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Model Uncertainty and Monetary Policy*

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Abstract

Model uncertainty has the potential to change importantly how monetary policy should be conducted, making it an issue that central banks cannot ignore. In this paper, I use a standard new Keynesian business cycle model to analyze the behavior of a central bank that conducts policy with discretion while fearing that its model is misspecified. My main results are as follows. First, policy performance can be improved if the discretionary central bank implements a robust policy. This important result is obtained because the central bank's desire for robustness directs it to assertively stabilize inflation, thereby mitigating the stabilization bias associated with discretionary policymaking. In effect, a fear of model uncertainty can act similarly to a commitment mechanism. Second, exploiting the connection between robust control and uncertainty aversion, I show that the central bank's fear of model misspecification leads it to forecast future outcomes under the belief that inflation (in particular) will be persistent and have large unconditional variance, raising the probability of extreme outcomes. Private agents, however, anticipating the policy response, make decisions under the belief that inflation will be more closely stabilized, that is, more tightly distributed, than under rational expectations. Third, as a technical contribution, I show how to solve an important class of linear-quadratic robust Markov-perfect Stackelberg problems.

Keywords: *Model uncertainty, robustness, uncertainty aversion, time-consistency.*

JEL Classification: E52, E62, C61.

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1 Introduction

It is the nature of models to simplify reality. Unfortunately, this simplification goes hand-in-hand with model misspecification and model uncertainty; it weakens the foundations supporting model-based policy design and poses important challenges for central banks. To what extent does achieving robustness to model uncertainty require a sacrifice in policy performance? How does model uncertainty shape the beliefs that the central bank and private agents hold about future economic outcomes? Does a central bank's concern for model misspecification have a material effect on policy outcomes? These questions have important implications for monetary policy, and although central banks have always had to grapple with them, if not always explicitly, there is relatively little consensus about their answers.

I investigate these questions in the context of a new Keynesian business cycle model in which households, firms, and a central bank reside. The model is typical of those used to analyze monetary policy (Clarida, Galí, and Gertler, 1999) and is similar in spirit, if somewhat simpler, than Christiano, Eichenbaum, and Evans (2005) and Smets and Wouters (2003). Households smooth consumption relative to a habit stock, saving or borrowing against future income through the purchase or sale of one-period nominal bonds. Firms set prices to maximize profits, subject to a Calvo-style adjustment cost and inflation indexation. The central bank conducts policy with discretion, setting the return on the one-period nominal bond to optimize a policy loss function defined over macroeconomic aggregates, such as inflation and consumption.

To introduce model uncertainty, I follow Hansen and Sargent (2008) and assume that the central bank is skeptical of its model, fearing that it may be distorted by specification errors. Thus, the central bank in the economy I analyze designs policy while seeking robustness to unstructured perturbations about its approximating, or reference, model.[1] Importantly, in my analysis the central bank formulates its robust policy while taking into account that the distortions it fears also affect how private agents form expectations, similar to Woodford (2005). Although I employ robust control techniques, I take advantage of results in Hansen, Sargent, and Tallarini (1999) and Hansen, Sargent, Turmuhambetova, and Williams (2006) that relate the multiple models in robust control to the multiple priors in uncertainty aversion (Gilboa and Schmeidler, 1989) to reinterpret the solution to the robust control problem. A connection

[1] At the same time, I recognize that there are other ways to analyze monetary policy with uncertainty. See, for example, Levin, Wieland, and Williams (1999, 2003), Tetlow and von zur Muehlin (2001), Onatski and Stock (2002), Giannoni (2002), Levin and Williams (2003), Onatski and Williams (2003), and Svensson and Williams (2006).

between robust control and uncertainty aversion arises because the worst-case specification errors that emerge in the solution to the robust control problem manifest themselves in the form of worst-case shock processes. These worst-case shock processes can be interpreted as a set of worst-case beliefs, or a worst-case prior over future states, that is distorted relative to rational expectations.

An important result that emerges from my analysis is that robustness need not entail a decline in policy performance. To the contrary, a central bank that implements a robust policy may actually improve policy performance, and not just in extreme, low-probability states of nature, but on average. Although this result may seem surprising on the surface, it has a clear and intuitive explanation. When expectations are rational, time-inconsistency leads to a welfare-lowering stabilization bias in which inflation is understabilized and consumption is overstabilized relative to the Ramsey (commitment) policy (Dennis and Söderström, 2006). To the extent that a fear of model uncertainty directs the discretionary central bank to stabilize inflation more tightly, the desire for robustness can mitigate the stabilization bias and potentially raise welfare. Ordinarily, of course, a concern for reputation (Barro and Gordon, 1983), an optimal contract (Walsh, 1995), the appointment of an optimally conservative central banker (Rogoff, 1985), or the strategic delegation of policy objectives by a *benevolent* authority (Walsh, 2003) is required to improve on discretionary policymaking, but with robustness it is a *malevolent* planner that strategically designs the model (not the policy objectives), and the actions of the *malevolent* planner arise endogenously to reflect not the central bank's desire to raise welfare, but rather its fear of model misspecification.

My analysis also demonstrates that a central bank's fear of misspecification can distort importantly — and asymmetrically — its beliefs about likely future economic outcomes and the beliefs that private agents hold. Thus, where the central bank's worst-case beliefs emphasize the possibility that inflation may be persistent and have a large unconditional variance, anticipating the policy response, private agents' beliefs emphasize that inflation will be more closely stabilized, and more tightly distributed, than under rational expectations. In addition, because the central bank's worst-case beliefs assign greater probability to the tails of the inflation and consumption distributions than rational expectations do, and because outcomes in the tails of these distributions come at a disproportionately high cost, the robust policy responds more forcefully to shocks than the nonrobust policy and generates greater interest rate volatility as a consequence. For this reason, the central bank's fear of model misspecification can have important effects on policy outcomes.

Relatively few papers use robust control to analyze optimal monetary policy, and even fewer focus on discretionary policymaking. Leitemo and Söderström (2005) ask whether a greater desire for robustness makes monetary policy respond more aggressively to shocks and argue that the answer depends on the type of shock and the source of misspecification. Dennis, Leitemo, and Söderström (2006b) analyze robust monetary policy in an estimated small open economy model. They show that distortions to uncovered interest parity are both highly damaging and hard to detect. Giordani and Söderlind (2004) show how linear-quadratic techniques for obtaining time-consistent equilibria in rational expectations models can be applied to robust control problems, techniques that I modify and extend in this paper.[2] My analysis also relates to work by Dennis (2008) and Hansen and Sargent (2008, chapter 16); however, I analyze robust time-consistent policies, whereas they analyze robust Ramsey (commitment) policies.

The remainder of the paper is organized as follows. Section 2 summarizes the new Keynesian business cycle model that I use to study the effects of robustness. Section 3 describes my formulation of the robust Markov-perfect Stackelberg problem and presents its solution. Section 4 describes the connection between the robust Markov-perfect Stackelberg problem that I analyze and uncertainty aversion. Section 5 demonstrates how the central bank's desire for robustness distorts its expectation's operator and thereby influences monetary policy. Section 6 concludes.

2 The model

To illustrate how a fear of model misspecification can affect policy, I use a simple hybrid new Keynesian business cycle model as a laboratory. The model contains equations explaining inflation, π_t, and consumption, c_t, as a function of the short-term nominal interest rate, i_t, and two serially correlated shocks, s_t and d_t, and can be written as

$$\pi_t = \beta(1-\delta)\,\mathrm{E}_t\pi_{t+1} + \delta\pi_{t-1} + \kappa c_t + s_t, \tag{1}$$

$$c_t = (1-\gamma)\,\mathrm{E}_t c_{t+1} + \gamma c_{t-1} - \phi(i_t - \mathrm{E}_t\pi_{t+1}) + d_t, \tag{2}$$

$$s_t = \rho s_{t-1} + \sigma_\varepsilon \varepsilon_t, \tag{3}$$

$$d_t = \tau d_{t-1} + \sigma_\epsilon \epsilon_t. \tag{4}$$

[2] Specifically, unlike Giordani and Söderlind (2004), I assume that the leader fears that private-agent expectations are distorted by model misspecification and that private agents, while taking the leader's robust policy into account, use the approximating model to form expectations and make decisions.

Equation (1) describes a hybrid new Keynesian Phillips curve in which forward-dynamics arise through sticky prices and backward-dynamics enter through inflation indexation. The parameter $\delta \in \left[0, \frac{1}{2}\right]$ governs the importance of forward-looking expectations in price-setting, equaling zero under Calvo-pricing (Calvo, 1983), $\beta \in (0,1)$ represents the subjective discount factor, and $\kappa \in (0,\infty)$, the coefficient on consumption, is a function of the share of firms that set their price optimally each period. Equation (2) summarizes consumption behavior in an environment in which consumers have external habit formation (Abel, 1990). The parameter $\gamma \in \left[0, \frac{1}{2}\right]$ regulates the importance of habits while $\phi \in (0,\infty)$, the coefficient on the ex ante real interest rate, denotes the elasticity of intertemporal substitution. Supply and demand shocks, s_t and d_t, described by equations (3) and (4), respectively, each follow first-order autoregressive processes in which $\{\rho, \tau\} \in (0,1)$ and $\{\sigma_\varepsilon, \sigma_\epsilon\} \in (0,\infty)$, with the innovations $\begin{bmatrix} \varepsilon_t \\ \epsilon_t \end{bmatrix} \sim i.i.d.\,[\mathbf{0}, \mathbf{I}]$.

The short-term nominal interest rate, i_t, serves as the central bank's policy instrument. I assume that the central bank conducts policy with discretion, that it chooses $\{i_t\}_{t=0}^{\infty}$, and that its policy objective function, which it seeks to minimize, takes the form

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\pi_t^2 + \lambda c_t^2 + \mu i_t^2\right], \tag{5}$$

where $\lambda \in [0,\infty)$ and $\mu \in (0,\infty)$.

