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Abstract. This paper studies a multisector model of commodity markets with storage, solving the representative agent problem and obtaining the corresponding decentralized equilibrium. We describe the dynamics of the model, establishing geometric ergodicity, a Law of Large Numbers and a Central Limit Theorem.

1. Introduction

This paper studies a multisector version of Samuelson's (1971) classic model of commodity markets with storage. In both models, production consists of a random process for current output, and agents (speculators) have access to a storage technology which permits transfer of the commodities between time periods. Through storage, commodities are transferred towards periods where (expected) output is relatively low, and prices are correspondingly high.

Samuelson's primary interest was in characterizing equilibrium price processes in commodity markets. His technique identified market arbitrage and profit maximizing conditions with the first order conditions of an intertemporally maximizing representative agent, the utility function for whom is the integral of consumers' inverse demand function.[1]

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[1]Subsequently, Deaton and Laroque (1992) circumvented the representative agent construct, obtaining the same solution via a contraction mapping argument in the space of pricing functions. Their method is related to Coleman's policy iteration algorithm (1990). Our paper is closer to Samuelson's original approach.

Our study begins with the representative agent problem considered by Samuelson, but extended to a multisector setting in which realizations of current output are vectors in $\mathbb{R}^M_+$. The agent has preferences over the commodity space defined by a period utility function U over $\mathbb{R}^M_+$, and a discount factor ϱ. Storage is possible but costly, and the costs may differ across commodities. The resulting dynamic programming problem is solved using a weighted sup-norm approach.

Next we outline a decentralization in the spirit of Samuelson, where optimal storage for the representative agent coincides with equilibrium storage on the part of speculators. Equilibrium is defined in terms of market clearing, absence of arbitrage and profit maximization.

Our other results concern dynamics of the state process. Establishing global stability (ergodicity) is considerably more complicated than in the one-sector model, where policies are monotone.[2] Using the approach of Meyn and Tweedie (1993), we provide conditions under which the equilibrium process for the stock possesses a unique stationary distribution, and is globally stable in the sense that the marginal distribution of the stock converges at a geometric rate to the stationary distribution, independent of its initial condition.

Geometric ergodicity in turn leads to charaterization of the sample paths, which satisfy a Strong Law of Large Numbers and a Central Limit Theorem. In fact geometric ergodicity is a key property for modeling economic time series, and a considerable amount of research has been directed towards investigating its existence.[3]

1.1. **Empirical Studies.** Primary commodities account for a large percentage of trade in many economies, and the markets for these

[2]For example, the standard method of Stokey and Lucas (1989) does not apply.

[3]A good example is Kristensen (2005). We derive results similar to those found in Kristensen for certain time series models, but in an optimizing model with conditions stated only in terms of model primitives. Geometric ergodicity is also important for simulation-based estimation (see, e.g., Duffie and Singleton, 1993). Our results suggest that this multisector model is a suitable candidate for simulation-based estimation.

commodities have attracted widespread intervention by governments. Such policies—mainly aimed at stabilizing prices—have met with only mixed success, underlining the difficulty of manipulating endogenously determined variables.

Samuelson's one-sector model provides a scheme for analyzing the workings of commodity markets in which storage, total supply and price are all endogenous. It has been adopted as the theoretical foundations of many quantitative studies, including Deaton and Laroque (1992), Deaton and Laroque (1996) and Chambers and Bailey (1996).[4]

In treating a multisector version of the model, one of our aims is to provide the theoretical foundations for models which replicate commodity markets more closely by taking into account the joint determination of prices and quantities across related commodities. Prices and quantities are jointly determined because of contemporaneously correlated supply shocks and substitutability on the demand side.[5]

1.2. **Existing Research.** There are a number of additional studies related to Samuelson's commodity market model, all of which treat the one-sector case. Scheinkman and Schectman (1983) consider a similar model which includes supply responses to demand shocks. Wright and Williams (1991) combines theoretical and empirical models of storage in commodity markets. More recently, a version of the model with correlated shocks was used to investigate equilibrium forward price curves by Routledge, Seppi and Spatt (2000).

The dynamics of the single sector commodity pricing model were investigated in detail by Scheinkman and Schectman (1983). Confirming a conjecture of Samuelson (1971), they show that the process for the state

[4]See also Ng and Ruge-Murcia (2000), who add gestation lags in production, multiperiod forward contracts and convenience returns.

[5]In particular, the prices of substitutes are often closely integrated. A typical example is the markets for feed grains such as corn, sorghum, oats and barley. (In fact the US domestic prices of corn and sorghum are historically almost proportional, with the ratio determined by relative energy content.) It has also been argued that the prices of seemingly unrelated commodities are correlated even after controlling for relevant macroeconomic variables (Pindyck and Rotemburg, 1990).

(the stock of the commodity) converges asymptotically to a unique stationary distribution. Bobenrieth, Bobenrieth and Wright (2002) also investigate stability and properties of the stationary distribution.

1.3. **Outline.** Section 2 sets up the representative agent problem. Section 3 solves for and characterizes the associated optimal policy. Section 4 provides a decentralization. Section 5 considers dynamics. All proofs are given in Section 6.

2. Formulation of the Problem

We begin with a formulation of the multisector representative agent problem. In what follows, $\mathbb{R}_+^M$ is M copies of $\mathbb{R}_+ := [0, \infty)$. For x and y in $\mathbb{R}^M$ the relation $x \leq y$ means that $y - x \in \mathbb{R}_+^M$, while $x \ll y$ means that $y - x$ is interior to $\mathbb{R}_+^M$ (i.e., $y - x \in \operatorname{int} \mathbb{R}_+^M$). The notation $[x, y]$ denotes an order interval: $[x, y]$ is all $z \in \mathbb{R}^M$ such that $x \leq z \leq y$; (x, y) is all z with $x \ll z \ll y$, and so on. The symbol $\langle x, y \rangle$ is the inner product of x and y, and $\|x\| := \langle x, x \rangle^{1/2}$ is the Euclidean norm.

For differentiable $g \colon \mathbb{R}^M \to \mathbb{R}^N$, ∇g denotes the matrix of partial derivatives and $\nabla_m g$ is the m-th partial. We use λ to denote Lebesgue measure on $\mathbb{R}_+^M$, while $L_1(\mathbb{R}_+^M) = L_1(\mathbb{R}^M, \mathscr{B}(\mathbb{R}_+^M), \lambda)$ is the Lebesgue integrable functions on $\mathbb{R}_+^M$. Here $\mathscr{B}(\mathbb{R}_+^M)$ represents the Borel subsets of $\mathbb{R}_+^M$. By a distribution is meant a Borel probability measure on $\mathbb{R}_+^M$.

Consider a representative agent who at the start of time t owns a stock $X_t = (X_t^1, \ldots, X_t^M) \in \mathbb{R}_+^M$ of M different commodities.[6] This stock can be used for consumption $C_t = (C_t^m)_{m=1}^M$ and investment $I_t = (I_t^m)_{m=1}^M$. Both take values in $\mathbb{R}_+^M$ and satisfy $C_t + I_t \leq X_t$ for each t. Investment in I^m units of good m yields $\alpha^m I^m$ units next period, where $\alpha^m \in (0, 1)$ parameterizes storage cost, or depreciation. Hence investment $I_t = (I_t^m)_{m=1}^M$ at t yields the vector ΛI_t at $t+1$, where

$$\Lambda := \operatorname{diag}(\alpha^1, \ldots, \alpha^M) = \begin{pmatrix} \alpha^1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha^M \end{pmatrix}$$

[6]Throughout, the commodity index is a superscript, while time is the subscript.

In addition to investment ΛI_t carried over from the preceding period, the stock at $t+1$ is augumented by a random vector $W_{t+1} \in \mathbb{R}^M_+$:

$$X_{t+1} = \Lambda I_t + W_{t+1} \tag{1}$$

Assumption 2.1. The shocks $(W_t)_{t \geq 1}$ are independent and identically distributed, with common distribution φ. In addition,

$$\mu := \mathbb{E}\|W_t\| = \int \|z\| \varphi(dz) < \infty \tag{2}$$

The sequence of shocks $(W_t)_{t \geq 1}$ is defined on an arbitrary probability space $(\Omega, \mathscr{F}, \mathbb{P})$.[7] We let $\mathbb{E}$ denote expectation with respect to $\mathbb{P}$. Also, $(\mathscr{F}_t)_{t \geq 1}$ is the natural filtration generated by the sequence $(W_t)_{t \geq 1}$, with $\mathscr{F}_t := \sigma(W_1, \ldots, W_t)$. The initial condition $X_0 \in \mathbb{R}^M_+$ for the state is treated as given and indepedent of $(W_t)_{t \geq 1}$. Let ψ_0 be its distribution.

