CENTRE FOR APPLIED MACROECONOMIC ANALYSIS

The Australian National University

CAMA Working Paper Series **January, 2010**

EXPECTATIONS TRAPS AND COORDINATION FAILURES: SELECTING AMONG MULTIPLE DISCRETIONARY EQUILIBRIA

Richard Dennis
Federal Reserve Bank of San Francisco

Tatiana Kirsanova
University of Exeter

CAMA Working Paper 2/2010
http://cama.anu.edu.au

Expectations Traps and Coordination Failures: Selecting Among Multiple Discretionary Equilibria*

Richard Dennis† — Federal Reserve Bank of San Francisco

Tatiana Kirsanova‡ — University of Exeter

January 6, 2010

Abstract

Discretionary policymakers cannot manage private-sector expectations and cannot coordinate the actions of future policymakers. As a consequence, expectations traps and coordination failures can occur and multiple equilibria can arise. In order to utilize the explanatory power of models with multiple equilibria it is necessary to understand how an economy arrives to a particular equilibrium. In this paper, we employ notions of robustness, learnability, and the potential for policy errors to motivate and develop a suite of equilibrium selection criteria. Central among these criteria are whether the equilibrium is learnable by private agents and jointly learnable by private agents and the policymaker. We use two New Keynesian policy models to identify the strategic interactions that give rise to multiple equilibria and to illustrate our equilibrium selection methods. Importantly, although the Pareto-preferred equilibrium is invariably an equilibrium identified by standard numerical iterative solution methods, unless it is learnable by private agents, we find little reason to expect coordination on that equilibrium.

Keywords: *Discretionary policymaking, multiple equilibria, coordination, equilibrium seclection.*

JEL Classification: E52, E61, C62, C73.

*We would like to thank seminar participants at the University of Exeter and the Australian National University for comments. The views expressed in this paper do not necessarily reflect those of the Federal Reserve Bank of San Francisco or the Federal Reserve System.

†Address for Correspondence: Economic Research, Mail Stop 1130, Federal Reserve Bank of San Francisco, 101 Market St, CA 94105, USA. Email: richard.dennis@sf.frb.org.

‡Address for correspondence: University of Exeter, School of Business and Economics, Streatham Court, Rennes Drive Exeter EX4 4PU. Email: t.kirsanova@exeter.ac.uk.

1 Introduction

Discretionary policymakers can fall foul of expectations traps and coordination failures. When private agents are forward-looking their expectations, including those they hold about future policy, can influence importantly how policy today is conducted. The discretionary policymaker's Achilles heel is that when formulating policy it is unable to manage private sector expectations, and this inability, although essential for time-consistent policymaking, leaves ajar the door to multiple equilibria. When expectations cannot be managed, private agents can form expectations that, although unwelcome from the policymaker's perspective, lead private agents to react in a manner that traps the policymaker into implementing a policy that validates those expectations. The trap is closed when a policy that renders those unwelcome expectations without foundation is more costly and hence less attractive to the discretionary policymaker than a policy that accommodates them.

The fact that multiple equilibria produces by expectations traps and coordination failures can beset discretionary control problems is troublesome, yet hugely important. Troublesome, because efforts to solve or mitigate the time-consistency problem rely invariably on there being a unique discretionary equilibrium. A Rogoff-style (Rogoff, 1985) approach of delegating objectives to a discretionary policymaker (as per Jensen (2002) and Walsh (2003), among others) is unlikely to be successful unless it also solves the coordination problem. Similarly, to the extent that an optimal contract (Walsh, 1995) can successfully overcome the time-consistency problem, it too should address the coordination problem. Important, because it means that discretionary policy behavior can be considerably richer and more varied than is commonly appreciated. Moreover, because the mechanisms that lead to expectations traps and coordination failures involve strategic interactions between agents, they are not precluded by linear constraints and quadratic objectives. As a consequence, much research analyzing discretionary policymaking since Kydland and Prescott (1977) may have inadvertently considered only one of several equilibria, potentially overlooking essential aspects of discretionary policy behavior.

It is not unusual for economies to transition between periods of high and low inflation, a phenomenon that expectations traps have the potential to explain (Albanesi, Chari, and Christiano, 2003). Similarly, transitions from one equilibrium to another offers an explanation for policy regime changes, like those analyzed by Davig and Leeper (2006). Accordingly, an explanation for the change in U. S. inflation behavior between the 1970s and the 1980s could be that Volcker's appointment to Federal Reserve Chairman served to coordinate expectations and behavior, switching the economy from one discretionary equilibrium to another. However, in order to utilize the explanatory power of multiple equilibria it is necessary to first consider how an economy arrives at a particular equilibrium. In the words of Benhabib and Farmer (1999, pp. 438), "in any model with multiple equilibria one must address the issue of how an equilibrium comes about".

In this paper, we study multiple equilibria in infinite-horizon linear-quadratic discretionary control problems (Blake and Kirsanova, 2007). We describe the control problem facing the discretionary policymaker and, drawing on Oudiz and Sachs (1985) and Currie and Levine (1985, 1993), reinterpret the control problem as a dynamic game between policymakers at different points in time. An important aspect of this game is that, within a period, the policymaker is a Stackelberg leader with respect to private agents. Feedback equilibria to the discretionary control problem correspond to Markov-perfect Stackelberg-Nash equilibria to the dynamic game. We show how strategic interaction among current and future policymakers, operating through endogenous state variables and private sector expectations, leads to a form of strategic complementarity (Cooper and John, 1988) and makes expectations traps and coordination failures possible.

We approach the coordination problem inherent in equilibrium selection from three angles. First, in the spirit of Hansen and Sargent (2008), we consider the discretionary policymaker to be a robust agent that seeks to guard against a malignant nature that coordinates adversely the actions of all future policymakers and private agents. This angle of approach, leads us to examine minimax loss and minimax regret (Savage, 1951) as equilibrium selection criteria. Second, we consider learning as a coordinating mechanism for equilibrium selection (Evans, 1986), drawing on the large literature that employs learning to analyze coordination in rational expectations models.[1] With agents learning eductively, and allowing private agents and/or the policymaker to be learning, we develop three expectational stability conditions whose satisfaction determines whether private agents and/or the policymaker might reasonably learn and coordinate on a particular equilibrium. Among these three sets of stability conditions, we show that the key conditions are those indicating whether an equilibrium is learnable by private agents in isolation and by private agents and the policymaker jointly. Third, we consider whether a form of trembling-hand errors (Selten, 1975) might permit poli-

[1] See Guesnerie and Woodford (1992), Evans and Guesnerie (1993, 2003, 2005), and Evans and Honkapohja (2001), among others.

cymakers to coordinate on the Pareto-preferred equilibrium. Pursuing this idea, we examine whether the perception that policymakers in subsequent periods could make a sequence of (correlated) policy errors might induce the current-period policymaker to pursue the Pareto-preferred policy. Intuitively, if the required number of errors is "small", then coordination on the Pareto-preferred equilibrium might be feasible.

To illustrate equilibrium multiplicity and the equilibrium selection methods, we analyze two New Keynesian models. The first model is the government-debt model developed by Blake and Kirsanova (2007). The second model is a simplified version of the dynamic stochastic general equilibrium models developed by Christiano, Eichenbaum, and Evans (2005) and Smets and Wouters (2007). In each model, the task confronting the policymaker is to stabilize inflation without impacting unduly the real economy. Inflation, in these models, is determined by the expected path of real marginal costs, so the policy challenge is to generate an appropriate path for real marginal costs. Since inflation depends on the entire expected path for real marginal costs while the discretionary policymaker can choose only today's policy, the policy chosen today depends necessarily on expected future policy. At the same time, the decisions that future policymakers make depend materially on the economic circumstances that they find themselves in, and hence on the choices previous policymakers have made. This interaction between policymakers over time produces coordination failure.

In the first model, the equilibrium selection criteria suggest that policymakers might reasonably coordinate on the Pareto-preferred equilibrium. In the second model, however, the message is very different. Here, the equilibrium selection criteria suggest that policymakers will likely coordinate on the Pareto-worst equilibrium, an equilibrium in which the conduct of monetary policy is highly unattractive. Instead of responding to an adverse cost-push shock by raising the interest rate, and thereby lowering real marginal costs today, because it reasons that future policymakers will raise interest rates in response to a higher capital stock, today's policymaker actually lowers the interest rate in order to stimulate investment. This policy stabilizes inflation, even though higher investment raises real marginal costs today, because it increases the capital stock and thereby lowers real marginal costs in the future.

Our research is related to several other papers. Like Blake and Kirsanova (2007), we analyze multiple equilibria in the linear-quadratic context, however where they focus on the existence of multiple equilibria, we focus on equilibrium characteristics and on equilibrium selection. This paper is also related to Albanesi, Chari, and Christiano (2003), who show that a modified version of the Lucas and Stokey (1983) cash-credit model can have multiple discretionary equilibria when some firms have sticky prices, and to King and Wolman (2004), who show how multiple discretionary equilibria can arise with strategic-complementarity in firms' pricing. Although similar strategic interactions are at work, in contrast to Albanesi, Chari, and Christiano (2003) and King and Wolman (2004), in which non-linearity is necessary for multiplicity to occur, the multiplicity that we analyze survives linearization. Where the Stackelberg player in each of these models is a central bank, Ortigueira and Pereira (2009) analyze time-consistent policymaking when the Stackelberg player is a fiscal authority and also find multiple discretionary equilibria.