The monetary policy transmission mechanism largely operates as follows. Through price rigidity a rise in the nominal interest rate raises the ex ante real interest rate, which lowers current period demand as households seek to defer consumption. Responding to lower demand, firms that can change their price moderate their price increase, which damps inflation. Monetary policy also operates through inflation expectations, with higher interest rates lowering inflation expectations and, hence, also current inflation.

Although the model is stylized, its usefulness resides in the fact that it is simple enough to be easily understood, yet rich enough to illustrate the importance robustness plays in shaping policy and economic outcomes. For the simulations in Section 5, I calibrate the model on the basis that the data are observed quarterly. Drawing on an array of studies, but on work by Christiano, Eichenbaum, and Evans (2005), Smets and Wouters (2003), and Dennis (2006) in particular, I set $\beta = 0.99$, $\delta = 0.5$, $\kappa = 0.12$, $\gamma = 0.4$, and $\phi = 0.05$. With respect to the shock processes, I set $\rho = \tau = 0.5$, and $\sigma_\varepsilon = \sigma_\epsilon = 1.0$. In the policy objective function, I set $\lambda = 0.5$ and $\mu = 0.1$. In Section 6, I consider alternative values for these parameters as part of a sensitivity analysis.

3 Robust Markov-perfect Stackelberg problems

In this section I present, in general terms, a linear-quadratic decision problem in which a Stackelberg leader conducts policy with discretion while seeking robustness to model misspecification. After describing the robust Markov-perfect Stackelberg problem, I show how to solve for an equilibrium — the approximating equilibrium—in which the leader employs a policy designed strategically to guard against model misspecification, while the followers, who do not fear model misspecification, make decisions and form expectations using the approximating model, taking the leader's desire for robustness into account.

3.1 The problem

The economy consists of a Stackelberg leader, such as a central bank, a fiscal authority, or, more generally, a government, and one or more followers, such as households, firms, and other private agents. Both the leader and the followers are assumed to share an approximating model[3] that they believe comes closest to describing the economy. According to this approximating model, an $n \times 1$ vector of endogenous variables, $\mathbf{z}_t$, consisting of n_1 predetermined variables, $\mathbf{x}_t$, and n_2 $(n_2 = n - n_1)$ non-predetermined variables, $\mathbf{y}_t$, evolves over time according to

$$\mathbf{x}_{t+1} = \mathbf{A}_{11}\mathbf{x}_t + \mathbf{A}_{12}\mathbf{y}_t + \mathbf{B}_1\mathbf{u}_t + \mathbf{C}_1\boldsymbol{\varepsilon}_{t+1}, \tag{6}$$

$$\mathrm{E}_t\mathbf{y}_{t+1} = \mathbf{A}_{21}\mathbf{x}_t + \mathbf{A}_{22}\mathbf{y}_t + \mathbf{B}_2\mathbf{u}_t, \tag{7}$$

where $\mathbf{u}_t$ is a $p \times 1$ vector of policy control variables, $\boldsymbol{\varepsilon}_t \sim i.i.d.\,[\mathbf{0}, \mathbf{I}_{n_s}]$ is an $n_s \times 1$ $(n_s \leq n_1)$ vector of white-noise innovations, and E_t is the private sector's mathematical expectations operator conditional upon period t information. The matrices $\mathbf{A}_{11}$, $\mathbf{A}_{12}$, $\mathbf{A}_{21}$, $\mathbf{A}_{22}$, $\mathbf{B}_1$, and $\mathbf{B}_2$ are conformable with $\mathbf{x}_t$, $\mathbf{y}_t$, and $\mathbf{u}_t$, as necessary, and contain the structural parameters that govern preferences and technology. The matrix $\mathbf{C}_1$ is determined to ensure that $\boldsymbol{\varepsilon}_t$ has the identity matrix as its variance-covariance matrix.

If the approximating model is known to be correctly specified,[4] then the leader's problem is to choose its control variables $\{\mathbf{u}_t\}_0^\infty$ to minimize

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\mathbf{z}_t'\mathbf{W}\mathbf{z}_t + \mathbf{z}_t'\mathbf{U}\mathbf{u}_t + \mathbf{u}_t'\mathbf{U}'\mathbf{z}_t + \mathbf{u}_t'\mathbf{R}\mathbf{u}_t\right], \tag{8}$$

[3] This terminology follows Hansen and Sargent (2008). Elsewhere, notably Onatski and Williams (2003), Giordani and Söderlind (2004), Dennis, Leitemo, and Söderström (2006a,b) and Tillmann (2007), the term "reference model" is used.

[4] Standard references to the solution of this problem include Oudiz and Sachs (1985) and Backus and Driffill (1986); see Dennis (2007) for an alternative method.

where $\beta \in (0,1)$ is the discount factor and, recall, $\mathbf{z}_t \equiv \begin{bmatrix} \mathbf{x}_t' & \mathbf{y}_t' \end{bmatrix}'$, subject to equations (1) and (2), Markov-perfection, and a known $\mathbf{x}_0$, where the weighting matrices, $\mathbf{W}$ and $\mathbf{R}$, are assumed to be positive semidefinite and positive definite, respectively, and where the pair ($\mathbf{A} \equiv \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$, $\mathbf{B} \equiv \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}$) is assumed to be stabilizable (Kwakernaak and Sivan, 1972, chapter 6).[5]

However, although the approximating model describes most accurately the economy's structure, the leader is skeptical of the model, fearing that it may be misspecified. Moreover, the leader fears that private agents use the distorted model to form expectations.[6] To accommodate its fear, the leader introduces a vector of specification errors, $\mathbf{v}_{t+1}$, and surrounds the approximating model with the class of distorted models

$$\mathbf{x}_{t+1} = \mathbf{A}_{11}\mathbf{x}_t + \mathbf{A}_{12}\mathbf{y}_t + \mathbf{B}_1\mathbf{u}_t + \mathbf{C}_1\left(\mathbf{v}_{t+1} + \boldsymbol{\varepsilon}_{t+1}\right), \tag{9}$$

$$\mathrm{E}_t\mathbf{y}_{t+1} = \mathbf{A}_{21}\mathbf{x}_t + \mathbf{A}_{22}\mathbf{y}_t + \mathbf{B}_2\mathbf{u}_t, \tag{10}$$

where the sequence of specification errors, $\{\mathbf{v}_{t+1}\}_0^\infty$, is restricted to satisfy the boundedness constraint

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \mathbf{v}_{t+1}' \mathbf{v}_{t+1} \leq \eta, \quad \eta \in [0, \overline{\eta}). \tag{11}$$

To guard against the specification errors that it fears, the leader formulates policy subject to the distorted model and takes the position that the specification errors will be as damaging as possible, a position operationalized through the metaphor that $\{\mathbf{v}_{t+1}\}_0^\infty$ is chosen by a fictitious evil agent whose objectives are diametrically opposed to those of the leader. Accordingly, the leader's robust problem is to choose $\{\mathbf{u}_t\}_0^\infty$ to minimize equation (8) and for the evil agent to choose $\{\mathbf{v}_{t+1}\}_0^\infty$ to maximize equation (8), subject to equations (9), (10), and (11) and a known $\mathbf{x}_0$. The evil agent's role in this problem is simply to help the leader devise a robust policy. Following Hansen and Sargent (2008, chapter 2), the way forward is to apply the Luenberger (1969) Lagrange multiplier theorem to replace this constraint problem, involving equation (11), with an equivalent multiplier problem.

Recognizing that the state vector is given by $\mathbf{x}_t$, and that on the stable manifold the non-predetermined variables, $\mathbf{y}_t$, must be a linear function of the state vector, $\mathbf{x}_t$, I conjecture that

[5] Note that although these assumptions about $\mathbf{W}$, $\mathbf{R}$, and the stabilizability of ($\mathbf{A}$, $\mathbf{B}$) are standard for linear-quadratic control problems (see Anderson, Hansen, McGrattan, and Sargent (1996) for example), I do not claim that any of them, specifically the assumptions about $\mathbf{R}$, are either necessary or sufficient for the existence or uniqueness of equilibrium.

[6] Whether the followers actually use the distorted model to form expectations is not the point. The point is that the leader fears that they do and designs its policy accordingly.

the followers' expectations of future non-predetermined variables are given by

$$\mathrm{E}_t \mathbf{y}_{t+1} = \mathbf{H} \mathrm{E}_t \mathbf{x}_{t+1}, \tag{12}$$

where $\mathbf{H}$ is yet to be determined. Substituting equation (12) into equation (10) and combining the resulting expression with equation (9) yields

$$\mathbf{A}_{21}\mathbf{x}_t + \mathbf{A}_{22}\mathbf{y}_t + \mathbf{B}_2\mathbf{u}_t = \mathbf{H}\mathrm{E}_t \left[\mathbf{A}_{11}\mathbf{x}_t + \mathbf{A}_{12}\mathbf{y}_t + \mathbf{B}_1\mathbf{u}_t + \mathbf{C}_1 \left(\mathbf{v}_{t+1} + \boldsymbol{\varepsilon}_{t+1}\right)\right]. \tag{13}$$

