bias towards expansion (something we have already seen in the duration statistics)

In order to understand the results on the transition probabilities it is worth computing $E(\psi_t|\psi_{t-1})$ for the SETAR model and the linear model. These are presented in Figure 9 as a cross plot of $E(\psi_t|\psi_{t-1})$ against the values of ψ_{t-1} in the data set. Also on the same graph are the matched data points. We computed $E(\psi_t|\psi_{t-1})$ using a non-parametric estimator from 30000 simulated data points and a Gaussian kernel. It is apparent that a few extreme observations have had an effect upon the estimates of the SETAR model. The largest negative growth rate in the sample was in 1958/1 at a rate of -2.73%, but it was followed in 1958/2 with a positive growth rate of .58%. As seen in the figure this very rare occurrence becomes a population characteristic of the calibrated SETAR model and, whenever a very large negative growth rate occurs in the simulations, it induces a very rapid "bounce-back" effect.

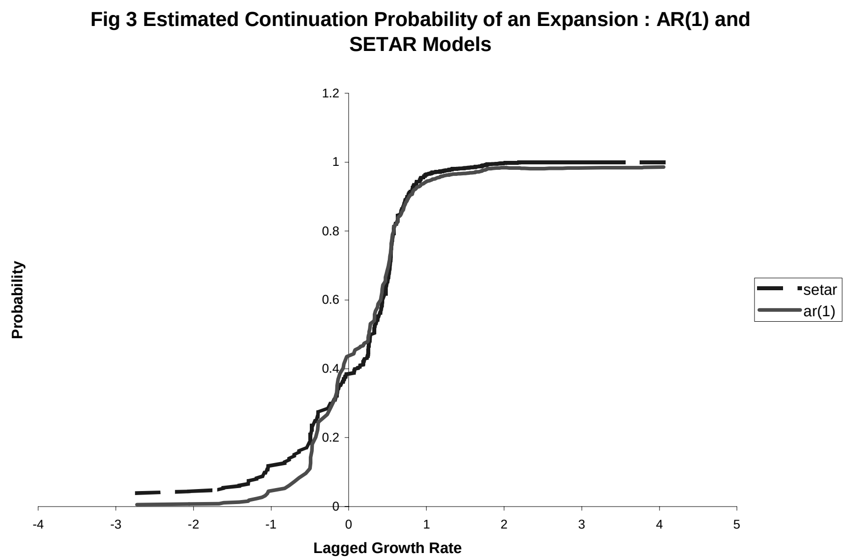

Figure 8:

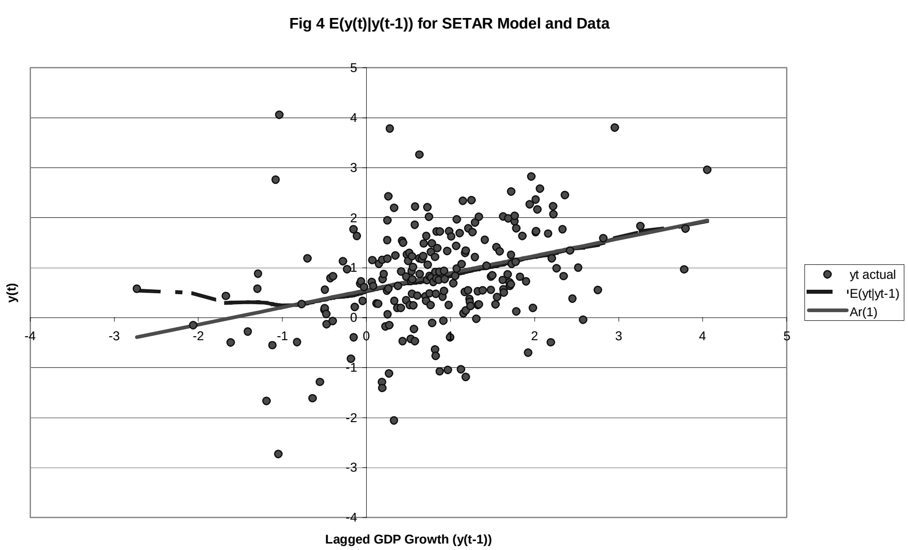

Figure 9:

5.2.2 A Stock Market Application

A similar analysis was performed with the stock market index constructed from the data used by Maheu and McCurdy. In that paper a Markov Switching model with duration dependence in the latent states was fitted to the series (this is the model DDMS-DD in their Table 4). Since it was the preferred model we simulate it to produce the bull and bear market characteristics in Table 3. Also in the table is an EGARCH model that was fitted to the same data. A cursory comparison of the simulated features with those in the data suggests that the DDMS-DD model produces bull market expansions that are too short and too weak and falls far short of capturing the shape of bull markets. The EGARCH model has a slightly better match to all features, but, overall, there is clearly room for improvement in selecting statistical models for the S&P.