The agent is identified by a period utility function U on $\mathbb{R}^M_+$ and a discount factor $\varrho \in (0,1)$.

Assumption 2.2. The utility function $U \colon \mathbb{R}^M_+ \to \mathbb{R}_+$ is strictly concave, strictly increasing and continuous everywhere on $\mathbb{R}^M_+$.

To formulate the optimization problem, let $\mathscr{I}$ denote the set of investment policies $i \colon \mathbb{R}^M_+ \to \mathbb{R}^M_+$ that are Borel measurable and satisfy the feasibility constraint $i(x) \leq x$. The agent seeks an $i \in \mathscr{I}$ which maximizes her expected discounted utility. In other words, she solves

$$\max_{i \in \mathscr{I}} \mathbb{E} \left[\sum_{t=0}^{\infty} \varrho^t U(X_t - i(X_t)) \right] \tag{3}$$

$$\text{subject to} \quad X_{t+1} = \Lambda i(X_t) + W_{t+1}, \quad X_0 \sim \psi_0 \tag{4}$$

Let v be the value function associated with this dynamic programming problem. That is,

$$v(x) := \sup_{i \in \mathscr{I}} v_i(x), \quad \text{where } v_i(x) := \mathbb{E} \left[\sum_{t=0}^{\infty} \varrho^t U(X_t - i(X_t)) \right] \tag{5}$$

Here $(X_t)_{t \geq 0}$ obeys the recursion in (4), but with $X_0 = x$. We call $i \in \mathscr{I}$ optimal if it attains the supremum in (5) for every $x \in \mathbb{R}^M_+$.

[7] Thus, $\mathbb{P}\{W_t \in \cdot\} = \mathbb{P} \circ W_t^{-1} = \varphi$ on $\mathscr{B}(\mathbb{R}^M_+)$.

3. Optimality

Since U and the state variable are potentially unbounded on $\mathbb{R}^M_+$, it is not immediately clear that the expressions in (3) and (5) are well defined. We use a weighted norm approach to establish existence of the value function and the validity of the Bellman equation.[8]

To begin, let us introduce the auxillary $\mathbb{R}^M_+$-valued linear process

$$Y_{t+1} = \Lambda Y_t + W_{t+1}, \quad Y_0 = x \tag{6}$$

and the function $\kappa \colon \mathbb{R}^M_+ \to \mathbb{R}$ defined by the infinite sum

$$\kappa(x) := 1 + \sum_{t=0}^{\infty} \delta^t \mathbb{E} U(Y_t) \qquad x \in \mathbb{R}^M_+$$

where δ is any constant in $(\varrho, 1)$. Note that for this δ we have $\varrho/\delta < 1$.

Lemma 3.1. *The function κ is finite, increasing and continuous everywhere on $\mathbb{R}^M_+$. For any $i \in \mathscr{I}$ we have $v_i \leq \kappa$ on $\mathbb{R}^M_+$. As a result, the value function v is well-defined, and the ratio v/κ is bounded.*

We call $w \colon \mathbb{R}^M_+ \to \mathbb{R}$ κ-bounded if w/κ is bounded; that is, if

$$\|w\|_\kappa := \|w/\kappa\|_\infty := \sup_{x \in \mathbb{R}^M_+} |w(x)/\kappa(x)| < \infty$$

The function $w \mapsto \|w\|_\kappa$ is a norm on the set of all κ-bounded functions on $\mathbb{R}^M_+$. Define $b_\kappa \mathscr{B} \mathbb{R}^M_+$ to be the set of κ-bounded Borel measurable function on $\mathbb{R}^M_+$, and $b_\kappa c \mathbb{R}^M_+$ to be those functions which are in addition continuous. The collection of functions $b_\kappa \mathscr{B} \mathbb{R}^M_+$ endowed with the norm $\| \cdot \|_\kappa$ is a Banach space. Using continuity of κ, it can be shown that $b_\kappa c \mathbb{R}^M_+$ is a $\| \cdot \|_\kappa$-closed subset of $b_\kappa \mathscr{B} \mathbb{R}^M_+$.

The Bellman operator $T \colon b_\kappa \mathscr{B} \mathbb{R}^M_+ \to b_\kappa \mathscr{B} \mathbb{R}^M_+$ is defined by

$$Tw(x) = \sup_{0 \leq \xi \leq x} \left\{ U(x - \xi) + \varrho \int w(\Lambda \xi + z) \varphi(dz) \right\} \quad x \in \mathbb{R}^M_+ \tag{7}$$

[8]Alternatively, one may assume that the shocks are bounded, in which case standard dynamic programming arguments apply. This is valid but unnecessarily excludes a number of shock distributions routinely used in econometric modeling. Our approach draws on Hernández-Lerma and Lasserre (1999, Chapter 8).

The operator T is a contraction of modulus $\gamma := \varrho/\delta < 1$ on $b_\kappa \mathscr{B}\mathbb{R}^M_+$ with respect to the $\| \cdot \|_\kappa$-norm:

Proposition 3.1. *T is a well-defined map from $b_\kappa \mathscr{B}\mathbb{R}^M_+$ to itself, and*

$$\|Tw - Tu\|_\kappa \leq \gamma \|w - u\|_\kappa \qquad w, u \in b_\kappa \mathscr{B}\mathbb{R}^M_+$$

If w is continuous then so is Tw, and hence T sends $b_\kappa c\mathbb{R}^M_+$ into itself.

We can now give Bellman's equation for the value function, and the resulting characterization of the optimal policy.

Theorem 3.1. *The value function v is the unique fixed point of T in $b_\kappa \mathscr{B}\mathbb{R}^M_+$, and for each $w \in b_\kappa \mathscr{B}\mathbb{R}^M_+$ we have $\|T^n w - v\|_\kappa \to 0$ as $n \to \infty$. In addition, v is continuous, strictly increasing and strictly concave on $\mathbb{R}^M_+$. A unique optimal policy $I \in \mathscr{I}$ exists. It is continuous, and satisfies*

$$I(x) = \operatorname*{argmax}_{0 \leq \xi \leq x} \left\{ U(x - \xi) + \varrho \int v(\Lambda \xi + z) \varphi(dz) \right\} \quad x \in \mathbb{R}^M_+ \tag{8}$$

Figure 1 shows (an approximation to) the value function for the two commodity case, where $\alpha^1 = \alpha^2 = \varrho = 0.9$, $U(x, y) = x^{0.4} y^{0.4}$ and $W = (e^\xi, e^\eta)$ with (ξ, η) independent standard normal. The approximation was carried out by iterating the Bellman operator, starting at $U \in b_\kappa c\mathbb{R}^M_+$. The sequence converges to v at rate $O(\gamma^n)$.[9]

Next we obtain additional properties of the optimal policy via first order and envelope conditions. Some care is required, as optimal investment is not always interior. To establish the desired properties we require that consumption is nonzero on the interior of $\mathbb{R}^M_+$—a rather plausible condition. To this end the following restriction on U is added:

[9] Each iterate was approximated using a continuous piecewise affine function constructed as the infimum of 324 supporting hyperplanes. This technique is related to the method proposed by Santos and Vigo (1998), who suggest approximating value functions by continuous piecewise affine functions. (Our algorithm for constructing this approximation is somewhat different.)

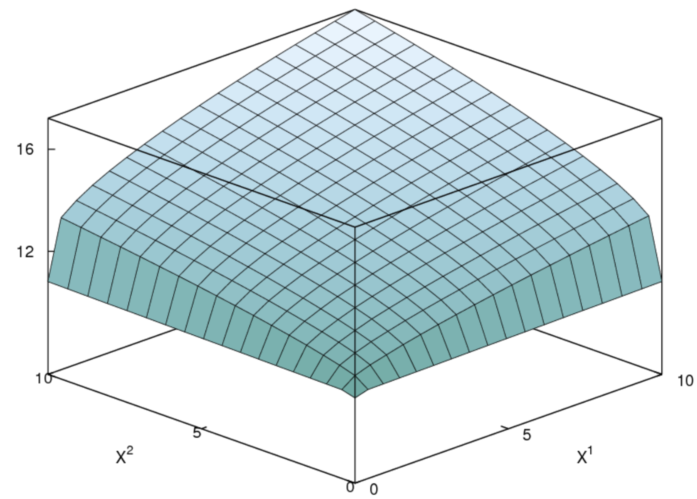

FIGURE 1. Value Function

Assumption 3.1. U is differentiable on $\operatorname{int}\mathbb{R}^M_+$. Further, given any $x \gg 0$ and any c on the boundary $\partial\mathbb{R}^M_+ := \mathbb{R}^M_+ \setminus \operatorname{int}\mathbb{R}^M_+$ of $\mathbb{R}^M_+$, there exists a nonnegative direction vector $d \in \mathbb{R}^M_+$ such that $c + d \leq x$ and

$$\lim_{\theta \downarrow 0} \frac{U(c + \theta d) - U(c)}{\theta} = \infty. \tag{9}$$

In words, the directional derivative at c in the direction d is infinite.