Now, given the leader's fear that the followers use the distorted model to form expectations, implying $\mathrm{E}_t \mathbf{v}_{t+1} \neq \mathbf{0}$, and the fact that the specification errors are measurable with respect to period t information, implying $\mathrm{E}_t \mathbf{v}_{t+1} = \mathbf{v}_{t+1}$, equation (13) leads to

$$\begin{aligned} \mathbf{x}_{t+1} &= \overline{\mathbf{A}}\mathbf{x}_t + \overline{\mathbf{B}}\mathbf{u}_t + \overline{\mathbf{C}}\mathbf{v}_{t+1} + \mathbf{C}_1\boldsymbol{\varepsilon}_{t+1}, &(14)\\ \mathbf{y}_t &= \mathbf{J}\mathbf{x}_t + \mathbf{K}\mathbf{u}_t + \mathbf{L}\mathbf{v}_{t+1}, &(15) \end{aligned}$$

where

$$\begin{aligned} \mathbf{J} &= \left[\mathbf{A}_{22} - \mathbf{H}\mathbf{A}_{12}\right]^{-1}\left[\mathbf{H}\mathbf{A}_{11} - \mathbf{A}_{21}\right], &(16)\\ \mathbf{K} &= \left[\mathbf{A}_{22} - \mathbf{H}\mathbf{A}_{12}\right]^{-1}\left[\mathbf{H}\mathbf{B}_{1} - \mathbf{B}_{2}\right], &(17)\\ \mathbf{L} &= \left[\mathbf{A}_{22} - \mathbf{H}\mathbf{A}_{12}\right]^{-1}\mathbf{H}\mathbf{C}_1, &(18)\\ \overline{\mathbf{A}} &= \mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{J}, &(19)\\ \overline{\mathbf{B}} &= \mathbf{B}_{1} + \mathbf{A}_{12}\mathbf{K}, &(20)\\ \overline{\mathbf{C}} &= \mathbf{C}_{1} + \mathbf{A}_{12}\mathbf{L}. &(21) \end{aligned}$$

Equation (14) describes the behavior of the state variables, while equation (15) can be viewed as the reaction function for the aggregate private sector. Notice that this reaction function depends on $\mathbf{v}_{t+1}$ as well as $\mathbf{u}_t$, reflecting the fact that the evil agent is also a Stackelberg leader with respect to the aggregate private sector.

Partitioning $\mathbf{W} = \begin{bmatrix} \mathbf{W}_{11} & \mathbf{W}_{12} \\ \mathbf{W}_{21} & \mathbf{W}_{22} \end{bmatrix}$ and $\mathbf{U} = \begin{bmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \end{bmatrix}$ conformable with $\mathbf{x}_t$ and $\mathbf{y}_t$, I use equation (15) to eliminate the non-predetermined variables from the policy loss function. Following this substitution, recognizing that the value function takes the form $V(\mathbf{x}_t) = \mathbf{x}_t'\mathbf{V}\mathbf{x}_t + d$, writing the optimization problem recursively, and applying Luenberger's Lagrange multiplier theorem, the leader's robust multiplier problem can be written as

$$\mathbf{x}_t'\mathbf{V}\mathbf{x}_t + d \underset{\mathbf{u}_t\ \mathbf{v}_{t+1}}{\equiv_{\min\max}} \mathbf{x}_t'\overline{\mathbf{W}}\mathbf{x}_t + \mathbf{x}_t'\overline{\mathbf{U}}\widetilde{\mathbf{u}}_t + \widetilde{\mathbf{u}}_t'\overline{\mathbf{U}}'\mathbf{x}_t + \widetilde{\mathbf{u}}_t'\overline{\mathbf{R}}\widetilde{\mathbf{u}}_t + \beta\mathrm{E}_t\left(\mathbf{x}_{t+1}'\mathbf{V}\mathbf{x}_{t+1} + d\right), \tag{22}$$

where $\widetilde{\mathbf{u}}_t \equiv \begin{bmatrix} \mathbf{u}_t \\ \mathbf{v}_{t+1} \end{bmatrix}$ and where

$$\begin{aligned} \overline{\mathbf{W}} &\equiv \mathbf{W}_{11} + \mathbf{W}_{12}\mathbf{J} + \mathbf{J}^{'}\mathbf{W}_{21} + \mathbf{J}^{'}\mathbf{W}_{22}\mathbf{J}, & (23) \\ \overline{\mathbf{U}} &\equiv \begin{bmatrix} \mathbf{K}^{'}\mathbf{W}_{21} + \mathbf{K}^{'}\mathbf{W}_{22}\mathbf{J} + \mathbf{U}_1^{'} + \mathbf{U}_2^{'}\mathbf{J} \\ \mathbf{L}^{'}\mathbf{W}_{21} + \mathbf{L}^{'}\mathbf{W}_{22}\mathbf{J} \end{bmatrix}, & (24) \\ \overline{\mathbf{R}} &\equiv \begin{bmatrix} \mathbf{R} + \mathbf{K}^{'}\mathbf{W}_{22}\mathbf{K} + \mathbf{K}^{'}\mathbf{U}_2 + \mathbf{U}_2^{'}\mathbf{K} & \mathbf{K}^{'}\mathbf{W}_{22}\mathbf{L} + \mathbf{U}_2^{'}\mathbf{L} \\ \mathbf{L}^{'}\mathbf{U}_2 + \mathbf{L}^{'}\mathbf{W}_{22}\mathbf{K} & \mathbf{L}^{'}\mathbf{W}_{22}\mathbf{L} - \beta\theta\mathbf{I} \end{bmatrix}, & (25) \end{aligned}$$

$\theta \in [\underline{\theta}, \infty)$, subject to equation (14) and a known $\mathbf{x}_0$. Because the objectives of the leader and the evil agent are perfectly misaligned (they play a zero-sum game), the solution to this minmax problem can be obtained by solving the simultaneous choice problem (Hansen and Sargent, 2008, chapter 7). The first-order condition with respect to $\widetilde{\mathbf{u}}_t$ for the simultaneous choice problem gives rise to the decision rule

$$\begin{aligned} \widetilde{\mathbf{u}}_t &= -\begin{bmatrix} \mathbf{F}^{\mathbf{u}} \\ \mathbf{F}^{\mathbf{v}} \end{bmatrix} \mathbf{x}_t, \\ &= -\mathbf{F}\mathbf{x}_t, & (26) \end{aligned}$$

where

$$\mathbf{F} = \left[\overline{\mathbf{R}} + \beta\overline{\mathbf{B}}^{'}\mathbf{V}\overline{\mathbf{B}}\right]^{-1}\left[\overline{\mathbf{U}}^{'} + \beta\overline{\mathbf{B}}^{'}\mathbf{V}\overline{\mathbf{A}}\right], \tag{27}$$

with the value function and the solution for the non-predetermined variables, $\mathbf{y}_t$, updated according to

$$\begin{aligned} \mathbf{V} &= \beta\left(\overline{\mathbf{A}} - \overline{\mathbf{B}}\mathbf{F}\right)^{'}\mathbf{V}\left(\overline{\mathbf{A}} - \overline{\mathbf{B}}\mathbf{F}\right) + \overline{\mathbf{W}} - \mathbf{F}\overline{\mathbf{U}} - \overline{\mathbf{U}}^{'}\mathbf{F} + \mathbf{F}^{'}\overline{\mathbf{R}}\mathbf{F}, & (28) \\ d &= \frac{\beta}{(1-\beta)} tr\left[\mathbf{C}_1^{'}\mathbf{V}\mathbf{C}_1\right], \end{aligned}$$

and

$$\mathbf{H} = \mathbf{J} - \mathbf{K}\mathbf{F}^{\mathbf{u}} - \mathbf{L}\mathbf{F}^{\mathbf{v}}, \tag{29}$$

respectively. The equations above provide the basis for calculating numerically the worst case equilibrium, the vehicle through which the approximating equilibrium is obtained.

3.1.1 Worst-case equilibrium

In the worst-case equilibrium, reflecting the leader's worst-case fears, the approximating model is misspecified according to the worst-case distortion and the followers use the distorted model to form expectations. To obtain the worst-case equilibrium, I solve for the fix-point of equations (16) through (20), (23) through (25), and (27) through (29); then the law of motion for the state variables, the non-predetermined variables, the leader's control variables, and the worst-case specification errors are given by

$$\mathbf{x}_{t+1} = (\mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{H} - \mathbf{B}_1\mathbf{F}^{\mathbf{u}} - \mathbf{C}_1\mathbf{F}^{\mathbf{v}})\,\mathbf{x}_t + \mathbf{C}_1\boldsymbol{\varepsilon}_{t+1}, \tag{30}$$

$$\mathbf{y}_t = \mathbf{H}\mathbf{x}_t, \tag{31}$$

$$\mathbf{u}_t = -\mathbf{F}^{\mathbf{u}}\mathbf{x}_t, \tag{32}$$

$$\mathbf{v}_{t+1} = -\mathbf{F}^{\mathbf{v}}\mathbf{x}_t, \tag{33}$$

respectively.

3.1.2 Approximating equilibrium

In the approximating equilibrium, although the leader employs its robust decision rule, the approximating model is not misspecified and the followers, who are not robust decisionmakers, form expectations using the approximating model. The first step in obtaining the approximating equilibrium is to solve for the worst-case equilibrium, which supplies the leader's robust decision rule. Then, in a second step, I solve for the fix-point of

$$\widehat{\mathbf{J}} = \left[\mathbf{A}_{22} - \widehat{\mathbf{H}}\mathbf{A}_{12}\right]^{-1}\left[\widehat{\mathbf{H}}\mathbf{A}_{11} - \mathbf{A}_{21}\right], \tag{34}$$

$$\widehat{\mathbf{K}} = \left[\mathbf{A}_{22} - \widehat{\mathbf{H}}\mathbf{A}_{12}\right]^{-1}\left[\widehat{\mathbf{H}}\mathbf{B}_1 - \mathbf{B}_2\right], \tag{35}$$

$$\widehat{\mathbf{H}} = \widehat{\mathbf{J}} - \widehat{\mathbf{K}}\mathbf{F}^{\mathbf{u}}. \tag{36}$$

The solution to this second fix-point problem recovers how the followers' expectations are formed. Then, in the approximating equilibrium, the state variables, the non-predetermined variables, and the leader's control variables are given by

$$\mathbf{x}_{t+1} = \left(\mathbf{A}_{11} + \mathbf{A}_{12}\widehat{\mathbf{H}} - \mathbf{B}_1\mathbf{F}^{\mathbf{u}}\right)\mathbf{x}_t + \mathbf{C}_1\boldsymbol{\varepsilon}_{t+1}, \tag{37}$$

$$\mathbf{y}_t = \widehat{\mathbf{H}}\mathbf{x}_t, \tag{38}$$

$$\mathbf{u}_t = -\mathbf{F}^{\mathbf{u}}\mathbf{x}_t, \tag{39}$$

respectively.[7]

[7] Alternatively, in many instances the approximating equilibrium can be obtained by solving

$$\begin{bmatrix} \mathbf{x}_{t+1} \\ \mathrm{E}_t\mathbf{y}_{t+1} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} - \mathbf{B}_1\mathbf{F}^{\mathbf{u}} & \mathbf{A}_{21} \\ \mathbf{A}_{21} - \mathbf{B}_2\mathbf{F}^{\mathbf{u}} & \mathbf{A}_{22} \end{bmatrix}\begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \end{bmatrix} + \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{0} \end{bmatrix}[\boldsymbol{\varepsilon}_{t+1}]$$

for its rational expectations equilibrium. Where the rational expectations equilibrium is unique, the relationship between $\mathbf{y}_t$ and $\mathbf{x}_t$ and the law of motion for the state variables obtained by solving this equation are given by equations (37) and (38), respectively.