The remainder of this paper is structured as follows. In Section 2, we describe the linear-quadratic discretionary control problem, provide a game theoretic interpretation, define a symmetric Markov-perfect Stackelberg-Nash equilibrium, and show how such equilibria can be obtained. In Section 3, we outline how expectational stability criteria associated with learning, notions of trembling-hand errors, and robustness, in the form of minimax loss and minimax regret, can be used to select among multiple equilibria. In Section 4, we analyze two New Keynesian policy models, show that they each possess multiple equilibria, and illustrate how the selection criteria can be employed. Section 5 concludes.

2 The discretionary control problem

In this section, we outline the control problem facing a discretionary policymaker. We then reinterpret this control problem as a non-cooperative dynamic game and show that the standard optimal discretionary policy is a symmetric Markov-perfect Nash equilibrium of a dynamic game in which the policymaker is a Stackelberg leader and private agents are followers. To make explicit the game's leadership structure, we call this equilibrium a symmetric Markov-perfect Stackelberg-Nash equilibrium. Finally, we show that solving for a symmetric Markov-perfect Stackelberg-Nash equilibrium in this game requires solving a particular fix-point problem, and we discuss two iterative numerical methods for doing this.

2.1 Constraints and objectives

The economic environment is one in which n_1 predetermined variables, $\mathbf{x}_t$, and n_2 nonpredetermined variables, $\mathbf{y}_t$, $t = 0, 1, ..., \infty$, evolve over time according to

$$\mathbf{x}_{t+1} = \mathbf{A}_{11}\mathbf{x}_t + \mathbf{A}_{12}\mathbf{y}_t + \mathbf{B}_1\mathbf{u}_t + \mathbf{v}_{\mathbf{x}t+1}, \tag{1}$$

$$\mathrm{E}_t\mathbf{y}_{t+1} = \mathbf{A}_{21}\mathbf{x}_t + \mathbf{A}_{22}\mathbf{y}_t + \mathbf{B}_2\mathbf{u}_t, \tag{2}$$

where $\mathbf{u}_t$ is a $p \times 1$ vector of control variables, $\mathbf{v}_{\mathbf{x}t} \sim i.i.d.\,[\mathbf{0}, \mathbf{\Sigma}]$ is an $v \times 1$ $(1 \leq v \leq n_1)$ vector of white-noise innovations, and E_t is the mathematical expectations operator conditional upon period t information. Equations (1) and (2) capture aggregate constraints and technologies and the behavior (aggregate first-order conditions) of private agents. For their part, private agents are comprised of households and firms who are ex ante identical, respectively, infinitely lived, and atomistic. The matrices $\mathbf{A}_{11}$, $\mathbf{A}_{12}$, $\mathbf{A}_{21}$, $\mathbf{A}_{22}$, $\mathbf{B}_1$, and $\mathbf{B}_2$ are conformable with $\mathbf{x}_t$, $\mathbf{y}_t$, and $\mathbf{u}_t$ as necessary and contain the structural parameters that govern preferences and technologies. Importantly, the matrix $\mathbf{A}_{22}$ is assumed to have full rank.

In addition to private agents, the economy is populated by a large player, a policymaker. For each period t, the period-t policymaker's objectives are described by the loss function

$$L_t = \mathrm{E}_t \sum_{k=t}^{\infty} \beta^{(k-t)} \left[\mathbf{z}_k' \mathbf{W} \mathbf{z}_k + 2\mathbf{z}_k' \mathbf{U} \mathbf{u}_k + \mathbf{u}_k' \mathbf{Q} \mathbf{u}_k \right], \tag{3}$$

where $\beta \in (0,1)$ is the discount factor and $\mathbf{z}_k = \left[\begin{array}{cc} \mathbf{x}_k' & \mathbf{y}_k' \end{array} \right]'$. We assume that the weighting matrices $\mathbf{W}$ and $\mathbf{Q}$ are symmetric and, to ensure that the loss function is convex, that the matrix $\left[\begin{array}{cc} \mathbf{W} & \mathbf{U} \\ \mathbf{U}' & \mathbf{Q} \end{array} \right]$ is positive semi-definite.[2] We assume that the policymaker is a Stackelberg leader and that private agents are followers; we further assume that the policymaker does not have access to a commitment technology and that policy is conducted under discretion.[3] With policy conducted under discretion, each period the policymaker sets its control variables, $\mathbf{u}_t$, to minimize equation (3), taking the state, $\mathbf{x}_t$, and the decision rules for all future agents as given. Since the policymaker is a Stackelberg leader, the period-t policy decision is formulated taking equation (2) as well as equation (1) into account.

The control problem described above has many of the characteristics of an infinite horizon non-cooperative dynamic game, and is commonly viewed as such. Following Oudiz and Sachs (1985), Currie and Levine (1985), and Cohen and Michel (1988), the strategic players in the game are the (infinite) sequence of policymakers with private agents behaving competitively. Although private agents are not strategic players they are not inconsequential. Private agents are important because private-sector expectations are the conduit through which strategic

[2] It is standard to assume that the weighting matrices, $\mathbf{W}$ and $\mathbf{Q}$, are symmetric positive semi-definite and symmetric positive definite, respectively (see Anderson, Hansen, McGrattan, and Sargent (1996), for example). However, since many economic applications involve a loss function that places no penalty on the control variables, we note that the requirement of $\mathbf{Q}$ being positive definite can be weakened to $\mathbf{Q}$ being positive semi-definite if additional assumptions about other system matrices are met (Clements and Wimmer, 2003).

[3] Events within a period occur as follows. After observing the state, $\mathbf{x}_t$, decisions are made first by the incumbent policymaker and subsequently by private agents. At the end of the period the shocks $\mathbf{v}_{\mathbf{x}t+1}$ are realized.

interaction between current and future policymakers occurs. In this decision problem, policy behavior is described by a policy strategy, private-agent behavior is described by a private sector strategy, the expectations operator (E_t) and policy loss (payoff) are induced by the policy and private sector strategies, and the equilibrium that we seek to analyze is a symmetric Markov-perfect Stackelberg-Nash equilibrium.

2.2 Some useful definitions and equilibrium concepts

In the previous section we emphasized that the discretionary control problem can be modeled as a non-cooperative dynamic game, with the decisions of the policymaker and of private agents taking the form of strategies. Further, we noted that because the policymaker is assumed to be a Stackelberg leader the discretionary equilibrium that we are interested in is a symmetric Markov-perfect Stackelberg-Nash equilibrium. We now make these terms precise.

Definition 1 *A policy strategy $\mathbf{S}$ is a sequence of policy rules $\{\mathbf{F}_t\}_0^\infty$, where $\mathbf{F}_t$ is a function that maps $\{\mathbf{x}_t\}_0^t$ to $\mathbf{u}_t$. A policy strategy is said to be a Markov policy strategy if and only if each policy rule $\mathbf{F}_t$ is a function that maps $\mathbf{x}_t$ to $\mathbf{u}_t$. We denote by $\mathbf{S}_{-t}$ the sequence of policy rules $\{\mathbf{F}_s\}_0^\infty$ excluding $\mathbf{F}_t$.*

Definition 2 *A private sector strategy $\mathbf{T}$ is a sequence of decision rules $\{\mathbf{H}_t\}_0^\infty$, where $\mathbf{H}_t$ is a function that maps $\{\mathbf{x}_t\}_0^t$ to $\mathbf{y}_t$. A private sector strategy is said to be a Markov private sector strategy if and only if each decision rule $\mathbf{H}_t$ is a function that maps $\mathbf{x}_t$ to $\mathbf{y}_t$. We denote by $\mathbf{T}_{-t}$ the sequence of decision rules $\{\mathbf{H}_s\}_0^\infty$ excluding $\mathbf{H}_t$.*

Definition 3 *A policy strategy $\mathbf{S}$ is a Stackelberg-Nash equilibrium if for every decision period t: i) $\mathbf{F}_t$ minimizes equation (3) subject to equations (1) and (2) and $\mathbf{x}_t$ known, taking $\mathbf{S}_{-t}$ and $\mathbf{T}_{-t}$ as given; and ii) $\mathbf{H}_t$ satisfies equations (1) and (2), taking $\mathbf{S}$ and $\mathbf{T}_{-t}$, as given.*

Definition 4 *A policy strategy $\mathbf{S}$ is a perfect Stackelberg-Nash equilibrium if for every decision period t and any history $\{\mathbf{F}_s, \mathbf{H}_s\}_0^{t-1}$: i) $\mathbf{F}_t$ minimizes equation (3) subject to equations (1) and (2) and $\mathbf{x}_t$ known, taking $\mathbf{S}_{-t}$ and $\mathbf{T}_{-t}$ as given; and ii) $\mathbf{H}_t$ satisfies equations (1) and (2), taking $\mathbf{S}$ and $\mathbf{T}_{-t}$ as given.*

A perfect Stackelberg-Nash equilibrium is time-consistent because it is subgame perfect. However, the strategies that characterize equilibrium are not necessarily Markov strategies and, as a consequence, trigger-strategy equilibria, and other equilibria supported by threats and punishments are not ruled out. The sustainable equilibria studied by Chari and Kehoe (1990), Ireland (1997), and Kurozumi (2008) as well as the "reputational" equilibria examined by Barro and Gordon (1983) are all examples of perfect Stackelberg-Nash equilibria.