Example 3.1. Consider the Cobb-Douglas utility function $U(c) = \prod_{m=1}^M (c^m)^{a_m}$, where $a := \sum_{m=1}^M a_m < 1$. This function satisfies all of the conditions of Assumptions 2.2 and 3.1. The only element of this claim which requires proof is the interiority condition in Assumption 3.1. For the proof, pick any $c \in \partial\mathbb{R}^M_+$ and any $x \gg 0$ with $c \leq x$. Let $d = x - c$. Evidently $d \in \mathbb{R}^M_+$ and $c + d \leq x$. Moreover $U(c) = 0$ and, for $\theta \leq 1$,

$$U(c + \theta d) = U((1 - \theta)c + \theta x) \geq U(\theta x) = \theta^a U(x)$$

$$\therefore \quad \frac{U(c + \theta d) - U(c)}{\theta} \geq \frac{U(\theta x)}{\theta} = \theta^{a-1} U(x)$$

Since $x \gg 0$ we have $U(x) > 0$, and as $a < 1$ the right hand side converges to infinity when $\theta \downarrow 0$.

Proposition 3.2. *Under the stated assumptions optimal consumption is interior, in the sense that if* $x \gg 0$ *then* $x - I(x) \gg 0$. *In addition, the value function* v *is differentiable on* $\operatorname{int} \mathbb{R}^M_+$ *and*

$$\nabla v(x) = \nabla U(x - I(x)) \qquad x \in \operatorname{int} \mathbb{R}^M_+ \tag{10}$$

Given differentiability of v, I satisfies the two first order conditions

$$\alpha^m \varrho \int \nabla_m v(\Lambda I(x) + z)\varphi(dz) \leq \nabla_m U(x - I(x)) \tag{11}$$

$$\alpha^m \varrho \int \nabla_m v(\Lambda I(x) + z)\varphi(dz) < \nabla_m U(x - I(x)) \Rightarrow I^m(x) = 0 \tag{12}$$

for all $m = 1, \ldots M$. Here I^m is the m-th component function of I, in that $I(x) = (I^m(x))_{m=1}^M \in S$.

Figures 2 gives the optimal investment policy functions $I^1(x^1, x^2)$ and $I^2(x^1, x^2)$ at top and bottom respectively. The are obtained by solving (8), with v the approximate value function given in Figure 1. The parameters are the same as above: $\alpha = \varrho = 0.9$, $U(x, y) = x^{0.4}y^{0.4}$ and $W = (e^\xi, e^\eta)$ with (ξ, η) independent standard normal.

4. Speculative Prices

In this section we study a decentralized market in which equilibrium storage by speculators is shown to be equal to the optimal investment policy of the representative agent in Section 3. Equilibrium prices correspond to the representative agent's marginal utility of consumption.

The market has M commodities, the vector of spot prices for which is given at time t by $p_t = (p_t^m)_{m=1}^M \in \mathbb{R}^M_+$. Demand for the commodities comes from both consumers and speculators.[10] Demand from consumers is determined by a fixed demand curve. Let P be the inverse demand curve. The component functions of P are denoted by P^m. That is, $P(x) = (P^m(x))_{m=1}^M \in \mathbb{R}^M_+$. This is to be understood as

[10]Here "consumers" are usually best thought of as producers rather than final consumers. (While this is obviously the case for products such as coal, crude oil, base metals and lumber, it also applies to agricultural commodities such as corn, which is used as feed grain, industrial alcohol and fuel ethanol.)

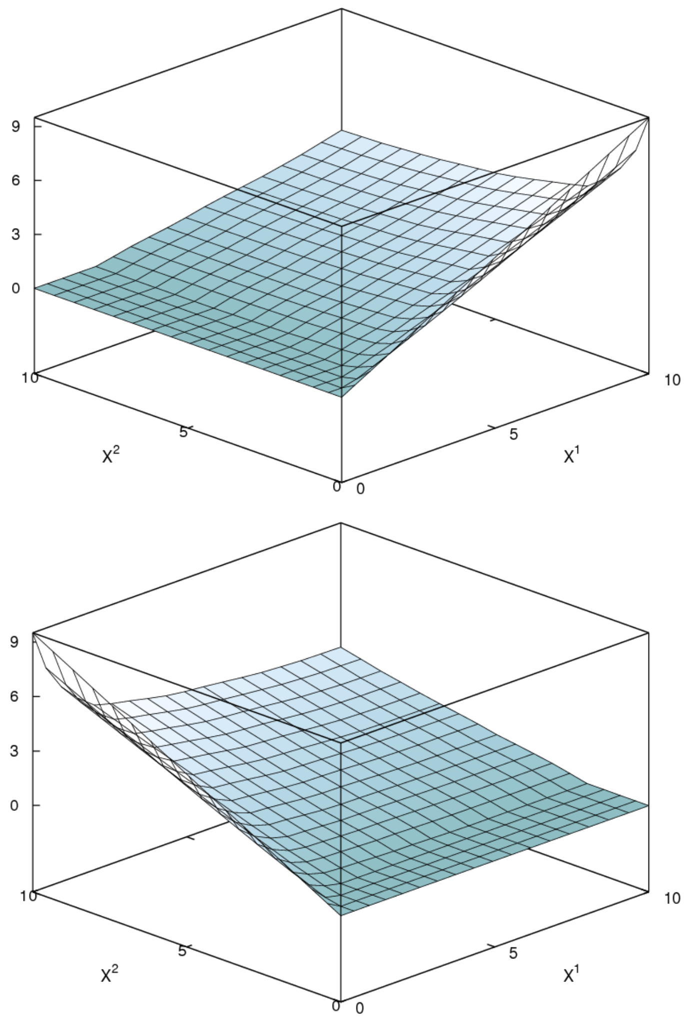

FIGURE 2. Investment in Commodities 1 (top) and 2 (bottom)

the vector of prices at which consumers demand commodity bundle x. We suppose that P can be adequately modeled by setting $P = \nabla U$ for some suitable choice of U, where U satisfies Assumptions 2.2 and 3.1.[11]

[11]This assumption is, in some senses, rather strict. In particular, the fact that U must be concave restricts the class of functions that can be attained by ∇U. Whether or not the class is too small in applications is best viewed an empirical question rather than a theoretical one, and is left for future research.

Speculators are able to store the commodities between periods: A vector $I_t = (I_t^m)_{m=1}^M \in \mathbb{R}_+^M$ carried over from time t yields ΛI_t at $t+1$.[12] Aggregate supply X_t is the sum of this quantity and a "harvest" W_t:

$$X_t = \Lambda I_{t-1} + W_t.$$

As the storage decision (speculative investment) is made prior to observing W_{t+1} we require that I_t is $\mathscr{F}_t$-measurable.

Speculators are assumed to be risk neutral. The risk-free interest rate r is taken to be constant, and we set $\varrho := (1+r)^{-1}$. Letting $\mathbb{E}_t = \mathbb{E}[\cdot \mid \mathscr{F}_t]$ denote time t expectation, nonexistence of arbitrage requires that

$$\varrho \mathbb{E}_t \langle p_{t+1}, \Lambda I_t \rangle - \langle p_t, I_t \rangle \leq 0 \quad \mathbb{P}\text{-a.s.}$$

for all $\mathscr{F}_t$-measurable I_t with $0 \leq I_t \leq X_t$. A necessary and sufficient condition for this restriction to hold is

$$\alpha^m \varrho \mathbb{E}_t p_{t+1}^m - p_t^m \leq 0 \quad \mathbb{P}\text{-a.s.}, \quad m = 1, \ldots, M \tag{13}$$

Profit maximization gives the additional restriction

$$\alpha^m \varrho \mathbb{E}_t p_{t+1}^m - p_t^m < 0 \text{ implies } I_t^m = 0, \quad m = 1, \ldots, M \tag{14}$$

Equilibrium requires, in addition to (13) and (14), that

$$p_t = P(X_t - I_t) \tag{15}$$

This is a market clearing condition stated in terms of prices.

We seek a set of prices and quantities which satisfy (13)–(15). The following theorem identifies such an equilibrium via the optimal policy of the representative agent defined in Section 3.