4 Robustness and uncertainty aversion

Using a big "$\mathbf{X}$" little "$\mathbf{x}$" notation, the law of motion for the state variables in the worst-case equilibrium, equation (30), can be re-expressed as

$$\begin{aligned} \mathbf{x}_{t+1} &= \left(\mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{H} - \mathbf{B}_1\mathbf{F}^{\mathbf{u}}\right)\mathbf{x}_t + \mathbf{C}_1\left(\mathbf{F}^{\mathbf{v}}\mathbf{X}_t + \varepsilon_{t+1}\right), \\ \mathbf{X}_{t+1} &= \mathbf{M}\mathbf{X}_t + \mathbf{C}_1\varepsilon_{t+1}, \end{aligned} \tag{40}$$

where $\mathbf{M} \equiv \mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{H} - \mathbf{B}_1\mathbf{F}^{\mathbf{u}} - \mathbf{C}_1\mathbf{F}^{\mathbf{v}}$ and $\mathbf{x}_t = \mathbf{X}_t$ in equilibrium. Thus, the worst-case law of motion for the state variables is one in which the shock processes appear distorted, with their conditional mean twisted, or slanted, relative to the approximating model.

Equation (40) suggests a connection between robust control and the maxmin expected utility framework developed by Gilboa and Schmeidler (1989) to describe behavior they refer to as uncertainty aversion.[8] Gilboa and Schmeidler (1989) assume that beliefs about the likelihood of future states are so vague that they are represented by a set of prior densities rather than by a single prior density. The relationship between uncertainty aversion and robust control is considered in Hansen, Sargent, Turmuhambetova, and Williams (2006), who document conditions under which the multiple models in the robust control framework is behaviorally equivalent in the equilibrium to the multiple priors in the Gilboa and Schmeidler (1989) framework.

The arguments in Hansen, Sargent, Turmuhambetova, and Williams (2006) suggest that the solution to the robust control problem described in Section 3.1 can be obtained equivalently by solving the problem in which the Stackelberg leader chooses $\{\mathbf{u}_t\}_0^\infty$ to minimize, and an evil agent chooses point-wise the probabilities in the probability density function, $p^*\left(\mathbf{x}_{t+1}|\mathbf{x}_t\right)$, associated with the expectations operator, E_t^*, to maximize

$$\mathrm{E}_t^* \sum_{t=0}^{\infty} \beta^t \left[\mathbf{z}_t^{'}\mathbf{W}\mathbf{z}_t + \mathbf{z}_t^{'}\mathbf{U}\mathbf{u}_t + \mathbf{u}_t^{'}\mathbf{U}\mathbf{z}_t + \mathbf{u}_t^{'}\mathbf{R}\mathbf{u}_t\right], \tag{41}$$

subject to

$$\mathbf{x}_{t+1} = \mathbf{A}_{11}\mathbf{x}_t + \mathbf{A}_{12}\mathbf{y}_t + \mathbf{B}_1\mathbf{u}_t + \mathbf{C}_1\varepsilon_{t+1}, \tag{42}$$

$$\mathrm{E}_t^*\mathbf{y}_{t+1} = \mathbf{A}_{21}\mathbf{x}_t + \mathbf{A}_{22}\mathbf{y}_t + \mathbf{B}_2\mathbf{u}_t, \tag{43}$$

Markov-perfection, and a known $\mathbf{x}_0$. In addition, the difference between the distorted conditional probability density function, $p^*\left(\mathbf{x}_{t+1}|\mathbf{x}_t\right)$, and the rational expectations conditional

[8] See also Epstein and Wang (1994), who extend the Gilboa and Schmeidler (1989) analysis to intertemporal models.

probability density function, $p(\mathbf{x}_{t+1}|\mathbf{x}_t)$, is constrained to satisfy

$$\beta\sum_{t=0}^{\infty}\beta^t\left[\int\limits_{\mathbf{x}_{t+1}} p^*(\mathbf{x}_{t+1}|\mathbf{x}_t)\ln\left(\frac{p^*(\mathbf{x}_{t+1}|\mathbf{x}_t)}{p(\mathbf{x}_{t+1}|\mathbf{x}_t)}\right)d\mathbf{x}_{t+1}\right]\leq\omega,\quad \omega\in[0,\overline{\omega}), \tag{44}$$

where ω in equation (44) plays the same role as η in equation (11).[9] Of course, it must also be the case that $\int\limits_{\mathbf{x}_{t+1}} p^*(\mathbf{x}_{t+1}|\mathbf{x}_t)\,d\mathbf{x}_{t+1}=1, \forall\ \mathbf{x}_t\in\Re^{n_2}$. Equation (44) is a (discounted) relative entropy condition (Kullback and Leibler, 1951), in which the expectation of a (log-) likelihood ratio is taken with respect to a distorted probability density.

The connection between robust control and uncertainty aversion suggests an alternative interpretation of the worst-case equilibrium. In an important sense, the worst-case equilibrium can be viewed as a tool, or as a vehicle, for generating the worst-case prior density, with decisions then made in view of this worst-case prior density. More generally, the connection between robust control and uncertainty aversion facilitates analyzing robust control problems in terms of the effect a fear of model uncertainty has on the beliefs held by the various agents residing in the model. For this reason, it is the properties of the probability density functions that underlie the rational expectations equilibrium, the worst-case equilibrium, and the approximating equilibrium that I characterize and discuss when I analyze the new Keynesian business cycle model.

5 Robust monetary policy

In this section I apply the tools developed above to the hybrid new Keynesian model summarized in Section 2, analyzing and exploring the effect a central bank's desire for robustness can have on expectations, monetary policy, and the broader economy. I solve the central bank's robust decision problem, examine the nature of the specification errors that it fears, and document how these specification errors distort the expectation operators that the central bank and private agents use to form expectations. Following Section 3.1, I consider the case where the central bank fears that private agents use the distorted model to form expectations. I report the worst-case shock processes, I show how robustness distorts the expectation operators that agents use, and I document the relationship between robustness and policy loss.

[9] In the approximating model, and hence in the worst-case equilibrium, the number of innovations, n_s, will generally be less than n. With the state vector, $\mathbf{x}_t$, consisting of shocks, $\mathbf{s}_t$, and predetermined variables, $\mathbf{p}_t$, $p(\mathbf{x}_{t+1}|\mathbf{x}_t)$ is given by $p(\mathbf{x}_{t+1}|\mathbf{x}_t)=|\mathbf{D}|\,p(\mathbf{s}_{t+1}|\mathbf{x}_t)$, where $\mathbf{D}$ is a Jacobian of transformation. The solution to the robust control problem provides the Jacobians relevant for the worst-case equilibrium and the approximating equilibrium.

Before introducing robustness, it is useful to construct a benchmark by solving the nondistorted problem in which all expectations are formed rationally. For the parameterization provided earlier, the central bank's optimal discretionary policy can be described by the state-contingent decision rule[10]

$$i_t = 5.180 s_t + 6.133 d_t + 0.937 \pi_{t-1} + 0.800 c_{t-1}. \tag{45}$$

The optimal policy is to raise the nominal interest rate in response to adverse supply shocks and stimulatory demand shocks, thereby mitigating their contemporaneous impact on inflation and consumption, and to tighten policy in response to (past) higher inflation and consumption, thereby returning the economy to steady state more quickly. A notable feature of equation (45) is that its feedback coefficients are large, revealing aggressive policy responses even under rational expectations.[11]

5.1 Robustness

To introduce robustness a value for θ must be provided. Following standard practice, I set θ to generate a particular detection error probability, here 0.1.[12] As described earlier, the central bank designs policy fearing that its approximating model is misspecified and fearing, further, that private agents use the misspecified model to form expectations. For their part, private agents are not concerned about model misspecification, and, in the approximating equilibrium, their expectations are formed using the approximating model.

Applying the solution method developed in Section 3.1, the worst-case shock processes are summarized by equations (46) and (47):

$$s_{t+1} = 0.545 s_t + 0.038 d_t + 0.007 \pi_{t-1} + 0.004 c_{t-1} + \varepsilon_{t+1}, \tag{46}$$

$$d_{t+1} = 0.041 s_t + 0.544 d_t + 0.007 \pi_{t-1} + 0.005 c_{t-1} + \epsilon_{t+1}. \tag{47}$$

[10] Equilibrium is given by the set $\{\mathbf{F}, \mathbf{H}, \mathbf{M}, \mathbf{N}, \mathbf{V}\}$, summarizing the decision rules for the central bank and private agents, the law of motion for the state vector, and the matrix in the value function. However, I note in passing that all of the policy rules reported in this paper are implementable. In other words, if the central bank were to implement policy according to $\mathbf{F}$ and private agents were allowed to reform expectations, then in the unique stable rational expectations equilibrium, private-agent decision rules are governed by $\mathbf{H}$ and the law of motion for the state vector is governed by $\mathbf{M}$ and $\mathbf{N}$.