Definition 5 *A policy strategy* $\mathbf{S}$ *is a Markov-perfect Stackelberg-Nash equilibrium if restricting* $\mathbf{S}$ *to be a Markov policy strategy and* $\mathbf{T}$ *to be a Markov private sector strategy, for every time period* t *and any history of Markov policy and decision rules* $\{\mathbf{F}_s, \mathbf{H}_s\}_0^{t-1}$*: i)* $\mathbf{F}_t$ *minimizes equation (3) subject to equations (1) and (2) and* $\mathbf{x}_t$ *known, taking* $\mathbf{S}_{-t}$ *and* $\mathbf{T}_{-t}$ *as given; and ii)* $\mathbf{H}_t$ *satisfies equations (1) and (2), taking* $\mathbf{S}$ *and* $\mathbf{T}_{-t}$ *as given.*

Definition 6 *A policy strategy* $\mathbf{S}$ *is a symmetric Markov-perfect Stackelberg-Nash equilibrium if and only if: i)* $\mathbf{S}$ *is a Markov-perfect Stackelberg-Nash equilibrium in which* $\mathbf{F}_t = \mathbf{F}$, $\forall\ t$*; and ii)* $\mathbf{T}$ *is a Markov private sector strategy in which* $\mathbf{H}_t = \mathbf{H}$, $\forall\ t$.

Although the discretionary control problem described in section 2.1 is standard in the monetary policy literature, there are other notions of discretion in the literature. These different notions of discretion are associated either with different dynamic games or with different equilibrium concepts. For example, Cohen and Michel (1988) define and describe both the Stackelberg game, in which the policymaker leads private agents and an alternative game in which the policymaker and private agents choose simultaneously. de Zeeuw and van der Ploeg (1991) analyze open-loop, closed-loop, and feedback equilibria for non-cooperative dynamic games, considering both simultaneous-move and sequential-move games. Blake (2004) shows how to solve for open-loop time-consistent equilibria when the policymaker leads private agents. Ortigueira and Pereira (2009) analyze time-consistent fiscal policy in a model where the government is a Stackelberg leader with respect to private agents when setting taxes, but not when issuing debt. Judd (1998, chapter 16) analyzes time-consistent fiscal policy assuming that the private sector and the government move simultaneously; see also Ortigueira (2006) and Ortigueira and Pereira (2009). In these applications, the government plays against a coalition of private agents while sharing the same objective function, making the game a cooperative one. An alternative, non-cooperative, formulation would posit different objectives for the private sector and the government (Chow, 1997, chapter 6).

2.3 Characterizing equilibrium

For the decision problem summarized by equations (1)—(3), we now describe the equilibrium conditions that characterize a symmetric Markov-perfect Stackelberg-Nash equilibrium, focusing on equilibria for which the decision rules are linear in the state vector.

First, if a symmetric Markov-perfect Stackelberg-Nash equilibrium exists, then in this equilibrium the behavior of the policymaker and private agents in all states, $\mathbf{x}_t$, and in all decision periods, $t = 0, ..., \infty$, is described by the linear rules

$$\mathbf{u}_t = \mathbf{F}\mathbf{x}_t, \tag{4}$$

$$\mathbf{y}_t = \mathbf{H}\mathbf{x}_t, \tag{5}$$

respectively. In this equilibrium, the law-of-motion for the predetermined variables is given by

$$\mathbf{x}_{t+1} = \mathbf{M}\mathbf{x}_t + \mathbf{v}_{\mathbf{x}t+1},$$

where the spectral radius of $\mathbf{M}$ is less than $\beta^{-\frac{1}{2}}$. Further, since the loss function is quadratic and the constraints are linear, the payoff to the policymaker in period t that corresponds to these rules is summarized by the quadratic state-contingent value function

$$V(\mathbf{x}_t) = \mathbf{x}_t^{'}\mathbf{V}\mathbf{x}_t + d,$$

where $\mathbf{V}$ is symmetric positive semi-definite. Importantly, because the policy rule, $\mathbf{F}$, and the decision rule, $\mathbf{H}$, in a symmetric Markov-perfect Stackelberg-Nash equilibrium apply in all states, the subgames one needs to consider when solving for a symmetric Markov-perfect Stackelberg-Nash equilibrium are those indexed only by time.

Second, if a symmetric Markov-perfect Stackelberg-Nash equilibrium exists for the subgame beginning in period $t+1$, then one can condition the subgame beginning in period t on the $\overline{\mathbf{H}}$, $\overline{\mathbf{F}}$, $\overline{\mathbf{M}}$, $\overline{\mathbf{V}}$, and $\overline{d}$ that characterize the equilibrium of the subgame beginning in period $t+1$. Thus, the decision problem facing the policymaker in the subgame beginning in period t is to choose a rule for setting $\mathbf{u}_t$ in order to minimize

$$\begin{aligned}\mathbf{x}_t^{'}\mathbf{V}\mathbf{x}_t + d = {} & \mathbf{x}_t^{'}\mathbf{W}_{11}\mathbf{x}_t + \mathbf{x}_t^{'}\mathbf{W}_{12}\mathbf{y}_t + \mathbf{y}_t^{'}\mathbf{W}_{21}\mathbf{x}_t + \mathbf{y}_t^{'}\mathbf{W}_{22}\mathbf{y}_t + 2\mathbf{x}_t^{'}\mathbf{U}_1\mathbf{u}_t + 2\mathbf{y}_t^{'}\mathbf{U}_2\mathbf{u}_t + \mathbf{u}_t^{'}\mathbf{Q}\mathbf{u}_t \\ & + \beta \mathrm{E}_t\left(\mathbf{x}_{t+1}^{'}\overline{\mathbf{V}}\mathbf{x}_{t+1} + \overline{d}\right),\end{aligned} \tag{6}$$

subject to equations (1) and (2) and

$$\mathbf{u}_{t+1} = \overline{\mathbf{F}}\mathbf{x}_{t+1}, \tag{7}$$

$$\mathbf{y}_{t+1} = \overline{\mathbf{H}}\mathbf{x}_{t+1}, \tag{8}$$

and $\mathbf{x}_t$ known. Importantly, although $\overline{\mathbf{H}}$ and $\overline{\mathbf{V}}$ are functions of $\overline{\mathbf{F}}$, the problem's structure means that $\overline{\mathbf{F}}$ does not have a separate, explicit, effect on the current period payoff, $V(\mathbf{x}_t) =$

$\mathbf{x}_t^{'}\mathbf{V}\mathbf{x}_t + d$. Consequently, as this decision problem is formulated, equation (7) does not bind as a separate constraint.

Using equation (8) to form $\mathrm{E}_t\mathbf{y}_{t+1}$, substituting the resulting expression into equation (2), and exploiting equation (1), we obtain the aggregate private sector reaction function

$$\mathbf{y}_t = \mathbf{J}\mathbf{x}_t + \mathbf{K}\mathbf{u}_t, \tag{9}$$

where

$$\mathbf{J} = \left(\mathbf{A}_{22} - \overline{\mathbf{H}}\mathbf{A}_{12}\right)^{-1}\left(\overline{\mathbf{H}}\mathbf{A}_{11} - \mathbf{A}_{21}\right), \tag{10}$$

$$\mathbf{K} = \left(\mathbf{A}_{22} - \overline{\mathbf{H}}\mathbf{A}_{12}\right)^{-1}\left(\overline{\mathbf{H}}\mathbf{B}_{1} - \mathbf{B}_{2}\right). \tag{11}$$

Provided $rank\,(\mathbf{K}) \neq \mathbf{0}$, equation (9) implies that the period-t policymaker is a Stackelberg leader with respect to the period-t private sector. Then, substituting equation (9) into equations (6) and (1), the decision problem facing the policymaker in the subgame beginning in period t is to choose a rule for setting $\mathbf{u}_t$ in order to minimize

$$\mathbf{x}_t^{'}\mathbf{V}\mathbf{x}_t + d = \mathbf{x}_t^{'}\widehat{\mathbf{W}}\mathbf{x}_t + 2\mathbf{x}_t^{'}\widehat{\mathbf{U}}\mathbf{u}_t + \mathbf{u}_t^{'}\widehat{\mathbf{Q}}\mathbf{u}_t + \beta\mathrm{E}_t\left(\mathbf{x}_{t+1}^{'}\overline{\mathbf{V}}\mathbf{x}_{t+1} + \overline{d}\right), \tag{12}$$

subject to

$$\mathbf{x}_{t+1} = \widehat{\mathbf{A}}\mathbf{x}_t + \widehat{\mathbf{B}}\mathbf{u}_t + \mathbf{v}_{\mathbf{x}t+1}, \tag{13}$$

where

$$\widehat{\mathbf{W}} = \mathbf{W}_{11} + \mathbf{W}_{12}\mathbf{J} + \mathbf{J}^{'}\mathbf{W}_{21} + \mathbf{J}^{'}\mathbf{W}_{22}\mathbf{J}, \tag{14}$$

$$\widehat{\mathbf{U}} = \mathbf{W}_{12}\mathbf{K} + \mathbf{J}^{'}\mathbf{W}_{22}\mathbf{K} + \mathbf{U}_1 + \mathbf{J}^{'}\mathbf{U}_2, \tag{15}$$

$$\widehat{\mathbf{Q}} = \mathbf{Q} + \mathbf{K}^{'}\mathbf{W}_{22}\mathbf{K} + 2\mathbf{K}^{'}\mathbf{U}_2, \tag{16}$$

$$\widehat{\mathbf{A}} = \mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{J}, \tag{17}$$