Theorem 4.1. *Let X_0 be given, and let I be the optimal investment policy defined in Theorem 3.1. The sequence $(I_t, p_t, X_t)_{t \geq 0}$ identified by*

$$I_t = I(X_t), \quad p_t = P(X_t - I_t), \quad X_{t+1} = \Lambda I_t + W_{t+1} \tag{16}$$

is a market equilibrium in the sense that (13)–(15) *all hold.*

[12] Speculators are assumed to be of measure one, in the sense that individual investment I_t is equal to aggregate market investment I_t.

In the proof we use the notation $p(x) := P(x - I(x))$. This function is can be regarded as a rational expectations equilibrium pricing functional. It represents the vector of equilibrium spot prices in the commodity market when supply is $x \in \mathbb{R}^M_+$, and consists of the M component functions $p^m \colon \mathbb{R}^M_+ \to \mathbb{R}_+$, so that $p(x) = (p^m(x))_{m=1}^M$. In view of $\nabla U = P$ and the envelope condition (10) we have

$$p(x) = P(x - I(x)) = \nabla U(x - I(x)) = \nabla v(x)$$

In particular, under the equilibrium pricing functional the spot price equals marginal utility of consumption.

Using this notation, we can restate the set of M first order conditions in (11) and (12) as

$$\alpha^m \varrho \int p^m(\Lambda I(x) + z) F(dz) \leq p^m(x) \tag{17}$$

and

$$\alpha^m \varrho \int p^m(\Lambda I(x) + z) F(dz) < p^m(x) \Rightarrow I^m(x) = 0. \tag{18}$$

Proof of Theorem 4.1. Since (15) is true by definition, we need only establish that (13) and (14) both hold. In view of (16) and the definition $p^m(x) = P^m(x - I(x))$,

$$\begin{aligned} \mathbb{E}_t p^m_{t+1} &= \mathbb{E}_t P^m(X_{t+1} - I(X_{t+1})) = \mathbb{E}_t p^m(X_{t+1}) \\ &= \mathbb{E}_t p^m(\Lambda I(X_t) + W_{t+1}) = \int p^m(\Lambda I(X_t) + z) \varphi(dz) \end{aligned}$$

so to prove (13) we must show that

$$\alpha^m \varrho \int p^m(\Lambda I(X_t) + z) \varphi(dz) \leq p^m(X_t).$$

But this is immediate from (17). And by a similar argument, (14) is immediate from (18). $\square$

5. Dynamics

In this section we turn to equilibrium dynamics of the commodity stock $(X_t)_{t\geq 0}$ given the optimal investment policy I defined in (8). The process is Markovian and obeys the stochastic recursive sequence

$$X_{t+1} = \Lambda I(X_t) + W_{t+1}, \quad (W_t)_{t\geq 0} \overset{\text{IID}}{\sim} \varphi \tag{19}$$

The sequence $(X_t)_{t\geq 0}$ can also be seen as the equilibrium time path given in (16), as discussed in Section 4.

Throughout this section we maintain Assumptions 2.2 and 2.1. Assumption 3.1 is not required. Instead, additional restrictions on the nature of the shock distribution are necessary:

Assumption 5.1. The distribution φ of the shock W can be represented by a density, which we again denote by φ. The density φ is continuous everywhere on $\mathbb{R}^M_+$ and positive on its interior.

Many standard distributions satisfy all of our assumptions, a useful example being the multivariate lognormal density. Heavy tailed densities are also possible, provided that μ in (2) remains finite. It would appear that the latter assumption is difficult to weaken substantially while maintaining our stability results.[13]

The dynamics in (19) can be encapsulated in the Markov density kernel

$$q(x, y) := \varphi(y - \Lambda I(x)) \qquad x, y \in \mathbb{R}^M_+ \tag{20}$$

Intuitively, $q(x, y)$ is the conditional density of X_{t+1} when $X_t = x$.[14] If $y \not\geq \Lambda I(x)$ then $y - \Lambda I(x) \notin \mathbb{R}^M_+$ and $\varphi(y - \Lambda I(x))$ is not defined. For such values of x and y we take $q(x, y) = 0$. Alternatively, one can regard φ as defined on all of $\mathbb{R}^M$ and equal to zero on $\mathbb{R}^M \setminus \mathbb{R}^M_+$.

[13]The assumption that φ is a density can perhaps be relaxed without losing the stability results given below. However, the density assumption is suitable for empirical applications and allows slightly more direct proofs, as well as a more explicit construction of the Markov process generated by the optimal policy.

[14]To see this, observe that for any $B \in \mathscr{B}(\mathbb{R}^M_+)$ the change of variable $z = y - \Lambda I(x)$ yields $\int \mathbb{1}_B(y)\varphi(y - \Lambda I(x))dy = \int \mathbb{1}_B(\Lambda I(x) + z)\varphi(z)dz = \mathbb{P}\{\Lambda I(x) + W \in B\} = \mathbb{P}\{X_{t+1} \in B \mid X_t = x\}$, where $\mathbb{1}_B$ denotes the indicator function of B.

Using standard arguments[15] we can deduce that if X_t has any distribution ψ_t (not necessarily a density), then X_{t+1} has a distribution represented by density ψ_{t+1}, where

$$\psi_{t+1}(y) = \int q(x,y)\psi_t(dx) \qquad y \in \mathbb{R}^M_+ \tag{21}$$

Let $\mathbf{M}$ be a map from the set of distributions on $\mathbb{R}^M_+$ into the set of densities on $\mathbb{R}^M_+$ defined by $\psi \mapsto \psi\mathbf{M}$,

$$\psi\mathbf{M}(y) = \int q(x,y)\psi(dx) \qquad y \in \mathbb{R}^M_+ \tag{22}$$

This map is called the Markov operator corresponding to q.[16] In light of (21), the marginal distributions (ψ_t) of (X_t) satisfy $\psi_{t+1} = \psi_t\mathbf{M}$. Iterating backwards we obtain $\psi_t = \psi_0\mathbf{M}^t$, where $\mathbf{M}^t$ is the t-th composition of $\mathbf{M}$ with itself, and, as above, ψ_0 is the distribution of X_0.

A distribution ψ^* on $\mathbb{R}^M_+$ is called stationary for the optimal process (19) if ψ^* is a fixed point of $\mathbf{M}$. The interpretation is that if $X_t \sim \psi^*$, then $X_{t+1} \sim \psi^*\mathbf{M} = \psi^*$, and hence probabilities are unchanged. Since $\mathbf{M}$ maps distributions into densities, any fixed point ψ^* of $\mathbf{M}$ must be a density (because ψ^* is the image of itself under $\mathbf{M}$). Hence in what follows we need concern ourselves only with stationary densities, rather than stationary distributions. For such a stationary density, the defining condition $\psi^*\mathbf{M} = \psi^*$ translates to

$$\int q(x,y)\psi^*(x)dx = \psi^*(y) \qquad y \in S \tag{23}$$

We measure the distance between densities φ and φ' according to their deviation with respect to the norm on $L_1(\mathbb{R}^M_+)$:

$$d_1(\psi,\psi') := \int |\psi(x) - \psi'(x)| d(x)$$

By Scheffè's Identity, we also have

$$d_1(\psi,\psi') = \sup_{|h|\leq 1} \left| \int h(x)\psi(x)dx - \int h(x)\psi'(x)dx \right| \tag{24}$$

[15] See, for example, Lasota and Mackey (1994) or Stachurski (2002).

[16] As is traditional, $\mathbf{M}$ acts on distributions to the left.

Here the supremum is with respect to all Borel measurable bounded functions with supremum norm less than 1.[17]

To state our results, we introduce two classes $\mathscr{H}_1$ and $\mathscr{H}_2$ of Borel measurable, real-valued functions on $\mathbb{R}^M_+$. Let s be any arbitrary but fixed constant in $[1, \infty)$. The first class $\mathscr{H}_1$ is those functions h satisfying

$$|h(x)| \leq \|x\| + s, \ \forall x \in \mathbb{R}^M_+.$$

The second class $\mathscr{H}_2 \subset \mathscr{H}_1$ is those functions h satisfying

$$h(x)^2 \leq \|x\| + s, \ \forall x \in \mathbb{R}^M_+.$$

We now come to the main stability result of the paper.