[11] These feedback coefficients, particularly on the shocks, depend importantly on the interest rate stabilization parameter, μ, and would be smaller were μ larger.

[12] Informally, a detection error probability is the probability that an econometrician observing equilibrium outcomes would infer incorrectly whether the approximating equilibrium or the worst-case equilibrium generated the data. Descriptions of how detection error probabilities can be calculated can be found in Hansen, Sargent, and Wang (2002), Hansen and Sargent (2008, chapter 9), Dennis (2005), Dennis, Leitemo, and Söderström (2006a,b), and Cateau (2006). Only Dennis, Leitemo, and Söderström (2006a,b) treat the case where the specification errors distort both the conditional mean and the conditional volatility of the shock processes.

These worst-case shock processes can be interpreted two ways. One interpretation is that they convey information about the location and behavior of specification errors that the central bank should be concerned about. According to this interpretation, the central bank is concerned that the demand and supply shocks may exhibit greater serial correlation than the approximating model asserts, that the demand and supply shocks might be correlated, and that the Phillips curve and the consumption Euler equation may omit terms involving lags of consumption and inflation. By revealing aspects of the model's structure to which monetary policy is particularly sensitive, the worst-case shock processes indicate areas of the model to which the central bank should devote resources to ensure that the specification is appropriate.

An alternative interpretation is that the worst-case shock processes reveal how the central bank's expectation operator is twisted, or slanted, by its fear of misspecification. According to this interpretation, the probability density function associated with the worst-case equilibrium describes how the central bank forms expectations and how it fears that private agents form expectations. Similarly, the probability density function associated with the approximating equilibrium describes how households and firms form expectations in the absence of misspecification, potentially differing from rational expectations through the influence of the central bank's robust policy. Note that the probability density function that the central bank employs does not coincide with the economy's data generating process in the approximating equilibrium, reflecting the central bank's enduring pessimism about its model. This interpretation exploits the connection between robust control and uncertainty aversion, and focuses attention on the probability density functions that underlie beliefs and expectation formation. It is the properties of these beliefs, represented by probability density functions, that I characterize and discuss in the remainder of this section.

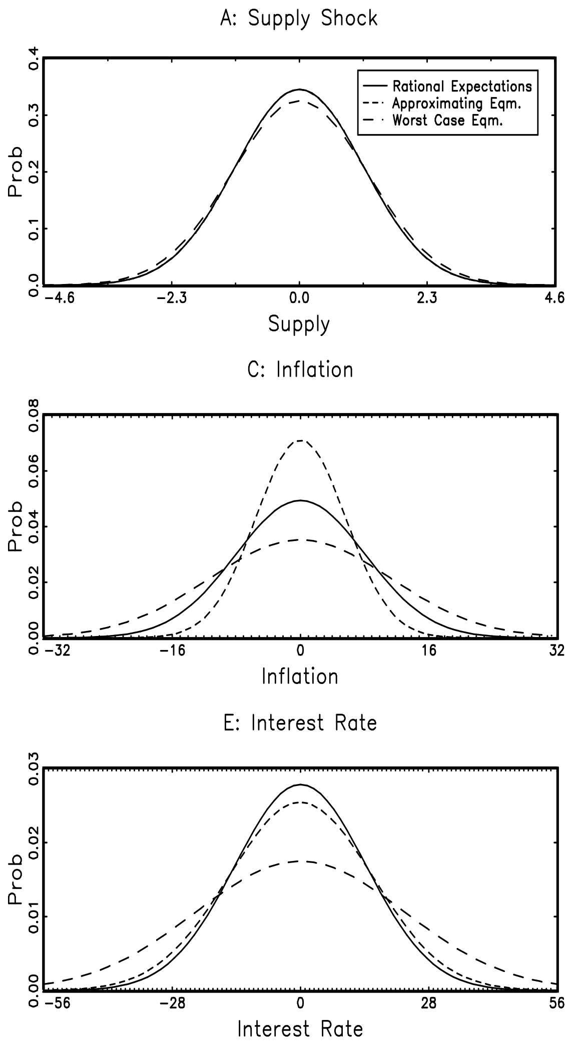

Fig. 1: Undistorted and distorted unconditional probability densities

Figure 1 displays the unconditional distributions of the supply shock, the demand shock, inflation, consumption, and the interest rate in the rational expectations equilibrium, the worst-case equilibrium, and the approximating equilibrium, under the assumption that the innovations, ε_t and ϵ_t, are joint $n.i.i.d.\,[\mathbf{0},\mathbf{I}]$ distributed.[13] Relative to rational expectations, the worst-case supply shock (panel A) and demand shock (panel B) each have greater unconditional variance. Although these distortions to the supply and demand shocks appear small, they have important effects on the worst-case distributions of inflation (panel C), consumption (panel D), and the interest rate (panel E). Specifically, the central bank's fear of misspecification causes it to assign greater probability to inflation and consumption outcomes that would

[13]The assumption of normality has been added purely to enable densities to be plotted; it is unnecessary for the solution method itself.

seem extreme under rational expectations. Similarly, with the policy for the robust central bank described by

$$i_t = 6.939s_t + 7.814d_t + 1.208\pi_{t-1} + 0.967c_{t-1}, \tag{48}$$

the interest rate's worst-case distribution also exhibits a much greater unconditional variance than the rational expectations distribution. Essentially, in terms of its unconditional expectations operator, the central bank obtains robustness by overweighting the probability it attaches to extreme inflation (in particular) and consumption outcomes, and this leads to an interest rate distribution that also assigns greater probability to extreme interest rate outcomes.

The central bank's desire to guard against extreme outcomes has important implications for the approximating equilibrium. By designing policy to guard against extreme inflation outcomes, the robust policy has a strong damping effect on the distribution of inflation (especially) and consumption in the approximating equilibrium. As shown in panel C, in the approximating equilibrium, inflation is distributed much more tightly about its unconditional mean than when expectations are rational, illustrating how the central bank's fear of misspecification leads it to "overstabilize" inflation. Similarly, the robust central bank also "overstabilizes" consumption (panel D), but at the cost of greater interest rate volatility (panel E).

Although the unconditional probability densities displayed in Figure 1 reveal the relationship between the central bank's pessimism and the probability it assigns to extreme outcomes, because they are unconditional they do not reveal how model uncertainty twists, or slants, the central bank's conditional expectations operator. To this end, for a given initial state,[14] Figure 2 presents the marginal probability density functions used to form one-quarter-ahead forecasts.

[14] Specifically, purely for illustrative purposes, in the initial period the supply shock, the demand shock, lagged inflation, and lagged consumption are all set to 1.0.