$$\widehat{\mathbf{B}} = \mathbf{B}_{1} + \mathbf{A}_{12}\mathbf{K}. \tag{18}$$

Conditional on $\overline{\mathbf{H}}$ and $\overline{\mathbf{V}}$ (and $\overline{\mathbf{F}}$), equations (12) and (13) describe a standard linear-quadratic dynamic programming problem. To guarantee existence of a solution, we need $\left(\widehat{\mathbf{A}}, \widehat{\mathbf{B}}\right)$ to be a controllable pair and $\left(\widehat{\mathbf{A}}, \widehat{\mathbf{W}}\right)$ to be a detectable pair (Laub, 1979; Anderson, Hansen, McGrattan, and Sargent, 1996). Suppose that, for a given $\mathbf{J}$ and $\mathbf{K}$, $\left(\widehat{\mathbf{A}}, \widehat{\mathbf{B}}\right)$ is a controllable pair and $\left(\widehat{\mathbf{A}}, \widehat{\mathbf{W}}\right)$ is a detectable pair, then the solution to the subgame beginning in period t has the form of rules (4) and (5), with

$$\mathbf{F} = -\left(\widehat{\mathbf{Q}} + \beta\widehat{\mathbf{B}}'\overline{\mathbf{V}}\widehat{\mathbf{B}}\right)^{-1}\left(\widehat{\mathbf{U}}' + \beta\widehat{\mathbf{B}}'\overline{\mathbf{V}}\widehat{\mathbf{A}}\right), \tag{19}$$

$$\mathbf{0} = \overline{\mathbf{H}}\mathbf{A}_{12}\mathbf{H} - \mathbf{A}_{22}\mathbf{H} + \overline{\mathbf{H}}\left(\mathbf{A}_{11} + \mathbf{B}_1\mathbf{F}\right) - \mathbf{A}_{21} - \mathbf{B}_2\mathbf{F}, \tag{20}$$

$$\mathbf{V} = \widehat{\mathbf{W}} + 2\widehat{\mathbf{U}}\mathbf{F} + \mathbf{F}'\widehat{\mathbf{Q}}\mathbf{F} + \beta\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right)'\overline{\mathbf{V}}\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right), \tag{21}$$

$$d = \beta tr\left(\mathbf{V}\mathbf{\Sigma}\right) + \beta\overline{d}. \tag{22}$$

From $\mathbf{F}$ and $\mathbf{H}$, the matrix $\mathbf{M}$ in the law-of-motion for the predetermined variables is then given by

$$\mathbf{M} = \mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{H} + \mathbf{B}_1\mathbf{F}. \tag{23}$$

Because $\overline{\mathbf{H}}$, $\overline{\mathbf{F}}$, $\overline{\mathbf{M}}$, $\overline{\mathbf{V}}$, and $\overline{d}$ represent a symmetric Markov-perfect Stackelberg-Nash equilibrium for the subgame beginning in period $t+1$, any fix-point of equations (19)—(23) in which $\mathbf{H} = \overline{\mathbf{H}}$, $\mathbf{F} = \overline{\mathbf{F}}$, $\mathbf{M} = \overline{\mathbf{M}}$, $\mathbf{V} = \overline{\mathbf{V}}$, and $d = \overline{d}$, such that $\mathbf{V}$ is symmetric positive semi-definite and $\left(\widehat{\mathbf{Q}} + \beta\widehat{\mathbf{B}}'\mathbf{V}\widehat{\mathbf{B}}\right)$ has full rank, is a symmetric Markov-perfect Stackelberg-Nash equilibrium for the subgame beginning in period t.

Before leaving this section is it worthwhile revisiting the relationship between the period-t and period-$t+1$ policymakers, a relationship that is obscured by the presence of $\overline{\mathbf{V}}$ in equations (6) and (12). By substituting equations (1) and (9) into equation (7) we obtain the reaction function for the period-$t+1$ policymaker

$$\begin{aligned} \mathbf{u}_{t+1} &= \overline{\mathbf{F}}\left[\left(\mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{J}\right)\mathbf{x}_t + \left(\mathbf{A}_{12}\mathbf{K} + \mathbf{B}_1\right)\mathbf{u}_t + \mathbf{v}_{\mathbf{x}t+1}\right], \\ &= \overline{\mathbf{F}}\left(\widehat{\mathbf{A}}\mathbf{x}_t + \widehat{\mathbf{B}}\mathbf{u}_t + \mathbf{v}_{\mathbf{x}t+1}\right) \end{aligned} \tag{24}$$

which, provided $rank\left(\widehat{\mathbf{B}}\right) \neq \mathbf{0}$, implies that the period-t policymaker is a Stackelberg leader with respect to the period-$t+1$ policymaker. Although equation (24) does not appear explicitly as a constraint in equations (12)—(18) its role and importance is contained within equation (21).

2.4 A standard iterative solution method

Although an array of root-solving methods could be used to solve equations (19)—(23), economic applications invariably employ the Backus and Driffill (1986) iterative method, which can be summarized as follows.

1. Guess values for $\overline{\mathbf{H}}$ and $\overline{\mathbf{V}}$ (and $\overline{\mathbf{F}}$)

2. Given $\overline{\mathbf{H}}$ and $\overline{\mathbf{V}}$ (and $\overline{\mathbf{F}}$), obtain values for $\mathbf{F}$ and $\mathbf{V}$ using equations (19) and (21), respectively, and a value for $\mathbf{H}$ using $\mathbf{H} = \mathbf{J} + \mathbf{KF}$.

3. If $\{\mathbf{H}, \mathbf{F}, \mathbf{V}\}$ are sufficiently close to $\left\{\overline{\mathbf{H}}, \overline{\mathbf{F}}, \overline{\mathbf{V}}\right\}$, then move to step 4. Otherwise set $\overline{\mathbf{H}} \leftarrow \mathbf{H}$, $\overline{\mathbf{F}} \leftarrow \mathbf{F}$, and $\overline{\mathbf{V}} \leftarrow \mathbf{V}$ (possibly with damping) and return to step 2.

4. Obtain $\mathbf{M}$ using equation (23) and d using $d = \frac{\beta}{1-\beta} tr\left(\mathbf{V\Sigma}\right)$.

Because the private sector decision rule $\mathbf{H}$ that satisfies the equation $\mathbf{H} = \mathbf{J} + \mathbf{KF}$ also satisfies equation (20) any fix-point found by this iterative method will be a symmetric Markov-perfect Stackelberg-Nash equilibrium.

2.5 Another iterative solution method

An alternative to the Backus and Driffill (1986) method is to use the previously developed Oudiz and Sachs (1985) method. The Oudiz and Sachs (1985) method requires solving a "finite-horizon" problem and then taking the solution of this problem in the limit as $T \uparrow \infty$ as the solution to the infinite-horizon problem. In practice, the Oudiz and Sachs (1985) approach is equivalent to the Backus and Driffill (1986) method described above, with the exception that it dictates *particular* values for $\overline{\mathbf{F}}$, $\overline{\mathbf{H}}$, and $\overline{\mathbf{V}}$ in step 1, values obtained by solving a terminal-period problem. Specifically, in step 1 the Oudiz and Sachs (1985) method suggests the initialization

$$
\begin{aligned}
\overline{\mathbf{F}} &= -\left(\mathbf{Q} + \overline{\mathbf{K}}'\mathbf{W}_{22}\overline{\mathbf{K}} + 2\mathbf{U}_2\overline{\mathbf{K}}\right)^{-1}\left(\overline{\mathbf{K}}'\mathbf{W}_{21} + \overline{\mathbf{K}}'\mathbf{W}_{22}\overline{\mathbf{J}} + \mathbf{U}_1' + \mathbf{U}_2'\overline{\mathbf{J}}\right), \\
\overline{\mathbf{H}} &= \overline{\mathbf{J}} + \overline{\mathbf{KF}}, \\
\overline{\mathbf{V}} &= \mathbf{W}_{11} + 2\overline{\mathbf{H}}'\mathbf{W}_{21} + \overline{\mathbf{H}}'\mathbf{W}_{22}\overline{\mathbf{H}} + 2\mathbf{U}_1\overline{\mathbf{F}} + 2\overline{\mathbf{H}}'\mathbf{U}_2\overline{\mathbf{F}} + \overline{\mathbf{F}}'\mathbf{Q}\overline{\mathbf{F}}, \\
\overline{d} &= 0,
\end{aligned}
$$

where

$$
\begin{aligned}
\overline{\mathbf{J}} &= \left(\mathbf{I} - \mathbf{A}_{22}\right)^{-1}\mathbf{A}_{21}, \\
\overline{\mathbf{K}} &= \left(\mathbf{I} - \mathbf{A}_{22}\right)^{-1}\mathbf{B}_2.
\end{aligned}
$$

By construction, the fix-point found by this iterative method will be a symmetric Markov-perfect Stackelberg-Nash equilibrium.

3 Equilibrium selection

The previous section characterized the conditions that must be satisfied in a symmetric Markov-perfect Stackelberg-Nash equilibrium. These conditions do not speak to the issues of existence and uniqueness of equilibrium, however. Indeed, although Svensson (2000), Dennis and Söderström (2006) and Dennis (2007) find that non-existence can arise when $\mathbf{Q}$ is not positive definite, the literature is largely silent on the question of existance.[4] On the question of uniqueness, the literature has essentially assumed that uniqueness would follow from the linear-quadratic nature of the optimization problem (Oudiz and Sachs, 1985; Söderlind, 1999). However, Blake and Kirsanova (2007) show via counter-example that this assumption is incorrect.