Theorem 5.1. *The following statements are true:*

(1) *The optimal process* (19) *has a unique stationary density* ψ^*.
(2) *The stationary density* ψ^* *satisfies* $\int \|x\| \psi^*(dx) < \infty$. *In particular, the steady state expected value in each sector is finite.*
(3) *If* ψ_0 *is any distribution with* $\int \|x\| \psi_0(dx) < \infty$, *then there is an* $M < \infty$ *and a* $\beta < 1$ *such that,* $\forall t \in \mathbb{N}$,

$$\sup_{h \in \mathscr{H}_1} \left| \int h(x) \psi_0 \mathbf{M}^t(x) dx - \int h(x) \psi^*(x) dx \right| \leq \beta^t M$$

We present several corollaries to the theorem:

Corollary 5.1. *Let* $(X_t)_{t \geq 0}$ *be the optimal process starting at* $x_0 \in \mathbb{R}^M_+$. *For any such* x_0, *the density* ψ_t *of* X_t *converges in* $L_1(\mathbb{R}^M_+)$ *to* ψ^* *at a geometric rate.*

For the next corollary some additional notation is useful. Let $(X^*_t)_{t \geq 0}$ be a stationary version of the process. That is, $X^*_{t+1} = \Lambda I(X^*_t) + W_{t+1}$ and $X^*_0 \sim \psi^*$. Now fix $h \in \mathscr{H}_1$ and consider the constants

$$m^*_h := \int h(x) \psi^*(x) dx = \mathbb{E} h(X^*_0)$$

$$v^*_h := \mathbb{E}[h(X^*_0) - m^*_h]^2 + 2 \sum_{t \geq 1} \mathbb{E}[h(X^*_0) - m^*_h][h(X^*_t) - m^*_h]$$

[17]From (24) it is easy to see that L_1 convergence of densities implies uniform (and hence weak) convergence of distribution functions.

Corollary 5.2. *Let ψ_0 be an arbitrary in initial condition and let $(X_t)_{t \geq 0}$ be the process starting at $X_0 \sim \psi_0$. If $h \in \mathscr{H}_1$, then m_h^* is finite, and*

$$\text{(LLN)} \qquad \frac{1}{n} \sum_{t=1}^{n} h(X_t) \to m_h^* \quad \mathbb{P}\text{-a.s. as } n \to \infty$$

If, in addition, $h \in \mathscr{H}_2$, then v_h^* is finite, and

$$\text{(CLT)} \qquad \frac{1}{n} \sum_{t=1}^{n} h(X_t) \to N(m_h^*, v_h^*) \quad \text{in distribution as } n \to \infty$$

The two most important consequences of Corollary 5.2 are as follows. First, for any event $B \in \mathscr{B}(\mathbb{R}_+^M)$ we have $\frac{1}{n}\sum_{t=1}^{n} \mathbb{1}_B(X_t) \to \psi^*(B)$, and hence the steady state probability $\psi^*(B)$ is approximately equal to the fraction of time that the equilibrium quantity spends in B as the time horizon tends to infinity. This is the standard concept of ergodicity. Second, expectations and probabilities vis-a-vis the stationary distribution can be computed by simulation, appealing (LLN). For such calculations, (CLT) provides (asymptotic) error bounds.

As an application of the second point, we compute an estimate ψ_n^* of the stationary density ψ^* via conditional Monte Carlo (Glynn and Henderson, 2001). Fix $y \in \mathbb{R}_+^M$. Since $x \mapsto q(x, y)$ is bounded it is an element of $\mathscr{H}_1$. By Corollary 5.2, then, we have

$$\psi_n^*(y) := \frac{1}{n} \sum_{t=1}^{n} q(X_t, y) \to \int q(x, y) \psi^*(x) dx \quad \mathbb{P}\text{-a.s. as } n \to \infty$$

By (23), the right hand side is precisely $\psi^*(y)$, so $\psi_n^*(y) \to \psi^*(y)$ almost surely for all y. Figure 3 displays an instance of ψ_n^* for the same parameters as in Figure 1, where $n = 2000$.

6. Proofs

This section collects all remaining proofs. Throughout the proofs we adopt the new notation $\alpha := \max_{1 \leq m \leq M} \alpha^m$. As the largest eigenvalue, α is the spectral radius of Λ, and hence $\|\Lambda x\| \leq \alpha \|x\|$, $\forall x \in \mathbb{R}^M$.

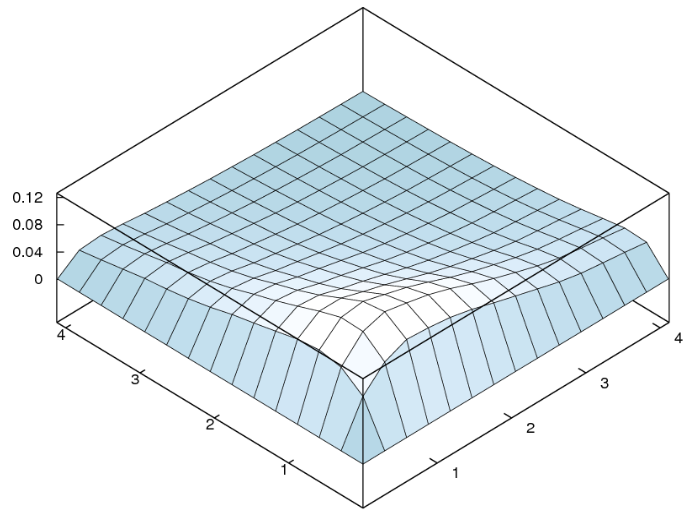

FIGURE 3. Approximation ψ_n^* of ψ^*

6.1. **Optimality.** Our first task is to prove Lemma 3.1. Recall our definition of the auxillary process $(Y_t)_{t\geq 0}$ by $Y_{t+1} = \Lambda Y_t + W_{t+1}$ with $Y_0 = x$. Alternatively, $Y_t = \Lambda^t x + \sum_{j=1}^t \Lambda^{t-j} W_j$. From this expression one can verify the claim in Lemma 3.1 that κ is finite, increasing and continuous. Indeed, since U is concave there exist positive constants b_0 and b_1 such that $U(x) \leq b_0 + b_1\|x\|$ for all $x \in \mathbb{R}^M_+$. Moreover, the matrix norm of Λ is just $\alpha = \max_{1\leq m\leq M} \alpha^m < 1$; and hence $\|\Lambda^k z\| \leq \alpha^k \|z\|$ for any $z \in \mathbb{R}^M$. Consequently,

$$\mathbb{E}U\left(\Lambda^t x + \sum_{j=1}^t \Lambda^{t-j} W_j\right)$$
$$\leq b_0 + b_1 \mathbb{E}\left\|\Lambda^t x + \sum_{j=1}^t \Lambda^{t-j} W_j\right\| \leq b_0 + b_1\|x\| + \frac{\mu}{1-\alpha} =: M(x)$$

Given this bound the finiteness of $\kappa(x) = 1 + \sum_{t=0}^\infty \delta^t \mathbb{E}U(Y_t)$ is immediate. In fact

$$\kappa(x) = 1 + \sum_{t=0}^\infty \delta^t \mathbb{E}U\left(\Lambda^t x + \sum_{j=1}^t \Lambda^{t-j} W_j\right) \leq 1 + \frac{M(x)}{1-\delta}$$

The assertion that κ is increasing follows from monotonicity of U. Continuity of κ follows from continuity of U and the Dominated Convergence Theorem.

The remainder of the proof of Lemma 3.1 is straightforward. Since U is increasing and $X_t - i(X_t) \leq X_t \leq Y_t$ pointwise on Ω for any $i \in \mathscr{I}$ we have $v_i \leq \kappa$. Since $v(x)$ is defined as $\sup_{i \in \mathscr{I}} v_i(x)$ and since $v_i(x) \leq \kappa(x)$ for every $i \in \mathscr{I}$ the function v exists and is dominated by κ. This completes the proof of Lemma 3.1.

Next we turn to the proof of Proposition 3.1. For this proof some extra notation is useful. In particular, for any appropriately integrable function h on $\mathbb{R}^M_+$ we define the new function $\mathbf{N}h$ by

$$\mathbf{N}h(x) := \int h(\Lambda x + z)\varphi(dz)$$

so that $\mathbf{N}h(x)$ is the expectation of $h(Y_t)$ given $Y_{t-1} = x$.[18] Evidently $h \leq h'$ implies $\mathbf{N}h \leq \mathbf{N}h'$, and $\mathbf{N}\mathbb{1} = \mathbb{1}$. We let $\mathbf{N}^t$ be the t-th iterate, in which case $\mathbf{N}^t h(x)$ is the expectation of $h(Y_t)$ given $Y_0 = x$. In particular, $\mathbf{N}^t U(x) = \mathbb{E}U(Y_t)$ for all t, and we can express κ as

$$\kappa(x) = 1 + \sum_{t=0}^{\infty} \delta^t \mathbb{E}U(Y_t) = 1 + \sum_{t=0}^{\infty} \delta^t \mathbf{N}^t U(x)$$

To prove that T is well-defined and contracting on $b_\kappa \mathscr{B}\mathbb{R}^M_+$ we need

Lemma 6.1. *For any $x \in \mathbb{R}^M_+$ the weight function κ satisfies*

$$\sup_{0 \leq \xi \leq x} \int \kappa(\Lambda \xi + z)\varphi(dz) \leq \frac{\kappa(x)}{\delta}$$

Proof. Pick any $x \in \mathbb{R}^M_+$. Since κ is increasing,

$$\sup_{0 \leq \xi \leq x} \int \kappa(\Lambda \xi + z)\varphi(dz) \leq \int \kappa(\Lambda x + z)\varphi(dz) = \mathbf{N}\kappa(x)$$

[18]The operator $\mathbf{N}$ corresponds to T in Stokey, Lucas and Prescott (1989, §8.1).