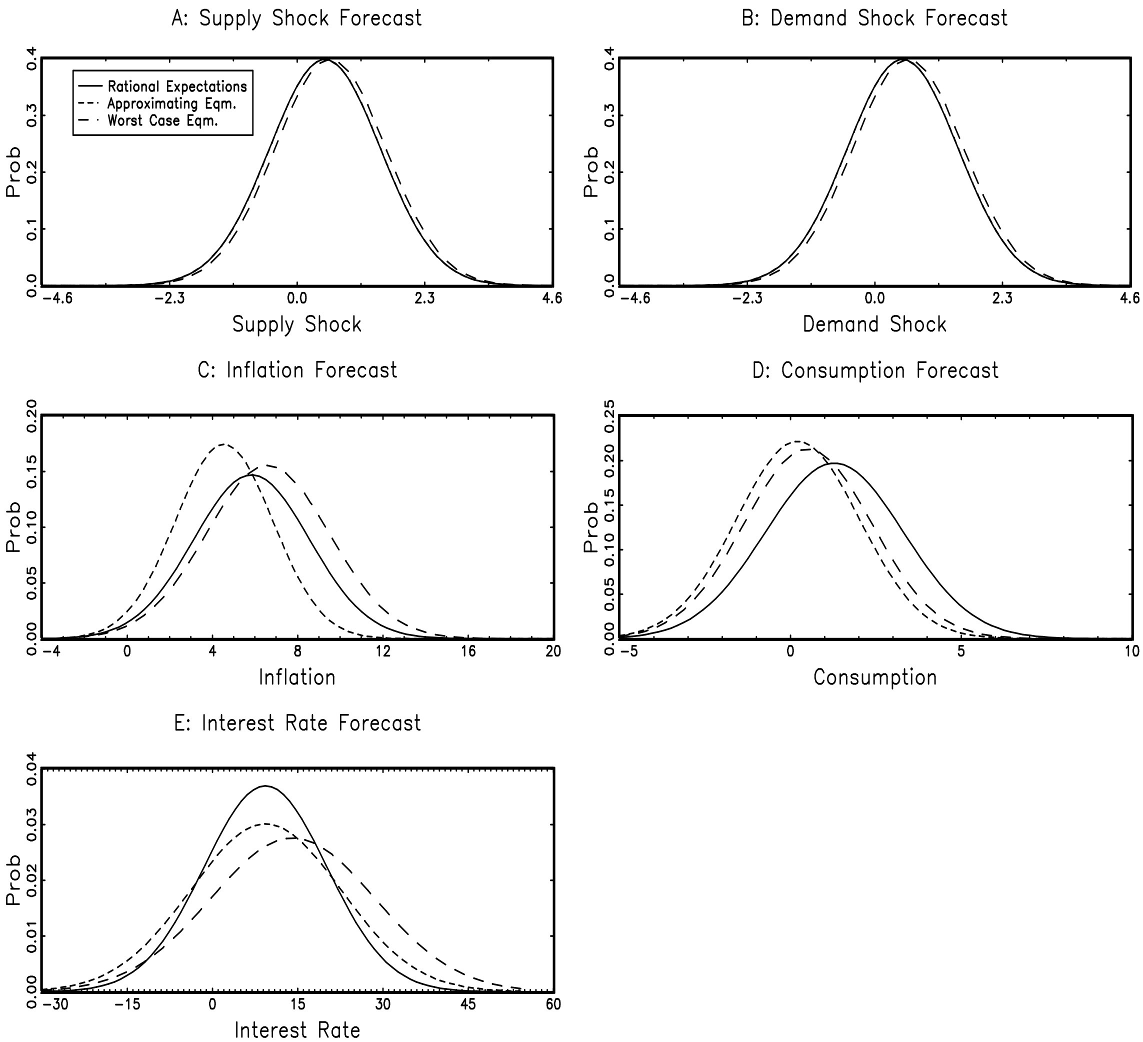

Fig. 2: Distorted and undistorted one-quarter-ahead forecasts

Figure 2 focuses on one-quarter-ahead forecast densities because the recursive nature of the robust optimization problem implies that it is these densities that are critical for the central bank's robust decision problem. Complementing Figure 1, panels A and B in Figure 2 show that, although the worst-case probability densities for the supply and the demand shocks are shifted to the right of those associated with the approximating equilibrium (which, of course, coincide with rational expectations), the distortions are reasonably modest. At the same time, these apparently small distortions to the shock distributions have a large impact on the one-quarter-ahead forecast densities for inflation (panel C), consumption (panel D) and the interest rate (panel E). Revealing a more subtle story than Figure 1, Figure 2 shows that the worst-case density for inflation is slanted to the right, with the central bank fearing higher inflation outcomes, and that the worst-case density for consumption is slanted to the left, with the central bank fearing lower consumption outcomes. Although it may seem more intuitive for the central bank to fear higher consumption outcomes, which would be inflationary, the probability densities are not unconstrained. Through the structure of the approximating model, because the central bank pessimistically expects higher inflation outcomes, it also expects higher interest rate outcomes, which leads it to expect lower consumption outcomes. Notice, however, that, unlike for consumption and the interest rate, where the distorted probability density function for future inflation is right-slanted, its counterpart in the approximating equilibrium is left-slanted.

Because the distribution of inflation is especially interesting and relevant, particularly since the central bank's role in the economy is to provide inflation with a nominal anchor, Figure 3 examines the probability density function for inflation at different forecast horizons.

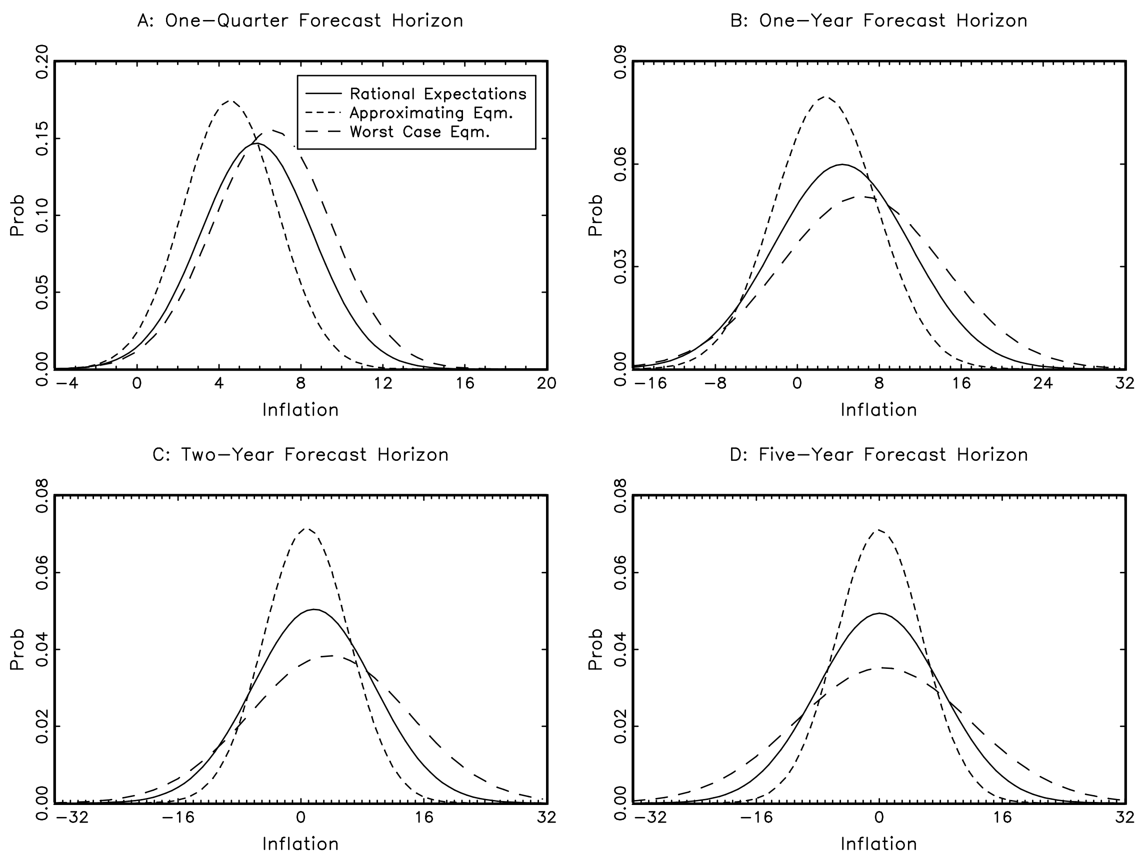

Fig. 3: Conditional probability densities for inflation

For the same initial state used for Figure 2, Figure 3 depicts the probability density functions required to forecast inflation one quarter ahead (panel A), one year ahead (panel B), two years ahead (panel C), and five years ahead (panel D) in the rational expectations equilibrium, the worst-case equilibrium, and the approximating equilibrium. For the worst-case equilibrium, panel A illustrates the extent to which the central bank's pessimism shifts rightward the one-quarter-ahead forecast density relative to rational expectations. However, this rightward shift, which raises the probability the central bank assigns to higher inflation outcomes, is evident in panels A to C. Equally evident in Figure 3 is the fact that the probability density functions associated with the approximating equilibrium, relative to rational expectations, are shifted to the left. The central bank's robust policy, which attaches pessimistically large probabilities to extreme inflation outcomes, skews inflation's distribution downward in the approximating equilibrium. As the forecast horizon lengthens, however, the distortions to inflation's conditional mean weaken and the conditional probability density function converges to the unconditional probability density function. In fact, inflation's conditional probability density function at the five-year horizon (Figure 3, panel D) essentially coincides with the unconditional probability density function shown in Figure 1, panel C.

5.2 Robustness, detectability, and policy loss

Figure 4 traces out the relationship between the robustness parameter and the probability of making a detection error, in panel A, and between the robustness parameter and the cost of robustness, in panel B. I measure the cost of robustness according to

$$C = 100 \times \frac{\left(L_{ap}^{d} - L_{re}^{d}\right)}{L_{re}^{d}}, \tag{49}$$

where L_{ap}^{d} denotes policy loss in the approximating equilibrium and L_{re}^{d} denotes policy loss in the rational expectations equilibrium. Panel A reveals that the probability of making a detection error is monotonically increasing in the robustness parameter, θ. Underlying this result is the fact that, as θ increases, greater weight is placed on the approximating model as being correct, the worst-case distortions are more tightly constrained, and the robust policy converges to the rational expectations policy. As a consequence, in the limit as $\theta \uparrow \infty$, data generated from the approximating equilibrium look increasingly like those generated from the worst case equilibrium and the probability of making a detection error converges to 0.5 (Hansen and Sargent, 2008, chapter 9).

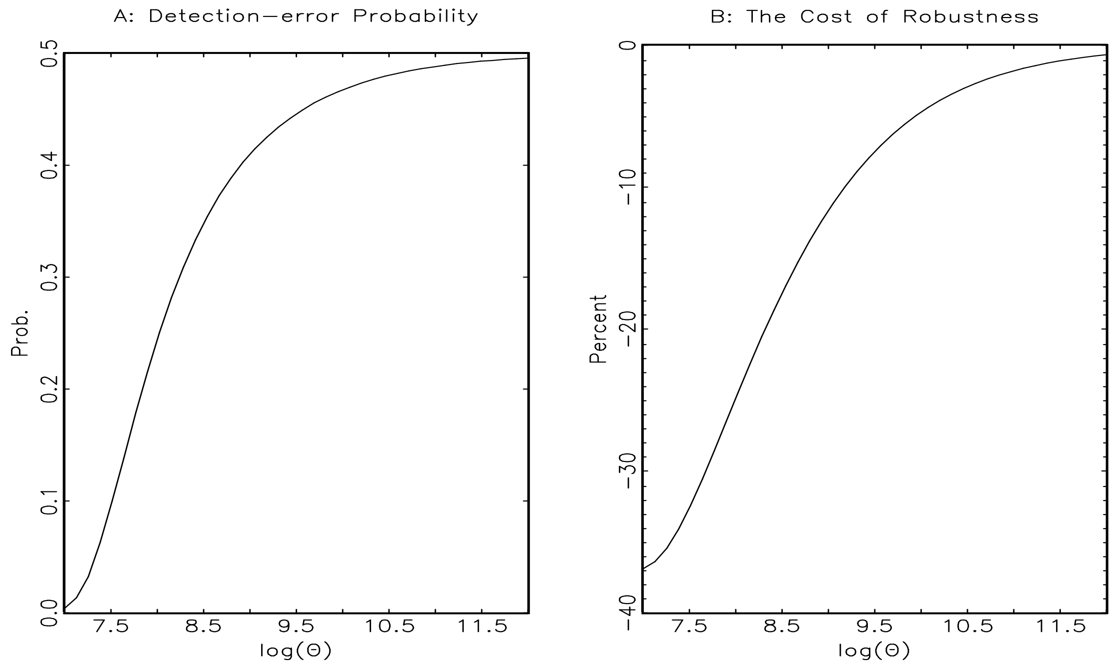

Fig. 4: Detectability and the cost of robustness

Panel B depicts the relationship between the robustness parameter and the cost of robustness.[15] What panel B reveals is that the central bank's desire for robustness actually causes policy loss to decline, not rise. In effect, even if specification errors are absent, the central bank is better off using the robust policy than using the rational expectations policy. Although this result may seem surprising at first, its genesis lies in the fact that monetary policy is conducted with discretion rather than with commitment.[16] Because private agents are forward-looking, the time-consistent policy with rational expectations is not optimal — it does not coincide with the optimal commitment policy — and other policies exist whose performance more closely approaches that of the optimal commitment policy.