It is worth noting that the DDMS-DD model features duration dependence in the states ξ_t of the MS model. Based on $\Pr(\xi_t = 1)$ Maheu and McCurdy state that 90% of the time the market is in a bull state. This contrasts with the conclusion (from the durations of Table 3) that $\Pr(S_t = 1) = .7$ (where S_t are the states found from the turning point rule and which produce Table 3) and so bull markets hold 70% of the time. Of course the difference is due to the fact that the latent states ξ_t have little in common with the S_t. Indeed, one does not need a model with duration dependence in the latent states in order to find duration dependence in the S_t.

Table 3 US Stock Market Characteristics,
Data and Non-Linear Models: 1854/6-1997/12

	Data	DDMS-DD	EGARCH
Dur Bear	4.4	5.3	5.2
Dur Bull	10.1	8.7	8.2
Amp Bull	61.8	44.7	44.7
Excess Bull	.10	.025	.025

5.3 Using the States as Regressors and Regressands

5.3.1 As Regressands

It is not uncommon to see the S_t used as a dependent variable in regressions, either linear or non-linear. To analyse the problems with this we need to allow the transition probabilities to vary over time so that (1) becomes

$$S_t = p_{01t} + (1 - p_{01t} - p_{10t})S_{t-1} + \eta_t, \tag{13}$$

and then it is necessary to ask how p_{01t}, p_{10t} might vary with some observable random variables z_t. One stream of the literature - see Estrella and Mishkin (1998), Birchenall et al (1999) and Chin et al (2000) - ignore the fact that S_t is not *i.d.* i.e. they ignore the relation above, and proceed by modelling $\Pr(S_t) = 1$ using a Probit or Logit framework, in which the single index underlying these models is made a function of some variables z_t. Clearly this will generally give an incorrect likelihood, as it ignores the fact that S_t is a Markov Chain.

It might be asked whether one could replace p_{01t} with a Probit function. This question has two dimensions to it. One involves the choice of functional form for the mapping between p_{01t}, p_{10t} and z_t.The other is what variables should appear in z_t. A Probit function is not an unreasonable choice as a solution to the functional form issue. Choice of variables is much more complex. In the articles above, either contemporaneous values of variables or a finite lag in them was used. But, if the relation is to come from a latent variable model, then one needs to apply the dating rules to the latent variable before observing what values of S_t emerge. As Harding and Pagan (2001) show, if the latent variable depends on z_t, most dating rules would mean that the complete past history of Δz_t would influence p_{01t} and p_{10t}. Thus, in general, it will be very difficult to derive a likelihood for S_t. If one knows the latent variable process, and the dating rule used to construct S_t, it is possible to find the likelihood by simulation methods, but these are very demanding requirements.

Motivated by (13), one possibility is to engage in direct modelling of the relation between S_t and the past history of S_t and z_t. One could use some non-parametric method for this purpose e.g. replace p_{01t} and p_{10t} with some neural netweek or spline functions of z_t. This is not an exact method but it may provide useful results.

5.3.2 As Regressors

There is an emerging tendency to utilize S_t as regressors e.g. in VARs as in Dueker (2005). Whether this is a valid operation depends upon how S_t was constructed. If S_t is constructed using NBER methods then we would have $S_t = f(y_{t+2}, y_{t+1}, y_t, y_{t-1}, y_{t-2})$, and so one could not use S_t as a regressor, since it is highly likely to correlated with the contemporaneous error term in a regression (unless y_t is s strongly exogenous variable). One would need to use S_{t-3} as a regressor to avoid a biassed estimator (or at least to instrument S_t). Indeed, if S_t is to measure a turning point a time t it is inevitable that information at points in time past t will be used in its construction. Difficulties arise if the S_t are just provided with little in the way of a precise account of how they are constructed but, as the example above shows, one needs to treat these variables with great caution.