If a model has multiple symmetric Markov-perfect Stackelberg-Nash equilibria, then an obvious and important question is whether the set of equilibria can be reduced and possibly made unique. In some cases one or more equilibria may seem implausible, counter-intuitive, or nonsensical, and one may want to discard those equilibria on these grounds. We do not employ this approach as a selection criterion because it is difficult, if not impossible, to know and specify in a universally accepted way what constitutes implausible, counter-intuitive, behavior. In other cases, one equilibrium may deliver superior performance (measured by loss), and one may want to choose that equilibrium on welfare grounds. However, using welfare, or loss, to select a particular equilibrium requires that agents be able to coordinate on that equilibrium.

In this section, we identify and discuss characteristics that may indicate whether a particular equilibrium is a likely candidate for coordination. We focus on three coordinating mechanisms: robustness, learning, and equilibrium-perturbations associated with trembling-hand errors. Of course there are many other approaches to equilibrium selection that one could take. For example, we acknowledge, but, for the reasons explained in Guesnerie and Woodford (1992), do not pursue the "minimum state variable" criterion (McCallum, 1983) or the "minimum variance" criterion (Taylor, 1977).[5]

[4] Sufficient conditions for the existence of a unique stabilizing equilibrium in the (related) optimal linear regulator problem can be found in Lancaster and Rodman (1995). However, these conditions do not translate automatically to the case in which some constraints are forward looking.

[5] Although linear-quadratic control problems are invariably approximations to non-linear counterparts, to keep our analysis general we tie our hands and do not exploit characteristics of the non-linear formulations to select among solutions to the linear-quadratic approximation.

3.1 Minimax loss and minimax regret

Unlike Hansen and Sargent (2008), who focus on guarding against adverse distortions to an approximating model, here we envisage the robust policymaker as one who wishes to guard against a malignant nature that coordinates mischievously the actions of all future policymakers and private agents. We envisage all policymakers as being robust decisionmakers and assume that each policymaker's desire for robustness is common knowledge. In this environment, the policy that the period-t policymaker finds robust will be the same as the policy that all other policymakers find robust, directing all policymakers to coordinate on the same (robust) policy.

Assume that the model has N symmetric Markov-perfect Stackelberg-Nash equilibria. Because the economic environment is one in which there is complete and perfect information, the existence and nature of all N equilibria is known to all agents. Moreover, the N equilibria can (invariably) be welfare ranked and, as a consequence, agents are not indifferent to which equilibrium prevails.

Let the policy rules associated with the N equilibria constitute the set of policy actions. We consider the expected loss to the period-t policymaker of choosing policy action $\mathbf{F}_i$, $i = 1, ..., N$, when private agents in period $s = t$ respond according to their reaction function and nature has coordinated private agents and policymakers in periods $s = t+1, ..., \infty$ on decision rule $\overline{\mathbf{H}}_j$ and policy action $\overline{\mathbf{F}}_j$, $j = 1, ..., N$, respectively. Because private agents respond according to their reaction function in period t, $\mathbf{H}$ and $\mathbf{M}$ in period t are given by (see equations (9)—(11) and (23))

$$
\begin{aligned}
\mathbf{H} &= \left(\overline{\mathbf{H}}_j \mathbf{A}_{12} - \mathbf{A}_{22}\right)^{-1} \left[\mathbf{A}_{21} + \mathbf{B}_2 \mathbf{F}_i - \overline{\mathbf{H}}_j \left(\mathbf{A}_{11} + \mathbf{B}_1 \mathbf{F}_i\right)\right], \\
\mathbf{M} &= \mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{H} + \mathbf{B}_1 \mathbf{F}_i,
\end{aligned}
$$

from which it follows that the state-contingent loss incurred by the period-t policymaker for each i and j, is

$$
L_{ij}\left(\mathbf{F}_i, \overline{\mathbf{F}}_j\right) = \mathbf{x}_t' \left(\widehat{\mathbf{W}} + 2\widehat{\mathbf{U}}\mathbf{F}_i + \mathbf{F}_i' \widehat{\mathbf{Q}} \mathbf{F}_i + \beta \mathbf{M}' \overline{\mathbf{V}}_j \mathbf{M}\right) \mathbf{x}_t + \frac{\beta}{1-\beta} tr\left[\overline{\mathbf{V}}_j \mathbf{\Sigma}\right].
$$

Using the probability density function for $\mathbf{x}_t$ in equilibrium j to average over the period-t state vector, the average loss incurred by the period-t policymaker is[6]

$$\begin{aligned} EL_{ij}\left(\mathbf{F}_i,\overline{\mathbf{F}}_j\right) &= \mathrm{E}_j\left[L_{ij}\left(\mathbf{F}_i,\overline{\mathbf{F}}_j\right)\right] \\ &= tr\left[\left(\widehat{\mathbf{W}}_i+2\widehat{\mathbf{U}}_i\mathbf{F}_i+\mathbf{F}_i^{\prime}\widehat{\mathbf{Q}}_i\mathbf{F}_i+\beta\mathbf{M}_i^{\prime}\overline{\mathbf{V}}_j\mathbf{M}_i\right)\mathbf{\Omega}_j\right]+\frac{\beta}{1-\beta}tr\left[\overline{\mathbf{V}}_j\mathbf{\Sigma}\right], \end{aligned}$$

where $\mathbf{\Omega}_j$ is the unconditional variance-covariance matrix of the state vector, $\mathbf{x}_t$, in equilibrium j.

The minimax loss policy action is the policy action $\mathbf{F}^* \in \{\mathbf{F}_1,...,\mathbf{F}_N\}$ that solves

$$\mathbf{F}^*=\arg\min_i\left\{\arg\max_j\left[EL_{ij}\left(\mathbf{F}_i,\overline{\mathbf{F}}_j\right)\right]\right\}.$$

An alternative criterion to minimax loss is minimax regret (Savage, 1951). According to the minimax regret criterion, a decisionmaker is thought to evaluate the opportunity cost (regret) of its actions and to choose the action that minimizes the maximum opportunity cost. From the expected payoffs associated with each policy action $\mathbf{F}_i$, when current-period private agents respond according to their reaction functions and nature has coordinated all future private agents on decision rule $\overline{\mathbf{H}}_j$ and all future policymakers on policy action $\overline{\mathbf{F}}_j$, we calculate expected regret

$$ER_{ij}\left(\mathbf{F}_i,\overline{\mathbf{F}}_j\right)=EL_{ij}\left(\mathbf{F}_i,\overline{\mathbf{F}}_j\right)-EL_{jj}\left(\mathbf{F}_j,\overline{\mathbf{F}}_j\right),$$

where, by construction, $ER_{ij}\left(\mathbf{F}_i,\overline{\mathbf{F}}_j\right)$ is non-negative $\forall\ i,j$.

The minimax regret policy action is the policy action $\widehat{\mathbf{F}} \in \{\mathbf{F}_1,...,\mathbf{F}_N\}$ that solves

$$\widehat{\mathbf{F}}=\arg\min_i\left\{\arg\max_j\left[ER_{ij}\left(\mathbf{F}_i,\overline{\mathbf{F}}_j\right)\right]\right\}.$$

3.2 Expectational stability

Evans (1986) motivates expectational stability as a selection criterion in rational expectations models with multiple equilibria. Loosely speaking, a rational expectations equilibrium is expectationally stable if, following small deviations to the expectation formation process, the system returns to that equilibrium under a "natural revision rule". The relevant revision rule emerges naturally from the thought process whereby agents undertake to revise how they form expectations based on how those expectations would effect the actual economy, seeking

[6] Although the period-t optimization is performed conditional on period t information, rather than evaluate loss for every $\mathbf{x}_t$ we instead integrate out (average over) the state vector, $\mathbf{x}_t$, and evaluate average loss so that the analysis is not contingent on any particular realization of the state vector. The integration is performed using the probability density function for $\mathbf{x}_t$ in equilibrium j because the analysis envisages a unilateral deviation by the period-t policymaker from equilibrium j.

to rationalize, or equate, a perceived law-of-motion with the actual law-of-motion. Although the revisions occur in meta-time, there is a close connection between expectational stability and real-time least-squares learnability of a rational expectations equilibrium (Marcet and Sargent, 1989; Evans and Honkapohja, 2001).

Like Evans (1986) and Evans and Guesnerie (2003, 2005), we view learning as a mechanism through which agents may coordinate on an equilibrium. Unlike these studies, however, the models we analyze are populated by both private agents and a policymaker, one or both of which may be learning. As a consequence, we analyze three learning problems and derive three expectational stability related conditions. In each case, the learning that we entertain is eductive in nature with agents revising their behavior in meta-time based on the outcomes of thought experiments. The notion of stability under learning that we consider is iterative expectational stability (IE-stability).[7]

Recall that a symmetric Markov-perfect Stackelberg-Nash equilibrium is characterized by $\{\mathbf{H}, \mathbf{F}, \mathbf{M}, \mathbf{V}, d\}$. Because $\mathbf{M}$ and d follow immediately and uniquely from $\mathbf{F}$, $\mathbf{H}$, and $\mathbf{V}$, we implement the partitioning $\{\{\mathbf{H}, \mathbf{F}, \mathbf{V}\}, \{\mathbf{M}, d\}\}$ and focus on $\{\mathbf{H}, \mathbf{F}, \mathbf{V}\}$ in what follows. Specifically, we consider:

1. Private sector learning, where we analyze whether private agents can learn $\mathbf{H}$, conditional on $\{\mathbf{F}, \mathbf{V}\}$.