As Dennis and Söderström (2006) document, in rational expectations models discretionary policies overstabilize consumption and understabilize inflation, relative to commitment policies, giving rise to a stabilization bias. This bias can be unwound by stabilizing inflation more aggressively and stabilizing consumption less aggressively, however the absence of a commitment mechanism makes this infeasible when expectations are rational. But model uncertainty imparts a deviation from rational expectations, which causes the central bank to implement

[15] Appendix A describes how the policy loss function is evaluated.

[16] Dennis and Ravenna (2007) obtain a related result from a model in which a central bank conducts policy while learning. The connection between the two results is that in each case policy is conducted with discretion and the central bank is only boundedly rational.

a policy that counteracts the likelihood of extreme inflation outcomes, partly mitigating the size of the stabilization bias. At the same time, as Figure 1 shows, the robust policy also stabilizes consumption more aggressively, and, as a consequence, whether robustness raises or lowers policy loss relative to the time-consistent rational expectations policy is likely to be parameter and model dependent, an issue to which I now turn.

6 The cost of robustness

In the absence of misspecification, the optimal commitment policy is (weakly) superior to all other policies, including robust policies. It follows immediately that a desire for robustness cannot improve policy loss when the central bank can commit. However, as shown above, when policy is conducted with discretion, stabilization bias provides an avenue whereby robust policies can improve upon nonrobust policies. In this section, I investigate the factors that govern this result and show that the finding that the cost of robustness can be negative holds for a wide range of parameter values.

I begin by presenting a simple model in which the central bank's desire for robustness is detrimental. Let inflation obey the forward-looking new Keynesian Phillips curve

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa c_t + s_t, \tag{50}$$

where, as earlier, c_t represents the consumption gap and $s_t \sim i.i.d[0, \sigma_\varepsilon^2]$ represents a cost push shock, and let the nonrobust decision problem for the central bank be to choose $\{c_t\}_0^\infty$ to minimize

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\boldsymbol{\pi}_t^2 + \lambda c_t^2 \right], \tag{51}$$

subject to equation (50) and s_0 known. Notice that the decision problem described by equations (50) and (51), which is equivalent to the problem in Woodford (2005), is a special case of the decision problem that I analyzed in Section 5.

Taking the parameterization from Woodford (2005), I set $\beta = 0.99$, $\kappa = 0.05$, $\lambda = 0.08$, and $\sigma_\varepsilon = 1.0$. After introducing a desire for robustness, Figure 5 displays the relationship between the robustness parameter, θ, and the probability of making a detection error, in panel A, and between the robustness parameter and the cost of robustness, in panel B.

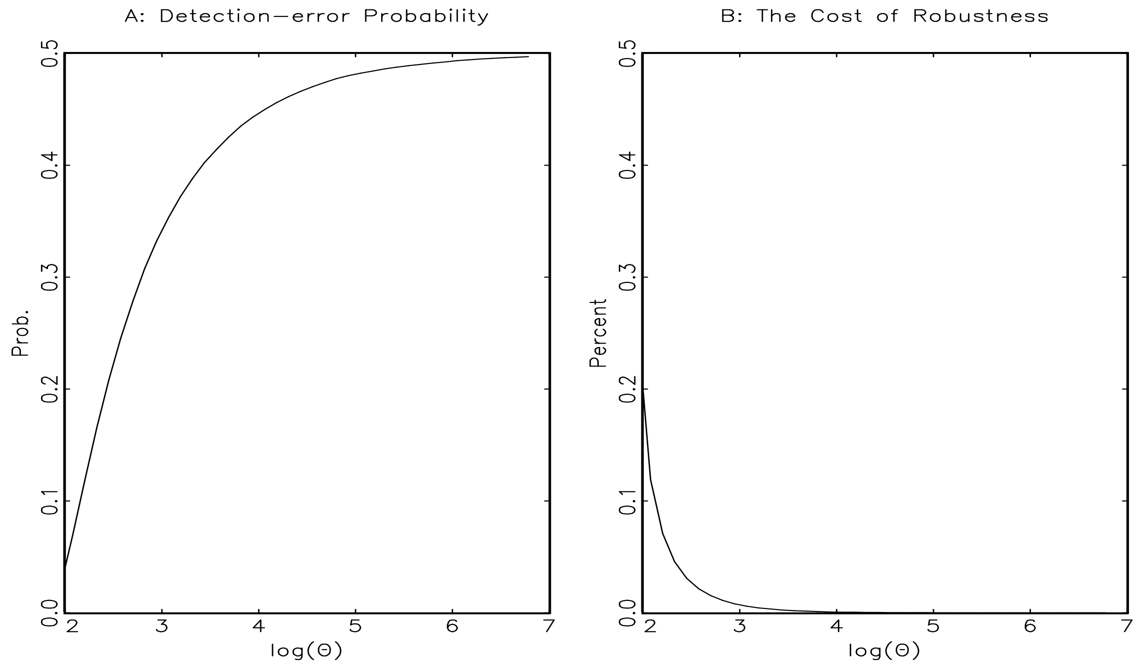

Fig. 5: Detectability and the cost of robustness (simple model)

Although the increase is small, no larger than 0.25 of a percentage point for the values of θ considered, Figure 5, panel B, shows that for this simple model a desire for robustness raises policy loss. Since the simple model is a special case of the hybrid new Keynesian model, Figure 5 establishes that robustness is does not improve policy loss for all parameterizations of the hybrid new Keynesian model. However, two questions immediately present themselves. First, which parameters in the hybrid new Keynesian model govern whether the cost of robustness is positive or negative? Second, is the negative cost of robustness exhibited in Figure 4 or the positive cost of robustness exhibited in Figure 5 the more general result?

To address these questions, I begin by extending the simple Phillips curve (equation (50)) to allow for endogenous inflation inertia and a serially correlated cost-push shock. With these additions, the simple model becomes

$$\pi_t = \beta(1-\delta)\,\mathrm{E}_t\pi_{t+1} + \delta\pi_{t-1} + \kappa c_t + s_t, \tag{52}$$

$$s_t = \rho s_{t-1} + \sigma_\varepsilon \varepsilon_t, \tag{53}$$

with $\delta = \rho = 0$ as benchmark values. Now, I consider independent variation in the two persistence parameters, ρ and δ, holding the detection-error probability constant at 0.25.[17]

[17] To prevent the detection-error probability from changing as I change the model parameters, I recalibrate θ for each parameterization. Note, however, that changes in θ, while important for magnitudes, do not influence

For this exercise, I vary (separately) ρ between 0.00 and 0.95 and δ between 0.00 and 0.50, keeping all other parameters at the benchmark values reported above. The results of this exercise are displayed in Figure 6 alongside a measure of the discretionary stabilization bias, which I construct according to

$$S = 100 \times \frac{\left(L_{re}^{c} - L_{re}^{d}\right)}{L_{re}^{d}}, \tag{54}$$

where L_{re}^{c} denotes policy loss under commitment and L_{re}^{d} denotes policy loss under discretion, both in the absence of robustness. Aside from special cases in which there is no time-consistency problem, S is unambiguously negative. Because the policy loss associated with the optimal commitment policy with rational expectations cannot be surpassed, equation (54) provides a lower bound for the cost of robustness.

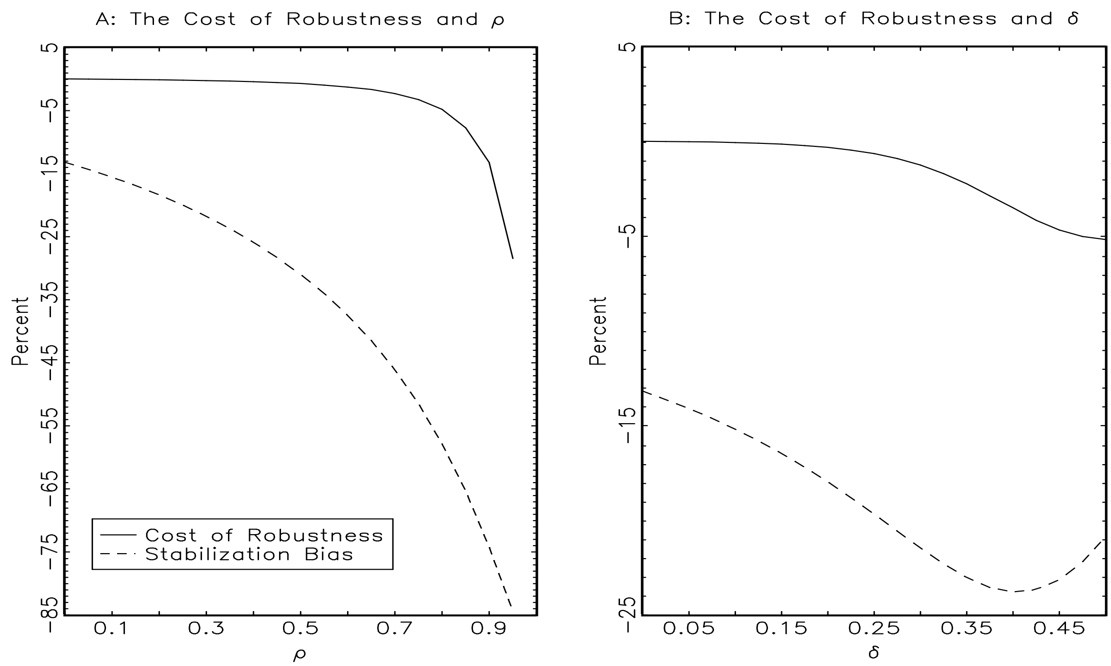

Fig. 6: Stabilization bias and the cost of robustness (simple model)

Panel A illustrates that adding persistence to the cost-push shock produces the result that robustness improves policy loss when policy is conducted with discretion. Similarly, panel B shows that adding endogenous persistence to the Phillips curve (raising δ) also produces the result that robustness improves policy loss. Together, panels A and B imply that persistence

whether the cost of robustness is positive or negative. The detection-error probability was set to 0.25 to ensure that results could be obtained for all parameterizations of the simple model.

in the Phillips curve, whether it be endogenous through δ or exogenous through ρ, is an important factor for the result that robustness can improve policy loss.