6 Relations Between Constructed States: Synchronization[11]

As mentioned at the beginning of the lecture we often want to look at the relation between a set of binary random variables e.g. we might want to know if business cycles or crises are synchronized,

6.1 Density Measures for Bivariate Cycles

It is useful to start a discussion of synchronization by concentrating upon the relations between the unconditional densities of two cycles S_{xt} and S_{yt}. It seems natural to define *strong perfect positive synchronization (SPPS)* as the case when the two random variables S_{xt} and S_{yt} are identical. Because of the binary nature of the random variables, necessary and sufficient conditions for this type of synchronization are

$$
\begin{aligned}
(a)\ \Pr(S_{yt} &= 1, S_{xt} = 0) = 0 \quad (14) \\
(b)\ \Pr(S_{yt} &= 0, S_{xt} = 1) = 0. \quad (15)
\end{aligned}
$$

[11]Material in this section comes from Harding and Pagan (2003b)

In the same vein *strong perfect negative synchronization (SPNS)* will obtain when

$$\Pr(S_{yt} = 0, S_{xt} = 1) = 1 \quad (16)$$
$$\Pr(S_{yt} = 1, S_{xt} = 0) = 1. \quad (17)$$

We will couch our discussion in terms of positive synchronization since it is easy to translate the requisite tests to the other case and, in most instances, it is positive synchronization that is of most interest. Cycles that are *strongly non-synchronized (SNS)* might then be regarded as the case when S_{xt} and S_{yt} are independent i.e. the joint probability function for S_{xt} and S_{yt} factorizes into the product of the marginal probability functions.

Because S_{yt} and S_{xt} are binary indicators it is easily seen that the probabilities in (14) and (15) can be expressed as expectations. Doing so yields the following moment conditions that need to hold under the two null hypotheses relating to synchronization mentioned above.

$$SPPS\ (a) \quad : \quad E(S_{yt}(1-S_{xt})) = E(S_{yt}) - E(S_{xt}S_{yt}) = 0 \quad (18)$$
$$SPSS\ (b) \quad : \quad E(S_{xt}(1-S_{yt})) = E(S_{xt}) - E(S_{xt}S_{yt}) = 0 \quad (19)$$
$$SNS \quad : \quad E(S_{xt}S_{yt}) - E(S_{xt})E(S_{yt}) = 0 \quad (20)$$

By subtracting the two conditions in $(SPPS)$ from each other one could get equivalent moment conditions

$$SPPS(i) \quad : \quad E(S_{yt}) - E(S_{xt}) = 0 \quad (21)$$
$$SPPS(ii) \quad : \quad E(S_{xt}) - E(S_{xt}S_{yt}) = 0 \quad (22)$$

These are useful since the first implies that the unconditional densities of S_{xt} and S_{yt} are identical while the second is a property of the conditional density. Indeed we can express $SPPS(ii)$ as

$$\mu_{S_x} - \sigma_{S_x}\sigma_{S_y}\rho_S + \mu_{S_x}\mu_{S_y} = 0, \quad (23)$$

where $\mu_{S_x} = E(S_{xt}), \mu_{S_y} = E(S_{yt})$ and ρ_S is the correlation coefficient between S_{xt} and S_{yt}. When $SPPS(i)$ holds $E(S_{yt}) = E(S_{xt}) = \mu_S$ and $\sigma^2_{S_x} = E(S_{xt})(1-E(S_{xt})) = \sigma^2_{S_y}$, so that (23) becomes

$$(1-\rho_S)\mu_S(1-\mu_S) = 0, \quad (24)$$

which implies that $\rho_S = 1$. Thus when testing perfect synchronization we can test if $\mu_{S_x} = \mu_{S_y}$ and $\rho_S = 1$. Although it is clear that, when $\rho_S = 1$, it has to be the case that $\mu_{S_x} = \mu_{S_y}$, our examples later show the value in performing the tests sequentially, since this is more informative about the reasons for any failure of perfect synchronization. When testing (SNS) we have $\sigma_{S_x}\sigma_{S_y}\rho_S = 0$, and so $\rho_S = 0$ is required. By concentrating upon $\hat{\rho}_S$ we are therefore able to provide a natural measure of the *degree* of synchronization.[12]