2. Policymaker learning, where we analyze whether the policymaker can learn $\{\mathbf{F}, \mathbf{V}\}$, conditional on $\{\mathbf{H}\}$.

3. Joint learning, where we analyze whether private agents and the policymaker can learn $\{\mathbf{H}, \mathbf{F}, \mathbf{V}\}$ jointly.

3.2.1 Preliminaries

To place the three learning problems in a unified framework, let us denote by $\mathbf{\Phi}$ the object(s) to be learned. Thus, in the case where only private agents are learning $\mathbf{\Phi} = \{\mathbf{H}\}$. Then, to determine whether $\mathbf{\Phi}$ is learnable we construct and analyze the T-map that relates a perception of $\mathbf{\Phi}$, denoted $\overline{\mathbf{\Phi}}$, to an actual $\mathbf{\Phi}$, $\mathbf{\Phi} = T\left(\overline{\mathbf{\Phi}}\right)$.

Definition 7 *A fix-point, $\mathbf{\Phi}^*$, of the T-map, $\mathbf{\Phi} = T\left(\overline{\mathbf{\Phi}}\right)$, is said to be IE-stable if*

$$\lim_{k\uparrow\infty} T^k\left(\overline{\mathbf{\Phi}}\right) = \mathbf{\Phi}^*,$$

[7] See Evans (2001) for a very useful discussion of adaptive versus eductive learning and of expectational stability (E-stability) versus iterative expectational stability (IE-stability).

for all $\overline{\mathbf{\Phi}} \neq \mathbf{\Phi}^*$*.*

It follows that $\mathbf{\Phi}^*$ is IE-stable if and only if it is a stable fix-point of the difference equation

$$\mathbf{\Phi}_{k+1} = T\left(\mathbf{\Phi}_k\right), \tag{25}$$

where index k denotes the step of the updating process. Similarly,

Definition 8 *A fix-point,* $\mathbf{\Phi}^*$*, of the T-map,* $\mathbf{\Phi} = T\left(\overline{\mathbf{\Phi}}\right)$*, is said to be locally IE-stable if*

$$\lim_{k\uparrow\infty} T^k\left(\overline{\mathbf{\Phi}}\right) = \mathbf{\Phi}^*,$$

for all $\overline{\mathbf{\Phi}}$ *about a neighborhood of* $\mathbf{\Phi}^*$*.*

Let the derivative of the T-map be denoted $DT\left(\mathbf{\Phi}^*\right)$, then it is straightforward to prove the following Lemma.

Lemma 1 *Assume that the derivative map,* $DT\left(\mathbf{\Phi}^*\right)$*, has no eigenvalues with modulus equal to* 1. *A fix-point,* $\mathbf{\Phi}^*$*, of the T-map,* $\mathbf{\Phi} = T\left(\overline{\mathbf{\Phi}}\right)$*, is locally IE-stable if and only if all eigenvalues of the derivative map,* $DT\left(\mathbf{\Phi}^*\right)$*, have modulus less than* 1*.*

Proof. Following Evans (1985), to analyze the local stability of equation (25) we linearize the equation about $\mathbf{\Phi}^*$. Using matrix calculus results from Magnus and Neudecker (1988, chapter 9) we obtain

$$d\left(vec\left(\mathbf{\Phi}_{k+1}\right)\right) = DT\left(\mathbf{\Phi}^*\right) d\left(vec\left(\mathbf{\Phi}_k\right)\right)$$

where $DT\left(\mathbf{\Phi}^*\right) = \partial\left(vec\left(T\left(\mathbf{\Phi}^*\right)\right)\right) / \partial\left(vec\left(\mathbf{\Phi}\right)\right)'$. Applying standard results for linear difference equations, if all of the eigenvalues of $DT\left(\mathbf{\Phi}^*\right)$ have modulus less than one, then $\mathbf{\Phi}^*$ is locally stable. In contrast, if one or more of the eigenvalues of $DT\left(\mathbf{\Phi}^*\right)$ have modulus greater than one, then $\mathbf{\Phi}^*$ is not locally stable. ■

3.2.2 Eductive learning by private agents

We begin with the case in which only private agents are learning and examine whether private agents can learn $\mathbf{H}$, given $\{\mathbf{F}, \mathbf{V}\}$. For a given policy rule, $\mathbf{u}_t = \mathbf{F}\mathbf{x}_t$, and a postulated private sector decision rule

$$\mathbf{y}_t = \overline{\mathbf{H}}\mathbf{x}_t,$$

the actual private sector decision rule takes the form

$$\mathbf{y}_t = \mathbf{H}\mathbf{x}_t,$$

where

$$\mathbf{H} = \left(\overline{\mathbf{H}}\mathbf{A}_{12} - \mathbf{A}_{22}\right)^{-1} \left[\mathbf{A}_{21} + \mathbf{B}_2\mathbf{F} - \overline{\mathbf{H}}\left(\mathbf{A}_{11} + \mathbf{B}_1\mathbf{F}\right)\right]. \tag{26}$$

Equation (26) describes the T-map, $T(\overline{\mathbf{H}})$, from $\overline{\mathbf{H}}$ to $\mathbf{H}$; it is, of course, equivalent to equation (20).

Lemma 2 *A symmetric Markov-perfect Stackelberg-Nash equilibrium is locally IE-stable under private sector learning if and only if all eigenvalues of*

$$-\left[\mathbf{I} \otimes (\mathbf{H}\mathbf{A}_{12} - \mathbf{A}_{22})\right]^{-1} \left[(\mathbf{A}_{11}+\mathbf{A}_{12}\mathbf{H} + \mathbf{B}_1\mathbf{F})' \otimes \mathbf{I}\right]$$

have modulus less than 1*.*

Proof. Applying standard matrix calculus rules to equation (26), the total differential can be written as

$$(\mathbf{H}\mathbf{A}_{12} - \mathbf{A}_{22})\, d\,(\mathbf{H}) + d\left(\overline{\mathbf{H}}\right) \mathbf{A}_{12}\mathbf{H} + d\left(\overline{\mathbf{H}}\right) (\mathbf{A}_{11} + \mathbf{B}_1\mathbf{F}) = \mathbf{0},$$

which after vectorizing can be rearranged to give

$$vec\left[d\,(\mathbf{H})\right] = -\left[\mathbf{I} \otimes (\mathbf{H}\mathbf{A}_{12} - \mathbf{A}_{22})\right]^{-1} \left[(\mathbf{A}_{11}+\mathbf{A}_{12}\mathbf{H} + \mathbf{B}_1\mathbf{F})' \otimes \mathbf{I}\right] vec\left[d\left(\overline{\mathbf{H}}\right)\right].$$

We apply Lemma 1 to obtain the required result. Note that invertability of $(\mathbf{H}\mathbf{A}_{12} - \mathbf{A}_{22})$ is virtually ensured by the assumption that $\mathbf{A}_{22}$ has full rank. ■

Because the eigenvalues of $\mathbf{M} = \mathbf{A}_{11}+\mathbf{A}_{12}\mathbf{H}+\mathbf{B}_1\mathbf{F}$ are all strictly less than $\beta^{-\frac{1}{2}}$, equilibria that are not locally IE-stable under private sector learning are those for which $(\mathbf{H}\mathbf{A}_{12} - \mathbf{A}_{22})$ is close to equaling the null matrix.

3.2.3 Eductive learning by the leader

We now turn to the case where the policymaker is learning, but private agents are not. Here we examine whether the policymaker can learn $\{\mathbf{F}, \mathbf{V}\}$, given $\{\mathbf{H}\}$. We show that although learning by policymakers is interesting and important in many contexts, here this local IE-stability criterion cannot discriminate among equilibria.

For a given private sector decision rule, $\mathbf{y}_t = \mathbf{H}\mathbf{x}_t$, and a postulated policy rule

$$\mathbf{u}_t = \overline{\mathbf{F}}\mathbf{x}_t,$$

and a postulated value function matrix $\overline{\mathbf{V}}$, the T-map $T(\overline{\mathbf{F}}, \overline{\mathbf{V}})$, from $\left\{\overline{\mathbf{F}}, \overline{\mathbf{V}}\right\}$ to $\{\mathbf{F}, \mathbf{V}\}$ is described by the following updating relationships

$$\mathbf{F} = -\left(\widehat{\mathbf{Q}} + \beta\widehat{\mathbf{B}}'\overline{\mathbf{V}}\widehat{\mathbf{B}}\right)^{-1} \left(\widehat{\mathbf{U}}' + \beta\widehat{\mathbf{B}}'\overline{\mathbf{V}}\widehat{\mathbf{A}}\right), \tag{27}$$

$$\mathbf{V} = \widehat{\mathbf{W}} + 2\widehat{\mathbf{U}}\mathbf{F} + \mathbf{F}'\widehat{\mathbf{Q}}\mathbf{F} + \beta\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right)' \overline{\mathbf{V}} \left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right), \tag{28}$$

where $\widehat{\mathbf{W}}$, $\widehat{\mathbf{U}}$, $\widehat{\mathbf{Q}}$, $\widehat{\mathbf{A}}$, and $\widehat{\mathbf{B}}$ are defined by equations (14)—(18) and do not depend on $\mathbf{F}$ or $\mathbf{V}$ (or on $\overline{\mathbf{F}}$ or $\overline{\mathbf{V}}$). Notice, that $\mathbf{F}$, given $\mathbf{H}$, is uniquely determined by $\mathbf{V}$, so the key to learning $\mathbf{F}$ is to learn $\mathbf{V}$. As a consequence, without loss of generality we can substitute equation (27) into equation (28) and analyze the the learning problem using the concentrated T-map $T(\overline{\mathbf{V}}) = \mathbf{V}$.