But from the definitions and the Dominated Convergence Theorem,

$$\begin{aligned}
\mathbf{N}\kappa(x) &= 1 + \mathbf{N} \sum_{t=0}^{\infty} \delta^t \mathbf{N}^t U(x) \\
&= 1 + \sum_{t=0}^{\infty} \delta^t \mathbf{N}^{t+1} U(x) \\
&= 1 + (1/\delta) \sum_{t=0}^{\infty} \delta^{t+1} \mathbf{N}^{t+1} U(x) \\
&\leq 1 + (1/\delta) \sum_{t=0}^{\infty} \delta^t \mathbf{N}^t U(x) \leq (1/\delta) + (1/\delta) \sum_{t=0}^{\infty} \delta^t \mathbf{N}^t U(x)
\end{aligned}$$

This last expression is just $(1/\delta)\kappa(x)$, and the proof is done. □

Using Lemma 6.1 we can show that the Bellman operator T does send $b_\kappa \mathscr{B}\mathbb{R}_+^M$ into itself—in particular, Tw is κ-bounded whenever w is. Indeed, for any $w \in b_\kappa \mathscr{B}\mathbb{R}_+^M$ we have

$$\begin{aligned}
|Tw(x)| &\leq \sup_{0 \leq \xi \leq x} \left| U(x - \xi) + \varrho \int w(\Lambda \xi + z) \varphi(dz) \right| \\
&\leq U(x) + \varrho \sup_{0 \leq \xi \leq x} \int |w(\Lambda \xi + z)| \varphi(dz) \\
&\leq U(x) + \varrho \|w\|_\kappa \sup_{0 \leq \xi \leq x} \int \kappa(\Lambda \xi + z) \varphi(dz) \\
&\leq U(x) + \frac{\varrho \|w\|_\kappa \kappa(x)}{\delta}
\end{aligned}$$

Since $U(x) \leq \kappa(x)$ it follows that for any $x \in \mathbb{R}_+^M$ we have

$$\frac{|Tw(x)|}{\kappa(x)} \leq 1 + \frac{\varrho \|w\|_\kappa}{\delta}$$

Thus Tw is κ-bounded, as was to be shown.

In order to prove that T is a contraction of modulus $\gamma = \varrho/\delta$ we use the following extention of Blackwell's sufficient condition, which is proved in Hernández-Lerma and Lasserre (1999, Proposition 7.2.9).

Lemma 6.2. *If T is monotone and, for any $c \in \mathbb{R}_+$ and $w \in b_\kappa \mathscr{B}\mathbb{R}_+^M$,*

$$T(w + c\kappa) \leq Tw + \gamma c \kappa \tag{25}$$

then T is a $\|\cdot\|_\kappa$-contraction of modulus γ on $b_\kappa \mathscr{B}\mathbb{R}_+^M$.

By monotonicity is meant that for any pair $w, w' \in b_\kappa \mathscr{B}\mathbb{R}_+^M$ with $w \leq w'$ we have $Tw \leq Tw'$. This property is easily shown and the proof is omitted. To verify (25), observe that

$$T(w + c\kappa)(x) =$$
$$\sup_{0 \leq \xi \leq x} \left\{ U(x - \xi) + \varrho \int w(\Lambda\xi + z)\varphi(dz) + c\varrho \int \kappa(\Lambda\xi + z)\varphi(dz) \right\}$$
$$\leq Tw(x) + c\varrho \sup_{0 \leq \xi \leq x} \int \kappa(\Lambda\xi + z)\varphi(dz)$$

In light of Lemma 6.1 we have

$$\sup_{0 \leq \xi \leq x} \int \kappa(\Lambda\xi + z)\varphi(dz) = \int \kappa(\Lambda x + z)\varphi(dz) \leq \frac{\kappa(x)}{\delta}$$

$$\therefore \quad T(w + c\kappa)(x) \leq Tw(x) + \frac{\varrho}{\delta} c\kappa(x)$$

Since $\gamma = \varrho/\delta$ the proof is complete.

The only claim in Proposition 3.1 that remains to be verified is that T maps the set of continuous κ-bounded functions $b_\kappa c\mathbb{R}_+^M$ into itself. In particular, we need to check that if w is continuous κ-bounded then Tw is continuous. To see this, pick any such w. In view of Berge's Theorem of the Maximum, the function

$$Tw(x) = \sup_{0 \leq \xi \leq x} \left\{ U(x - \xi) + \varrho \int w(\Lambda\xi + z)\varphi(dz) \right\}$$

will be continuous provided that

$$(x, \xi) \mapsto U(x - \xi) + \varrho \int w(\Lambda\xi + z)\varphi(dz)$$

is jointly continuous on $\{(x, \xi) : x \in \mathbb{R}_+^M, 0 \leq \xi \leq x\}$. The only nontrivial assertion is that if $(\xi_n) \subset \mathbb{R}_+^M$ with $\xi_n \to \xi$, then

$$\int w(\Lambda\xi_n + z)\varphi(dz) \to \int w(\Lambda\xi + z)\varphi(dz)$$

In view of continuity of w and the Dominated Convergence Theorem, it is sufficient to show that $|w(\Lambda\xi_n + z)|$ is dominated pointwise by some integrable function for all n. But if x is any vector with $\xi_n \leq x$ for all n, then for any $n \in \mathbb{N}$ and any $z \in \mathbb{R}^M_+$ we have

$$|w(\Lambda \xi_n + z)| \leq \|w\|_\kappa \kappa(\Lambda \xi_n + z) \leq \kappa(\Lambda x + z)$$

The integral of the right hand side is finite by Lemma 6.1. This completes the proof of Proposition 3.1.

Next we prove Theorem 3.1. Since T is a $\|\cdot\|_\kappa$-contraction on the Banach space $b_\kappa \mathscr{B}\mathbb{R}^M_+$ it follows that T has a unique fixed point $\bar{w} \in b_\kappa \mathscr{B}\mathbb{R}^M_+$ and $\|T^n w - \bar{w}\|_\kappa \to 0$ as $n \to \infty$ for any $w \in b_\kappa \mathscr{B}\mathbb{R}^M_+$. Moreover, $\bar{w} \in b_\kappa c\mathbb{R}^M_+$ and is therefore continuous, as $b_\kappa c\mathbb{R}^M_+$ is a closed subset of $b_\kappa \mathscr{B}\mathbb{R}^M_+$ on which T is invariant. The proof that $\bar{w}$ is in fact equal to the value function v is almost identical to the standard argument (i.e., the argument for bounded rewards) and is omitted.

Existence of a maximizer $I(x)$ for each x follows from continuity of the objective and compactness of the constraint. Continuity of I follows from Berge's Theorem of the Maximum. That v is strictly increasing and strictly concave can be proved by a small modification of the usual technique.[19]

Let us now consider the proof of Proposition 3.2. First we show that if $x \gg 0$ then $c(x) := x - I(x) \gg 0$. To this end, pick any $x \gg 0$ and let $c := c(x)$. Suppose instead that $c \in \partial \mathbb{R}^M_+$. By Assumption 3.1, there exists a $d \in \mathbb{R}^M_+$ such that $c + d \leq x$ and

$$\lim_{\theta \downarrow 0} \frac{U(c + \theta d) - U(c)}{\theta} = \infty.$$

For v the value function, define the new function g by

$$g(s) = \varrho \int v(\Lambda(x - s) + z)\varphi(dz)$$

[19]It is easy to show that T maps the set $\mathscr{C}$ of increasing concave functions in $b_\kappa c\mathbb{R}^M_+$ into itself. Moreover, a simple argument shows that $\|\cdot\|_\kappa$-convergence implies pointwise convergence, which in turn preserves monotonicity and concavity. Hence $\mathscr{C}$ is $\|\cdot\|_\kappa$-closed. As $T\colon \mathscr{C} \to \mathscr{C}$ and $\mathscr{C}$ is $\|\cdot\|_\kappa$-closed we have $v \in \mathscr{C}$. Finally, T maps elements of $\mathscr{C}$ into strictly increasing, strictly concave functions in $\mathscr{C}$, so v is strictly increasing and strictly concave (because $Tv = v$).