The discussion above also leads to the following quantities which might be the basis of test statistics,

$$\begin{aligned} SPPS(i) &: \hat{\mu}_{S_x} - \hat{\mu}_{S_y} \\ SPSS(ii) &: \hat{\rho}_S - 1 \\ SNS &: \hat{\rho}_S \end{aligned}$$

For later reference it should be noted that perfect synchronization between S_{yt} and S_{xt} only occurs when S_{yt} is identical to S_{xt}, and so one could have derived the moment conditions in (21) and (22) directly from that equality. This alternative interpretation is useful when looking at multivariate issues.

Rather than focus directly upon turning points a different way of measuring the degree of synchronization of cycles is to ask what fraction of time the cycles are in the same phase. This *concordance index,* which is the sample analog of $Pr\left(S_{xt} = S_{yt}\right)$, was advocated in Harding and Pagan (2002) and has the form (for two series y_t and x_t and a sample size of T)

$$\hat{I} = \frac{1}{T}\{\sum_{t=1}^{T} S_{xt}S_{yt} + \sum_{t=1}^{T}(1 - S_{xt})(1 - S_{yt})\}. \tag{25}$$

There are close connections between this index and those advanced in the meteorological literature to assess forecast accuracy, see Granger and Pesaran (2000). Artis et al. (1997) and Artis et al. (1999) use a modified version of $\hat{I}$ that is transformed to lie between zero and 100. It is easily seen that $\hat{I}$ is a monotonic function of $\hat{\rho}_S$ and that a value of $\hat{\rho}_S = 1$ corresponds to a concordance index of one, while $\hat{\rho}_S = -1$ implies a concordance index of zero. Hence it is natural to shift attention away from the former to the latter i.e. to focus upon the correlation between the two states S_{xt} and S_{yt}. Consequently, the tests based on $\hat{\rho}_S$ laid out in the previous section will be those employed

[12] Although it may be not the best such measure because the variables being correlated are highly non-normal.

in the paper, although it can sometimes be useful to reinterpret the value of $\hat{\rho}_S$ as a value for $\hat{I}$. [13]

ρ_S is also a useful index of the co-movement of cycles since it determines the extent to which the states S_{xt} and S_{yt} assume the same value at time t i.e. the extent to which one sees clustering of turning points around t. In general ρ_S is a function of the correlation between ΔS_{xt} and ΔS_{yt}, but it also depends upon the first two moments of the DGP of those series. Thus it is a more complete index of the co-movement of cycles than what is provided by the correlation between ΔS_{yt} and ΔS_{xt}.

6.2 Testing Synchronization

In the case of bivariate cycles we have proposed that $SPPS(i)$ be tested by considering whether

$$E(S_{yt} - S_{xt}) = 0.$$

This involves testing if two sample means are equal and is easily done. GMM methods can be employed to produce a robust standard error.

For testing non-synchronization we recommended the correlation between S_{xt} and S_{yt}, ρ_S. To estimate ρ_S we have the moment condition

$$E[\sigma_{S_x}^{-1}(S_{xt} - \mu_{Sx})\sigma_{S_y}^{-1}(S_{yt} - \mu_{Sy}) - \rho_S] = 0 \tag{26}$$

and the estimator generating equation is just

$$\frac{1}{T}\sum_{t=1}^{T} \hat{\sigma}_{S_x}^{-1}(S_{xt} - \hat{\mu}_{Sx})\hat{\sigma}_{S_y}^{-1}(S_{yt} - \hat{\mu}_{Sy}) - \hat{\rho}_S = 0. \tag{27}$$

Since we need to find estimates of the means and variances of S_{xt} and S_{yt} in order to compute $\hat{\rho}_S$, the estimated correlation coefficient is a sequential