Lemma 3 *All symmetric Markov-perfect Stackelberg-Nash equilibria are locally IE-stable under policymaker learning.*

Proof. Applying standard matrix calculus rules to equations (27) and (28), total differentials are given by

$$\left(\widehat{\mathbf{Q}} + \beta\widehat{\mathbf{B}}'\mathbf{V}\widehat{\mathbf{B}}\right) d\left(\mathbf{F}\right) + \beta\widehat{\mathbf{B}}' d\left(\overline{\mathbf{V}}\right)\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right) = \mathbf{0}, \tag{29}$$

$$2\left[\widehat{\mathbf{U}} + \mathbf{F}'\widehat{\mathbf{Q}} + \beta\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right)'\mathbf{V}\widehat{\mathbf{B}}\right] d\left(\mathbf{F}\right) + \beta\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right)' d\left(\overline{\mathbf{V}}\right)\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right) = \mathbf{I}d\left(\mathbf{V}\right). \tag{30}$$

Using equation (29) to solve for $d\left(\mathbf{F}\right)$ and substituting the resulting expression into equation (30) yields, upon rearranging,

$$\beta\left[-2\left(\widehat{\mathbf{U}} + \beta\widehat{\mathbf{A}}'\mathbf{V}\mathbf{B}\right)\left(\widehat{\mathbf{Q}} + \beta\widehat{\mathbf{B}}'\mathbf{V}\widehat{\mathbf{B}}\right)^{-1}\widehat{\mathbf{B}}' - 2\mathbf{F}'\widehat{\mathbf{B}}' + \left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right)'\right] d\left(\overline{\mathbf{V}}\right)\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right) = \mathbf{I}d\left(\mathbf{V}\right),$$

which, given equation (27), collapses to

$$\beta\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right)' d\left(\overline{\mathbf{V}}\right)\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right) = \mathbf{I}d\left(\mathbf{V}\right). \tag{31}$$

After vectorizing and recognizing that $\mathbf{M} = \widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}$, equation (31) can be written as

$$vec\left[d\left(\mathbf{V}\right)\right] = \beta\left(\mathbf{M}' \otimes \mathbf{M}'\right) vec\left[d\left(\overline{\mathbf{V}}\right)\right].$$

The matrix $\beta\left(\mathbf{M}' \otimes \mathbf{M}'\right)$ defines the derivative map $DT\left(\mathbf{V}\right)$. Applying Lemma 1, a symmetric Markov-perfect Stackelberg-Nash equilibria $\{\mathbf{H}, \mathbf{F}, \mathbf{M}, \mathbf{V}, d\}$ is a local IE-stable policy equilibrium if and only if all of the eigenvalues of $DT\left(\mathbf{V}\right)$ have modulus less than 1. Because the eigenvalues of $\mathbf{M}$ all have modulus less than $\beta^{-\frac{1}{2}}$ in all symmetric Markov-perfect Stackelberg-Nash equilibria the result follows. ■

3.2.4 Joint eductive learning

Finally, we analyze the case in which both private agents and the policymaker are learning. The postulated policy and decision rules are

$$\mathbf{y}_t = \overline{\mathbf{H}}\mathbf{x}_t,$$

$$\mathbf{u}_t = \overline{\mathbf{F}}\mathbf{x}_t,$$

and the postulated value function matrix is $\overline{\mathbf{V}}$. Then the actual policy and decision rules are given by

$$\begin{aligned}
\mathbf{H} &= \mathbf{J}+\mathbf{K}\mathbf{F}, && (32)\\
\mathbf{F} &= -\left(\widehat{\mathbf{Q}}+\beta\widehat{\mathbf{B}}'\overline{\mathbf{V}}\widehat{\mathbf{B}}\right)^{-1}\left(\widehat{\mathbf{U}}+\beta\widehat{\mathbf{B}}'\overline{\mathbf{V}}\widehat{\mathbf{A}}\right), && (33)\\
\mathbf{V} &= \widehat{\mathbf{W}}+2\widehat{\mathbf{U}}\mathbf{F}+\mathbf{F}'\widehat{\mathbf{Q}}\mathbf{F}+\beta\left(\widehat{\mathbf{A}}+\widehat{\mathbf{B}}\mathbf{F}\right)'\overline{\mathbf{V}}\left(\widehat{\mathbf{A}}+\widehat{\mathbf{B}}\mathbf{F}\right), && (34)
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{J} &= \left(\mathbf{A}_{22}-\overline{\mathbf{H}}\mathbf{A}_{12}\right)^{-1}\left(\overline{\mathbf{H}}\mathbf{A}_{11}-\mathbf{A}_{21}\right), && (35)\\
\mathbf{K} &= \left(\mathbf{A}_{22}-\overline{\mathbf{H}}\mathbf{A}_{12}\right)^{-1}\left(\overline{\mathbf{H}}\mathbf{B}_{1}-\mathbf{B}_{2}\right), && (36)
\end{aligned}$$

and $\widehat{\mathbf{W}}$, $\widehat{\mathbf{U}}$, $\widehat{\mathbf{Q}}$, $\widehat{\mathbf{A}}$, and $\widehat{\mathbf{B}}$ are defined by equations (14)—(18) and are functions of $\mathbf{J}$ and $\mathbf{K}$.

Given equations (35) and (36), equations (32)—(34) describe the T-map, $T\left(\overline{\mathbf{H}},\overline{\mathbf{F}},\overline{\mathbf{V}}\right)$, from $\left\{\overline{\mathbf{H}},\overline{\mathbf{F}},\overline{\mathbf{V}}\right\}$, to $\{\mathbf{H},\mathbf{F},\mathbf{V}\}$.

Lemma 4 *A symmetric Markov-perfect Stackelberg-Nash equilibrium is locally IE-stable under joint learning if and only if all eigenvalues of the matrix* $\mathbf{P}^{-1}\mathbf{L}$ *in*

$$vec\left[d\left(\mathbf{G}\right)\right]=\mathbf{P}^{-1}\ \mathbf{L}vec\left[d\left(\overline{\mathbf{G}}\right)\right],$$

where $vec\left[d\left(\mathbf{G}\right)\right]=\left[\begin{array}{ccc} vec\left[d\left(\mathbf{H}\right)\right]' & vec\left[d\left(\mathbf{F}\right)\right]' & vec\left[d\left(\mathbf{V}\right)\right]'\end{array}\right]'$ *and* $\mathbf{P}$ *and* $\mathbf{L}$ *are characterized below, have modulus less than* 1.

Proof. Total differentials of equations (32)—(36) about the point $\{\mathbf{H},\mathbf{F},\mathbf{V},\mathbf{J},\mathbf{K}\}$ are given by

$$\begin{aligned}
\mathbf{0} &= d\left(\mathbf{J}\right)+d\left(\mathbf{K}\right)\mathbf{F}+\mathbf{K}d\left(\mathbf{F}\right)-d\left(\mathbf{H}\right), && (37)\\
\mathbf{0} &= d\left(\overline{\mathbf{H}}\right)\widehat{\mathbf{A}}-\left(\mathbf{A}_{22}-\mathbf{H}\mathbf{A}_{12}\right)d\left(\mathbf{J}\right), && (38)\\
\mathbf{0} &= d\left(\overline{\mathbf{H}}\right)\widehat{\mathbf{B}}-\left(\mathbf{A}_{22}-\mathbf{H}\mathbf{A}_{12}\right)d\left(\mathbf{K}\right), && (39)\\
\mathbf{0} &= \beta\widehat{\mathbf{B}}'d\left(\overline{\mathbf{V}}\right)\mathbf{M}+\left(\widehat{\mathbf{Q}}+\beta\widehat{\mathbf{B}}'\mathbf{V}\widehat{\mathbf{B}}\right)d\left(\mathbf{F}\right)+2\left(\mathbf{K}'\mathbf{W}_{22}+\mathbf{U}_2'+\beta\widehat{\mathbf{B}}'\mathbf{V}\mathbf{A}_{12}\right)d\left(\mathbf{K}\right)\mathbf{F}\\
&\quad+\left(\mathbf{W}_{12}+\mathbf{J}'\mathbf{W}_{22}+\beta\widehat{\mathbf{A}}'\mathbf{V}\mathbf{A}_{12}\right)d\left(\mathbf{K}\right)+\left(\mathbf{K}'\mathbf{W}_{22}+\mathbf{U}_2'+\beta\widehat{\mathbf{B}}'\mathbf{V}\mathbf{A}_{12}\right)d\left(\mathbf{J}\right), && (40)\\
\mathbf{0} &= 2\left(\widehat{\mathbf{U}}+\mathbf{F}'\widehat{\mathbf{Q}}+\beta\mathbf{M}'\mathbf{V}\widehat{\mathbf{B}}\right)d\left(\mathbf{F}\right)+2\left(\mathbf{W}_{12}+\mathbf{H}'\mathbf{W}_{22}+\mathbf{F}'\mathbf{U}_2'+\beta\mathbf{M}'\mathbf{V}\mathbf{A}_{12}\right)d\left(\mathbf{J}\right)\\
&\quad+2\left(\mathbf{W}_{12}+\mathbf{H}'\mathbf{W}_{22}+\mathbf{F}'\mathbf{U}_2'+\beta\mathbf{M}'\mathbf{V}\mathbf{A}_{12}\right)d\left(\mathbf{K}\right)\mathbf{F}+\beta\mathbf{M}'d\left(\overline{\mathbf{V}}\right)\mathbf{M}-d\left(\mathbf{V}\right). && (41)
\end{aligned}$$