There is no difficulty in checking that g is well-defined and concave on $(-\infty, x]$. It follows that

$$h(\theta) := \frac{g(c + \theta d) - g(c)}{\theta}$$

is well-defined and (by concavity) decreasing on $(-\infty, 0) \cup (0, 1]$. As a result, the limit $\lim_{\theta \downarrow 0} h(\theta)$ exists and is finite.[20] Now since c is optimal, and since the alternative $c + \theta d$ is less than x and therefore feasible at x for $\theta \in (0, 1]$, we must have

$$U(c) + \varrho \int v(\Lambda(x - c) + z)\varphi(dz) \geq U(c + \theta d) + \varrho \int v(\Lambda(x - (c + \theta d)) + z)\varphi(dz)$$

Using the function g this can be rewritten as

$$U(c) + g(c) \geq U(c + \theta d) + g(c + \theta d)$$

Rearranging and dividing through by θ gives

$$\frac{U(c + \theta d) - U(c)}{\theta} \leq -\frac{g(c + \theta d) - g(c)}{\theta} = -h(\theta)$$

The left hand side diverges to infinity as $\theta \downarrow 0$, while the right hand side converges to a finite constant. This contradicts our assuption that $c \in \partial \mathbb{R}^M_+$, and we conclude that $c = x - I(x) \gg 0$.

To complete the proof of Proposition 3.2 we show that v is differentiable on $\operatorname{int} \mathbb{R}^M_+$ and satisfies the envelope condition $\nabla v(x) = \nabla U(x - I(x))$. We use the well-known techniques developed by Mirman and Zilcha (1975) and Benveniste and Scheinkman (1979, Lemma 1). In particular, if $x \in \operatorname{int} \mathbb{R}^M_+$ and w is any concave differentiable function defined on a neighborhood N of x and satisfying $w(x) = v(x)$ and $w(y) \leq v(y)$ for all $y \in N$, then v is differentiable at x and $\nabla v(x) = \nabla w(x)$.

Although investment is not interior, the interiority of consumption is sufficient for this technique to work. To see this, pick any $x_0 \gg 0$, and let $i_0 := I(x_0)$. Since $c(x_0) = x_0 - i_0 \gg 0$ we have $x_0 \gg i_0$, and there

[20]The value $h(\theta)$ increases monotonically as $\theta \downarrow 0$ and is bounded by $h(-1)$.

exists an open neighborhood N of x_0 with $N \subset \operatorname{int} \mathbb{R}_+^M$ and $i_0 \leq x$ for all $x \in N$. On the set N define

$$w(x) = U(x - i_0) + \varrho \int v(\Lambda i_0 + z)\varphi(dz)$$

Note that w is well-defined on N, as $i_0 \leq x$ for all $x \in N$. In addition, for each $x \in N$ investment i_0 is feasible, so

$$w(x) \leq v(x) = U(x - I(x)) + \varrho \int v(\Lambda I(x) + z)\varphi(dz)$$

Evidently w is concave and $v(x_0) = w(x_0)$. Finally, w is differentiable at x_0 with $\nabla w(x_0) = \nabla U(x_0 - I(x_0))$. Hence $\nabla v(x_0) = \nabla U(x_0 - I(x_0))$.

6.2. **Dynamics.** Now we turn to dynamics with a view to proving Theorem 5.1. Recall the definition of q in (20). With respect to this q we define q-small sets, aperiodicity and irreducibility.[21]

Definition 6.1. A set $C \in \mathbb{R}_+^M$ is called q-small if there exists a nontrivial $g \in L_1(\mathbb{R}_+^M)$ such that

$$\forall x \in C, \quad q(x, \cdot) \geq g \tag{26}$$

By nontrivial is meant that g is not the zero element in $L_1(\mathbb{R}_+^M)$. If such set C exists for q, and, moreover, $\int_C g > 0$, then the optimal process (X_t) is called aperiodic.[22]

Definition 6.2. The optimal process (X_t) is called irreducible if, $\forall x_0 \in \mathbb{R}_+^M$ and $\forall B \in \mathscr{B}(\mathbb{R}_+^M)$ with $\lambda(B) > 0$, the process (X_t) started at $X_0 \equiv x_0$ satisfies $\mathbb{P} \cup_{t \geq 1} \{X_t \in B\} > 0$.

Let $Vx) := \|x\| + s$, where $s \in [1, \infty)$ is an arbitrary but fixed constant, as defined in Section 5. By Meyn and Tweedie (1993), Theorem 16.1.2, if $(X_t)_{t \geq 0}$ is aperiodic, irreducible and possesses a q-small set C such that

$$\int V(y)q(x, y)dy \leq \gamma V(x) + b\mathbb{1}_C(x) \qquad x \in \mathbb{R}_+^M \tag{27}$$

[21]See Meyn and Tweedie (1993) for more details on these concepts.

[22]Our definitions of small sets and aperiodicity are slightly stronger than the standard definitions. See Meyn and Tweedie (1993, Chapter 5).

for some $\gamma < 1$ and $b < \infty$, then $(X_t)_{t \geq 0}$ is V-uniformly ergodic. In particular, there exists a unique stationary distribution (in this case a density) ψ^*; the density ψ^* satisfies $\int V(x)\psi^*(x)dx < \infty$; and, moreover, there is a $\beta < 1$ and $N < \infty$ such that

$$\sup_{|h| \leq V} \left| \int h(y)\delta_x \mathbf{M}^t(y)dy - \int h(y)\psi^*(y)dy \right| \leq \beta^t N V(x) \tag{28}$$

for all $t \in \mathbb{N}$ and all $x \in \mathbb{R}_+^M$. Here δ_x is the distribution concentrated at x, so that $\delta_x \mathbf{M}^t$ is the density of X_t when $X_0 \equiv x$.

Below we establish that $(X_t)_{t \geq 0}$ is V-uniformly ergodic. From V-uniform ergodicity the conclusions of Theorem 5.1 follow in a straightforward way. Parts (1) and (2) are immediate. To see that (3) is true, pick any distribution ψ_0 such that $\int \|x\|\psi_0(dx)$ is finite. Now take any $h \in \mathscr{H}_1$. Consider the term

$$\begin{aligned}&\left| \int h(y)\psi_0 \mathbf{M}^t(y)dy - \int h(y)\psi^*(y)dy \right| \\ &\qquad = \left| \int h(y) \left[\int \delta_x \mathbf{M}^t(y)\psi_0(dx) \right] dy - \int h(y)\psi^*(y)dy \right|\end{aligned}$$

Since $|h| \leq V$, (28) implies that the right hand side is dominated by

$$\int \left| \int h(y)\delta_x \mathbf{M}^t(y)dy - \int h(y)\psi^*(y)dy \right| \psi_0(dx) \leq \beta^t N \int V(x)\psi_0(dx)$$

$$\therefore \quad \left| \int h(y)\psi_0 \mathbf{M}^t(y)dy - \int h(y)\psi^*(y)dy \right| \leq \beta^t N \left(\int \|x\|\psi_0(dx) + s \right)$$

As h is any element of $\mathscr{H}_1$ the conclusion of Theorem 5.1 follows.

In summary, to prove Theorem 5.1, it is sufficient to establish that the optimal process (X_t) is irreducible, aperiodic and possesses a small set C such that (27) holds for $V(x) = \|x\| + s$. We begin with irreducibility:

Proposition 6.1. *The optimal process (X_t) is irreducible.*

Proof. Fix $x_0 \in \mathbb{R}_+^M$ and $B \in \mathscr{B}(\mathbb{R}_+^M)$ with $\lambda(B) > 0$. Let $\mathbf{1} \in \mathbb{R}_+^M$ be the vector of ones. Evidently one can select a strictly positive scalar a with the property $\lambda([a\mathbf{1}, \infty) \cap B) > 0$. To prove Proposition 6.1 we need the following two lemmas concerning a.