[13] A problem with looking at the value of $\hat{I}$ can be seen when when $\rho_S = 0$. Then $E(\hat{I}) = 1 + 2\mu_{S_x}\mu_{S_y} - \mu_{S_x} - \mu_{S_y}$ so that $E(\hat{I}) = .5$ only if $\mu_{S_x} = .5, \mu_{S_y} = .5$. Since μ_{S_x} is the probability of x_t being in an expansion, for the business cycle it is likely that it will be closer to .9 than .5. In that case $E(\hat{I}) \simeq .82$ and so one could easily think that the cycles are synchronized even though there is no relation between them. Of course a policy maker may not be too concerned with that fact, as they may only be interested in the fraction of time that (say) two economies are in the same phase and not the reason for it. But the example points to how what might appear to be a high degree of association between cycles can be quite misleading, as it is simply an artifact of expansions lasting for long periods of time relative to the sample. If one is to use $\hat{I}$ as a test statistic it is necessary to mean correct it, and that is essentially what happens when one uses $\hat{\rho}_S$.

method of moments estimator, to use Newey's (1984) term. The moment condition can be written as

$$E[m_t(\theta, S_{xt}, S_{yt}) - \rho_S] = 0, \tag{28}$$

where $\theta' = [\mu_{Sx}, \sigma_{S_x}, \mu_{Sy}, \sigma_{S_y}]$. Now, because $E\{\frac{\partial m_t}{\partial \theta}\} = 0$ under the null hypothesis that $\rho_S = 0$, the fact that θ has been estimated from the data does not impact upon the asymptotic distribution of $T^{1/2}(\hat{\rho}_S - \rho_S)$.

Testing for the second criterion used in perfect synchronization $(SPPS(ii))$ is a little more complex. When testing (SNS) it would be expected that $T^{1/2}\hat{\rho}_S$ would be asymptotically $N(0, v)$, and so $T^{1/2}\hat{v}^{-1/2}\hat{\rho}_S$ would be $N(0, 1)$ asymptotically. One cannot be entirely precise about the stationarity properties of the states S_{xt}, S_{yt} , since they depend upon the dating rule employed, but, for standard ones, like the NBER rule, these states follow stationary Markov Chains. It is conceivable that there do exist some dating rules for which this might not be true. However, the proposed $SPSS(ii)$ test involves testing on the boundary of the parameter space since $|\rho_S| \leq 1$. There is a literature on the distribution of $T^{1/2}\hat{v}^{-1/2}(\hat{\rho}_S - 1)$ in that case. As Chant (1974) and Andrews (2001) point out it is asymptotically a half normal. Since the series S_{xt} and S_{yt} are serially correlated the value of v will not be unity and will need to be estimated by using a robust covariance estimator. In this scalar case it is simply a matter of doing a one tail rather than two tail test. One could also generate p values numerically from the empirical density of $T^{1/2}\hat{v}^{-1/2}(1(\tilde{\rho}_S < 0)\tilde{\rho}_S - 1)$, where $\tilde{\rho}_S$ are drawn from an $N(1, T^{-1}\hat{v})$ density.

Although method of moments is an obvious way to perform estimation and inference about ρ_S it is often useful to recognize that $\hat{\rho}_S$ can be found from the regression

$$\hat{\sigma}_{S_y}^{-1} S_{yt} = a_1 + \rho_S \hat{\sigma}_{S_x}^{-1} S_{xt} + u_t, \tag{29}$$

since this makes clear difficulties that can arise with some procedures advocated in the existing literature. In particular, the critical role played by their implicit assumption that u_t is *i.i.d.* Thus both the market timing test of Pesaran and Timmermann (1992) and its close relative, Pearson's test of independence in a contingency table (see Artis et al (1997)), effectively make this assumption. Artis et.al. (1997) and Artis et.al. (1999), who work with transformations of the concordance index, derive statistics for independence of cycles that effectively assume the state S_{yt} to be *i.d.*

As one can see from the regression, when the null $\rho_S = 0$ holds, the error term inherits the serial correlation properties of S_{yt}. We have seen that S_{yt}

is strongly positively serially correlated and, as is well known, positive serial correlation sharply increases the chance of rejecting the null that $\rho_S = 0$, unless inferences are made robust to the serial correlation as well as to any heteroskedasticity in the errors, as can be easily done within the method of moments framework. Thus in applications below we report the t ratios for testing if $\rho_S = 0$ using the method of moments estimator and with inferences that do and do not make an allowance for serial correlation and heteroskedasticity. Notice that an advantage of the method of moments approach over the regression model is that we are making no assumptions about which of S_{yt} and S_{xt} are "exogenous".