Returning to the hybrid new Keynesian model, I now consider independent variations in ρ, τ, δ, and γ, again holding the detection-error probability constant at 0.25. I vary (separately) ρ between 0.00 and 0.95, τ between 0.00 and 0.95, δ between 0.00 and 0.50, and γ between 0.30 and 0.50, keeping the parameters that are not being changed at their benchmark values. The results of this exercise are displayed in Figure 7.

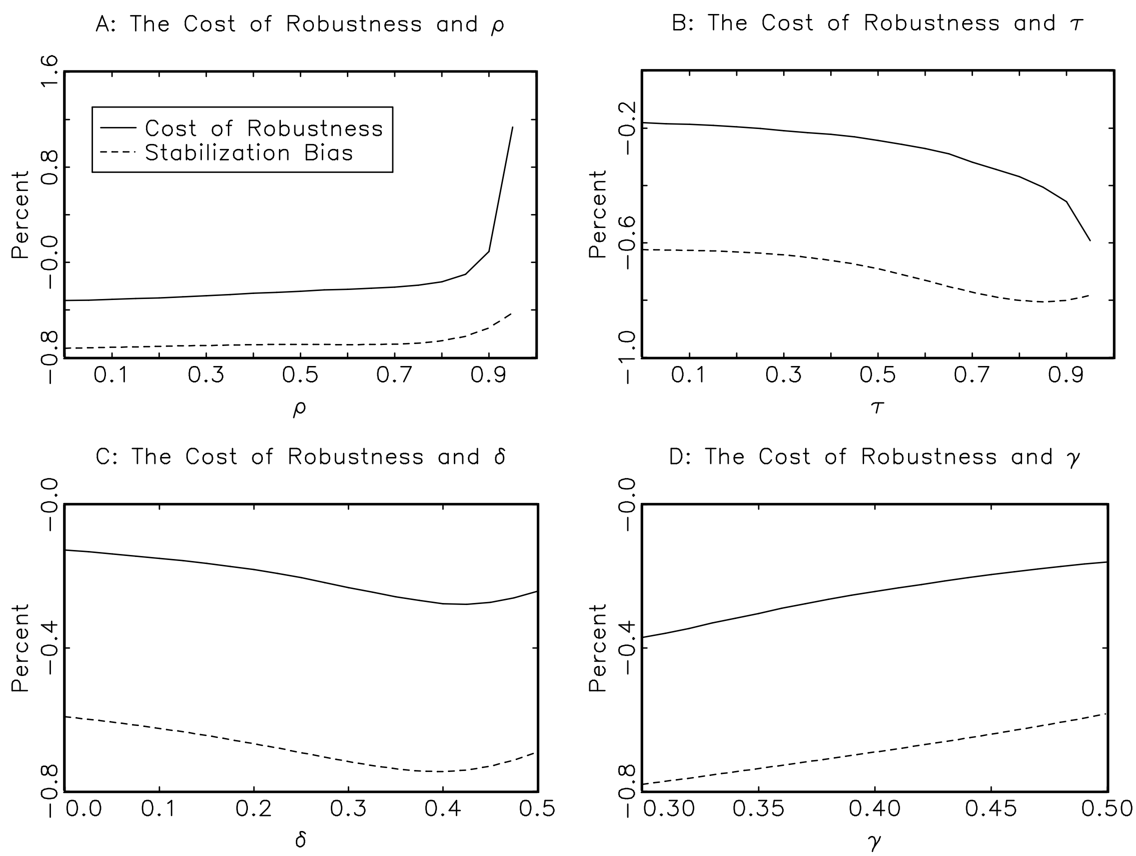

Fig. 7: Stabilization bias and the cost of robustness (hybrid NK model)

There are four important results to take away from Figure 7. First, the cost of robustness is negative for almost all parameter combinations considered. Only in Panel A, for very large values of ρ, is the cost of robustness positive. For example, if $\delta = 0.5$ and $\rho = 0.95$, implying considerable inflation persistence, then the cost of robustness is about 1.14 percent. Second, the cost of robustness and the stabilization bias are strongly correlated, particularly with respect to variation in δ and γ. Specifically, smaller values for the stabilization bias are associated with smaller values for the cost of robustness. Third, although the cost of robustness is generally negative, the cost of robustness generally falls well shy of the stabilization bias, which is to say that while robustness can lower policy loss it is certainly not a complete substitute for a commitment mechanism. Fourth, although the cost of robustness varies with all four of the parameters considered, consistent with Figure 6, it is most sensitive to variation in ρ, the persistence parameter in the cost-push shock.

Together, the results in Figures 6 and 7 suggest strongly that persistence in inflation, particularly persistence introduced through the cost-push shock, is closely associated with the finding that robustness can improve policy loss. Figures 6 and 7 also reveal that inflation persistence is a key factor governing the magnitude of the discretionary stabilization bias, consistent with Dennis and Söderström (2007). Importantly, the result that the cost of robustness is negative appears to hold for a wide range of parameter values in this model, suggesting that it may hold more generally among new Keynesian models, particularly those for which expectations are an important policy channel and the discretionary stabilization bias is large.

7 Conclusion

In this paper I develop a method for obtaining solutions to robust Markov-perfect Stackelberg problems in which the leader fears distortions to private-agents expectations. I apply this solution method to a stylized hybrid new Keynesian business cycle model to examine the effect a concern for model misspecification can have on the behavior and policy decisions of a central bank that conducts policy with discretion. Although I use robust control methods to generate the relevant equilibria, I exploit the connection between robust control and uncertainty aversion to focus my analysis on the properties of the probability densities that households, firms, and the central bank use to form expectations.

My analysis indicates that a concern for model uncertainty causes the central bank to make decisions on the basis of a distorted conditional expectations operator that emphasizes the possibility that inflation (in particular) and consumption may be more persistent than their approximating model acknowledges. Because the central bank fears that shocks to inflation and consumption will persist, it implements a policy that tends to stabilize inflation and consumption more tightly than the rational expectations policy. Through their effect on inflation, robust policies can improve on nonrobust policies by actually improving policy performance. This result arises because the central bank's concern for robustness moves it to stabilize inflation more tightly than would be credible were expectations rational, and this greater inflation stabilization partly offsets the higher variance of inflation associated with the discretionary stabilization bias. As a consequence, in this hybrid new Keynesian model and with discretionary policymaking, some degree of robustness to model uncertainty can be attained without sacrificing policy performance. Although the result that robustness can improve policy performance in the absence of a commitment mechanism is parameter dependent, it holds for a wide range of parameter values in the hybrid new Keynesian model that I analyze. In fact, the connection between the cost of robustness and the magnitude of the discretionary stabilization bias suggests that robustness is more likely to improve policy performance in models and for parameterizations where the time inconsistency problem is important and the stabilization bias is large.

Appendix A - Evaluating the loss function

Recall that the loss function takes the form

$$Loss\,[0,\infty] = \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\mathbf{z}_t^{'} \mathbf{W} \mathbf{z}_t + \mathbf{z}_t^{'} \mathbf{U} \mathbf{u}_t + \mathbf{u}_t^{'} \mathbf{U}^{'} \mathbf{z}_t + \mathbf{u}_t^{'} \mathbf{R} \mathbf{u}_t \right]. \tag{A1}$$

All of the equilibria considered in the paper can be expressed in the form

$$\mathbf{x}_{t+1} = \mathbf{M}\mathbf{x}_t + \mathbf{N}\boldsymbol{\varepsilon}_{t+1}, \tag{A2}$$
$$\mathbf{z}_t = \mathbf{G}\mathbf{x}_t, \tag{A3}$$
$$\mathbf{u}_t = \mathbf{F}^{\mathbf{u}}\mathbf{x}_t, \tag{A4}$$
$$\mathbf{v}_{t+1} = \mathbf{F}^{\mathbf{v}}\mathbf{x}_t. \tag{A5}$$

Given equations (A2) through (A5), the results in Appendices A1, A2, and A3 of Dennis (2007) can be applied to arrive at

$$Loss\,[0,\infty] = \mathbf{x}_0^{'} \mathbf{P} \mathbf{x}_0 + \frac{\beta}{(1-\beta)} tr \left[\mathbf{N}^{'} \mathbf{P} \mathbf{N} \right], \tag{A6}$$

where $\mathbf{P} = \widehat{\mathbf{W}} + \beta \mathbf{M}^{'} \mathbf{P} \mathbf{M}$, and $\widehat{\mathbf{W}} \equiv \mathbf{G}^{'} \mathbf{W} \mathbf{G} + \mathbf{G}^{'} \mathbf{U} \mathbf{F}^{\mathbf{u}} + \mathbf{F}^{\mathbf{u}'} \mathbf{U}^{'} \mathbf{G} + \mathbf{F}^{\mathbf{u}'} \mathbf{R} \mathbf{F}^{\mathbf{u}}$. From equation (A6), it follows that

$$\lim_{\beta \uparrow 1} (1-\beta) Loss\,[0,\infty] = tr \left[\mathbf{N}^{'} \mathbf{P} \mathbf{N} \right], \tag{A7}$$

which is invariant to the economy's initial state, $\mathbf{x}_0$.

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