Lemma 6.3. *If* $x \in (0, a\mathbf{1}]$, *then*

$$\int_B q(x,y)dy = \int_B \varphi(y - \Lambda I(x))dy > 0$$

Proof. For $y \in [a\mathbf{1}, \infty)$, the interiority of x implies that

$$y \geq a\mathbf{1} \geq x \gg \Lambda x \geq \Lambda I(x)$$

$$\therefore \quad \varphi(y - \Lambda I(x)) > 0$$

Since $[a\mathbf{1}, \infty) \cap B$ has positive Lebesgue measure, it follows that

$$\int_{B \cap [a\mathbf{1}, \infty)} \varphi(y - \Lambda I(x))dy > 0$$

$$\therefore \quad \int_B \varphi(y - \Lambda I(x))dy \geq \int_{B \cap [a\mathbf{1}, \infty)} \varphi(y - \Lambda I(x))dy > 0$$

□

Hence $X_t \in (0, a\mathbf{1}]$ implies $X_{t+1} \in B$ with positive probability.

Lemma 6.4. *There is an* $n \geq 0$ *such that* $\mathbb{P}\{X_n \in (0, a\mathbf{1}]\} > 0$, *where* (X_t) *is the process starting at* x_0.

Proof. Recall that $\alpha := \max_{1 \leq m \leq M} \alpha^m$. More generally, let $\|x\|_\infty := \max_{1 \leq m \leq M} x^m$ for any $x = (x^m)_{m=1}^M \in \mathbb{R}_+^M$. Note that $\|\Lambda x\|_\infty \leq \alpha \|x\|_\infty$ holds for any x. Note also that $\| \cdot \|_\infty$ is consistent with the ordering on $\mathbb{R}_+^M$, in the sense that $x \leq y$ implies $\|x\|_\infty \leq \|y\|_\infty$.

Since $a > 0$ and $\alpha < 1$, clearly we can choose an $n \geq 0$ and a $z_0 \gg 0$ such that

$$\alpha^n \|x_0\|_\infty + \|z_0\|_\infty \frac{1}{1 - \alpha} \leq a$$

Let E be the event $\cap_{t \leq n} \{W_t \leq z_0\}$. Evidently E has positive probability, so it suffices to prove that $X_n \leq a\mathbf{1}$ on E. To this end, observe that on E we have

$$X_t \leq \Lambda X_{t-1} + z_0, \quad t = 1, \dots, n$$

$$\therefore \quad X_n \leq \Lambda^n x_0 + \sum_{t=1}^n \Lambda^t z_0$$

$$\therefore \quad \|X_n\|_\infty \leq \alpha^n \|x_0\|_\infty + \|z_0\|_\infty \frac{1}{1 - \alpha}$$

$$\therefore \quad \|X_n\|_\infty \leq a$$
$$\therefore \quad X_n \leq a\mathbf{1}$$

The proof of Lemma 6.4 is now complete. ☐

It remains to complete the proof of Proposition 6.1. Clearly the process is irreducible if we can show that $\mathbb{P}\{X_{n+1} \in B\} > 0$, where n is defined in Lemma 6.4. But this must be so, because

$$\mathbb{P}\{X_{n+1} \in B\} \geq \mathbb{P}[\{X_n \in (0,a]\} \cap \{X_{n+1} \in B\}]$$
$$= \mathbb{P}[\mathbb{P}[\{X_n \in (0,a]\} \cap \{X_{n+1} \in B\} \,|\, \mathscr{F}_n]\,]$$

and

$$\mathbb{P}[\{X_n \in (0,a]\}\mathbb{P}[\{X_{n+1} \in B\} \,|\, \mathscr{F}_n]\,] = \mathbb{P}\left[\{X_n \in (0,a]\} \int_B q(X_n, y) dy\right]$$

This last term is strictly positive, because $\mathbb{P}\{X_n \in [0,a)\} > 0$ by Lemma 6.4, and on $\{X_n \in (0,a]\}$ the integral is strictly positive (Lemma 6.3). The proof is done. ☐

Next we address the existence of small sets.

Lemma 6.5. *All bounded Borel measurable subsets of* $\mathbb{R}^M_+$ *are* q*-small.*

Proof. Since measurable subsets of small sets are small, it suffices to prove that all sets of the form $C = [0,c]$, $c \gg 0$, are q-small. Pick any $c \gg 0$ and set $C := [0,c]$. Let α' be any number satisfying $\max_{1 \leq m \leq M} \alpha^m < \alpha' < 1$, and set

$$K := \{(x,y) \in \mathbb{R}^M_+ \times \mathbb{R}^M_+ : x \in C,\ \alpha' c \leq y \leq c\}$$

For $(x,y) \in K$ we have

$$y - \Lambda I(x) \geq y - \Lambda x \geq y - \Lambda c \geq \alpha' c - \Lambda c \gg 0$$

Since $\varphi(z) > 0$ whenever $z \gg 0$, it follows that $\varphi(y - \Lambda I(x)) > 0$. Combining this observation with the compactness of K and the continuity of φ, it follows that

$$\varepsilon := \min\{\varphi(y - \Lambda I(x)) : (x,y) \in K\}$$

exists and is strictly positive.

Let $g := \varepsilon \mathbb{1}_{[\alpha' c, c]}$. Since $\varepsilon > 0$, $c \gg 0$ and $\alpha' < 1$, the function g is nontrivial. We claim that g satisfies (26). To see this, pick any $x \in C$. If $y \notin [\alpha' c, c]$ then $g(y) = 0$, and (26) must hold. On the other hand, if $y \in [\alpha' c, c]$, then $(x, y) \in K$, and, by the definition of ε,

$$\varphi(y - \Lambda I(x)) \geq \varepsilon \geq g(y)$$

Either way we have $q(x, y) = \varphi(y - \Lambda I(x)) \geq g(y)$ as claimed. □

Lemma 6.6. *The optimal process (X_t) is aperiodic.*

Proof. Let C and g be as in the proof of Lemma 6.5. Evidently

$$\int_C g(y)dy \geq \int_{[\alpha' c, c]} g(y)dy = \varepsilon \lambda([\alpha' c, c]) > 0$$

where positivity follows from $\varepsilon > 0$, $\alpha' < 1$ and $c \gg 0$. □

To prove Theorem 5.1, it remains only to show that there exists a a q-small set C such that

$$\int V(y) q(x, y) dy \leq \gamma V(x) + b \mathbb{1}_C(x) \qquad x \in \mathbb{R}^M_+$$

for some $\gamma < 1$ and $b < \infty$, where $V(x) = \|x\| + s$. Using the change of variable $z = y - \Lambda I(x)$,

$$\int V(y) q(x, y) dy = \int V(y) \varphi(y - \Lambda I(x)) dy = \int V(\Lambda I(x) + z) \varphi(z) dz$$

From $V(x) = \|x\| + s$ this gives

$$\begin{aligned}
\int V(y) q(x, y) dy &= \int \|\Lambda I(x) + z\| \varphi(z) dz + s \\
&\leq \alpha \|x\| + \int \|z\| \varphi(z) dz + s \\
&\leq \alpha V(x) + b, \qquad b := \int \|z\| \varphi(z) dz + s
\end{aligned}$$

where, as before, $\alpha = \max_{1 \leq m \leq M} \alpha^m$. The constant b is finite by Assumption 2.1.

Let γ be any number in $(\alpha, 1)$. Choose a vector $c \gg 0$ such that

$$x \not\leq c \implies \alpha + \frac{b}{V(x)} \leq \gamma$$

It follows that if $x \nleq c$, then

$$\int V(y)q(x,y)dy \leq \alpha V(x) + b \leq \gamma V(x)$$

Defining $C := [0, c]$ now gives

$$\int V(y)q(x,y)dy \leq \gamma V(x) + b\mathbb{1}_C(x) \qquad x \in \mathbb{R}^M_+$$

as required. As C is q-small (Lemma 6.5) the proof is done.

Finally, let us turn to the proofs of Corollary 5.1 and Corollary 5.2.

Proof of Corollary 5.1. The proof is almost trivial: Let $x_0 \in \mathbb{R}^M_0$, and let $\psi_0 = \delta_{x_0}$. Evidently the conditions of Theorem 5.1 part (3) hold, and

$$\sup_{h \in \mathscr{H}_1} \left| \int h(x)\psi_0 \mathbf{M}^t(x)dx - \int h(x)\psi^*(x)dx \right| = O(\beta^t)$$

for some $\beta \in (0, 1)$. Since $\mathscr{H}_1$ contains all Borel measurable real-valued functions h with $|h| \leq 1$, it follows from (24) that $d_1(\psi_t, \psi^*) = O(\beta^t)$. □

Proof of Corollary 5.2. Since $(X_t)_{t \geq 0}$ has been shown to be V-uniformly ergodic, both the LLN and the CLT results are immediate from Meyn and Tweedie (1993, Theorem 17.0.1). □

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