The regression interpretation is also useful for looking at questions about whether the degree of synchronization has changed over time. It is possible to compute ρ_S recursively and to study its evolution over time. For formal testing of parameter stability one can utilize the methods in Sowell (1996).

6.3 Some Applications

Our two investigations of synchronization of cycles are with stock prices in different countries and the relation between the stock market and business cycles. In this investigation our focus is on the extent to which serial correlation and heteroscedasticity distort inferences about synchronization i.e. how one needs to account for the DGP of the S_t in this task.

6.3.1 Stock prices

Another example of cycles that are possibly synchronized relates to international stock market movements. We examine data on monthly stock price indices for three countries - Australia, the United Kingdom and the U.S. The data sets were used in Pagan (1998) and the rules to determine the phases of the cycles are described there (with a short description for the US data being available in Pagan and Sossonouv (2003)). Two sample periods are provided; from 1875/1- 1997/5 and the "post-WW2" period of 1945/1-1997/5. A striking feature of these data, seen in Table 4 , is that, while the means of the stock states ($\widehat{\mu}_S$) were quite different in the pre-WWII era, they became close in the post-WWII era, and we cannot reject the null hypothesis that they satisfy the necessary condition for perfect positive synchronization in that era. We can however reject the second of the $SPPS$ conditions in Table 5 since the robust t ratio for testing if $\rho_S^{Aus/UK}$ was unity would be 4.2 which, when referred to the half normal density, would provide a strong rejection. Nevertheless, even though not perfectly synchronized, there is strong evidence that the cycles are highly correlated, although the robust t ratios do dampen the strength of this evidence.

Table 4 Evidence on the Necessary Conditions for Perfect Synchronization Across Three Stock Markets					
	$\widehat{\mu}_S$				
	Australia	United Kingdom	United States	W_{PS}	p-value
1875/1-1997/5	0.68	0.56	0.61	9.5	0.009
1945/1-1997/5	0.67	0.64	0.64	0.9	0.65

Table 5 Concordance Indices and Correlations of Cycles In Equity Prices			
	UK/US	Aust/US	Aust/UK
1875/1-1997/5			
$\hat{I}$	0.66	0.61	.70
$\hat{\rho}_S$	0.29	0.16	.39
t	18.8	10.2	24.5
robust t	3.9	2.0	4.9
$\widehat{\gamma}$	1.6	0.4	1.7
1945/1-1997/5			
$\hat{I}$	0.67	0.69	0.79
$\hat{\rho}_S$	0.28	0.33	0.54
t	12.3	14.3	27.0
robust t	2.4	2.7	5.0
$\widehat{\gamma}$	1.1	0.3	2.0

6.3.2 Coherence between Stock Markets and the Business Cycle

We look at the relation between the business cycle and the stock market cycle. The business cycle dates are those taken from the NBER web page (S_{bt}) and the stock market bull and bear market dates (S_{et}) are those coming from the method and data in Pagan and Sussunov (2003) which are recorded in Ohn et al (2004). The data for the underlying equity price is different from that used by Maheu and McCurdy as it is based on the capital gain rather than the total return for the stock in each period.

We begin by comparing $E(S_{bt})$ and $E(S_{et})$. Over the complete period the two are not statistically different with a robust t ratio of 1.15, but in the Post-WW2 period $E(S_{bt}) = .84, E(S_{et}) = .70$, and the robust t ratio that they are the same is 3.09, so that there is a rejection of synchronization in the post-WW2 period, but one could accept it before the second world war. It is interesting to observe that the other component of the synchronization test, the correlation between the states, tells the opposite story. The correlation of the states across the whole period is .27. The standard t ratio that this is zero is 11.7 and the robust t ratio would be 5.0. But splitting the sample as pre and post WW2 it emerges that the correlation is .31 pre-War but half of that post-war. Indeed, the robust t ratio on the latter is just 2.1.

7 Conclusion

We have made the argument that one should never treat the constructed states S_t as *i.d.* random variables since they are very different in their nature to the binary states often modelled in micro-econometrics. One has to allow for the fact that they are essentially Markov Chains when engaging in a a broad range of estimation and inference methods. But, to date, the nature of the S_t has mostly been ignored, with the potential for quite mis-leading estimates and inferences. We have suggested some methods to deal with this fact but have not been able to deal with all of the problems arising in this literature.

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