Now, using equations (38) and (39) to solve for $d(\mathbf{J})$ and $d(\mathbf{K})$, respectively, and substituting these expressions into equations (37), (40), and (41) produces

$$\begin{aligned}
\mathbf{0} &= \mathbf{K}d(\mathbf{F}) + (\mathbf{A}_{22} - \mathbf{HA}_{12})^{-1} d\left(\overline{\mathbf{H}}\right)\mathbf{M} - d(\mathbf{H}), &(42)\\
\mathbf{0} &= \beta\widehat{\mathbf{B}}' d\left(\overline{\mathbf{V}}\right)\mathbf{M} + \left(\widehat{\mathbf{Q}} + \beta\widehat{\mathbf{B}}'\mathbf{V}\widehat{\mathbf{B}}\right) d(\mathbf{F})\\
&\quad + \left(\mathbf{W}_{12} + \mathbf{J}'\mathbf{W}_{22} + \beta\widehat{\mathbf{A}}'\mathbf{V}\mathbf{A}_{12}\right)(\mathbf{A}_{22} - \mathbf{HA}_{12})^{-1} d\left(\overline{\mathbf{H}}\right)\widehat{\mathbf{B}}\\
&\quad + 2\left(\mathbf{K}'\mathbf{W}_{22} + \mathbf{U}_2' + \beta\widehat{\mathbf{B}}'\mathbf{V}\mathbf{A}_{12}\right)(\mathbf{A}_{22} - \mathbf{HA}_{12})^{-1} d\left(\overline{\mathbf{H}}\right)\widehat{\mathbf{B}}\mathbf{F}\\
&\quad + \left(\mathbf{K}'\mathbf{W}_{22} + \mathbf{U}_2' + \beta\widehat{\mathbf{B}}'\mathbf{V}\mathbf{A}_{12}\right)(\mathbf{A}_{22} - \mathbf{HA}_{12})^{-1} d\left(\overline{\mathbf{H}}\right)\widehat{\mathbf{A}} &(43)\\
\mathbf{0} &= 2\left(\widehat{\mathbf{U}} + \mathbf{F}'\widehat{\mathbf{Q}} + \beta\mathbf{M}'\mathbf{V}\widehat{\mathbf{B}}\right) d(\mathbf{F}) + \beta\mathbf{M}' d\left(\overline{\mathbf{V}}\right)\mathbf{M} - d(\mathbf{V})\\
&\quad + 2\left(\mathbf{W}_{12} + \mathbf{H}'\mathbf{W}_{22} + \mathbf{F}'\mathbf{U}_2' + \beta\mathbf{M}'\mathbf{V}\mathbf{A}_{12}\right)(\mathbf{A}_{22} - \mathbf{HA}_{12})^{-1} d\left(\overline{\mathbf{H}}\right)\mathbf{M}, &(44)
\end{aligned}$$

where, again, the invertability of $(\mathbf{A}_{22} - \mathbf{HA}_{12})$ is virtually ensured by the assumption that $\mathbf{A}_{22}$ has full rank. By vectorizing and stacking equations (42)—(44) they can be written in the form

$$\mathbf{P}vec[d(\mathbf{G})] = \mathbf{L}vec\left[d\left(\overline{\mathbf{G}}\right)\right],$$

where

$$\mathbf{P} = \begin{bmatrix} \mathbf{I} & -\mathbf{K} & \mathbf{0}\\ \mathbf{0} & -\left(\widehat{\mathbf{Q}} + \beta\widehat{\mathbf{B}}'\mathbf{V}\widehat{\mathbf{B}}\right) & \mathbf{0}\\ \mathbf{0} & -2\left(\widehat{\mathbf{U}} + \mathbf{F}'\widehat{\mathbf{Q}} + \beta\mathbf{M}'\mathbf{V}\widehat{\mathbf{B}}\right) & \mathbf{I} \end{bmatrix},$$

and $\mathbf{L}$ is defined implicitly by equations (42)—(44). Because $\left(\widehat{\mathbf{Q}} + \beta\widehat{\mathbf{B}}'\mathbf{V}\widehat{\mathbf{B}}\right)$ has full rank in any symmetric Markov-perfect Stackelberg-Nash equilibrium, $\mathbf{P}$ too has full rank. The result follows. ■

Lemma 5 *The equilibrium identified by Oudiz and Sachs (1985) and all equilibria identified by Backus and Driffill (1986) are IE-stable under joint learning.*

Proof. The iterative numerical schemes employed by the Backus and Driffill (1986) and Oudiz and Sachs (1985) solution methods coincide with the learning scheme described by the T-map (32)—(34). As a consequence, these numerical solution methods apply direct numerical iterations on the non-linear T-map. If these numerical solution methods converge to a fix-point, then, by construction, the resulting equilibrium is IE-stable under joint learning.

■

Before leaving this section, we wish to emphasize that the IE-stability criteria associated with private sector learning and joint learning, although connected, are distinct. Joint learnability of an equilibrium neither implies nor is implied by private sector learnability of that equilibrium.

3.3 Trembling-hand errors

Building on sections 3.1 and 3.2, we now consider whether policymakers might be able to coordinate on the Pareto-preferred equilibrium. As earlier, assume that the model has N symmetric Markov-perfect Stackelberg-Nash equilibria that can be welfare ranked and whose existence and nature is known to all agents. Treating the policy rules associated with the N equilibria as a set of policy actions, because the equilibria are Nash, if policymakers in periods $s = t+1, ..., \infty$ are predicted to play $\overline{\mathbf{F}}_j$, $j = 1, ..., N$, then the period-t policymaker's best response is to also play $\mathbf{F}_j$. However, although it is never beneficial for the period-t policymaker to unilaterally deviate from Nash play, outside of the Pareto-preferred equilibrium the period-t policymaker can potentially benefit from deviations that involve multiple policymakers. With this in mind, we introduce a sequence of "trembling-hand" deviations from Nash play. Intuitively, the policy action in the Pareto-preferred equilibrium can be more easily coordinated upon if it can be supported by a relatively small group (or coalition) of deviating policymakers, a relatively small number of trembling-hand errors,[8] against all other policy actions.

Let equilibrium N represent the Pareto-preferred equilibrium. Consider the period-t policymaker's best response where the predicted future play is $\{\mathbf{F}_N^{t+1}, ..., \mathbf{F}_N^{t+p_j}, \mathbf{F}_j^{t+p_j+1}, \mathbf{F}_j^{t+p_j+2}, ...\}$, $j \neq N$, with private agents in periods $s = t, ..., \infty$ responding according to their reaction function. In this scenario, during periods $s = t+p_j+1, ..., \infty$ the policy rule and private-sector decision rules are given by $\mathbf{F}_j$ and $\mathbf{H}_j$, respectively. However, during periods $s = t, ..., t+p_j$ the policy rule is given by $\mathbf{F}_N$ and private agents respond according to their reaction function,

$$\mathbf{H}^s = \left(\mathbf{H}^{s+1}\mathbf{A}_{12} - \mathbf{A}_{22}\right)^{-1}\left[\mathbf{A}_{21} + \mathbf{B}_2\mathbf{F}_N - \mathbf{H}^{s+1}\left(\mathbf{A}_{11} + \mathbf{B}_1\mathbf{F}_N\right)\right]. \tag{45}$$

Given equation (45), the law-of-motion for the state vector during periods $s = t, ..., t+p_j$ is

$$\mathbf{M}^s = \mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{H}^s + \mathbf{B}_1\mathbf{F}_N.$$

[8] One might view the group of deviating policymakers to be small if it numbers less than a policymaker's average tenure. In the U. S., Federal Reserve chairmen are appointed to a four year term, but the average tenure is somewhat longer. In the U. K., monetary policy committee members have three-year contracts that overlap to prevent members from retiring simultaneously.

We know that if $p_j = 0$, then the period-t policymaker's best response is to play $\mathbf{F}_j$. However, as p_j increases, the period-t policymaker's best response can switch from $\mathbf{F}_j$ to $\mathbf{F}_N$. For each $\mathbf{F}_j$, we calculate the number of periods of multilateral deviation p_j required to switch the period-t policymaker's best response from $\mathbf{F}_j$ to $\mathbf{F}_N$. If p_j is relatively small for all $j \neq N$, then the Pareto-preferred equilibrium can be supported by a relatively small group of deviating policymakers, raising the likelihood that coordination on the Pareto-preferred equilibrium could occur.

Although the period-t policymaker's best response may switch from $\mathbf{F}_j$ to $\mathbf{F}_N$ as p_j increases, it need not. In fact, whether the period-t policymaker's best response switches from $\mathbf{F}_j$ to $\mathbf{F}_N$ as p_j increases turns on whether the Pareto-preferred equilibrium is locally IE-stable under private sector learning.

Lemma 6 *The period-t policymakers best response will switch from $\mathbf{F}_j$ to $\mathbf{F}_N$ in the limit as $p_j \uparrow \infty$ if and only if the Pareto-preferred equilibrium is locally IE-stable under private sector learning.*

Proof. Consider equation (45). If equilibrium N is locally IE-stable under private sector learning, then, $\mathbf{H}^s \rightarrow \mathbf{H}^N$ in the limit as $p_j \uparrow \infty$, which implies $\mathbf{M}^s \rightarrow \mathbf{M}^N$ and $\mathbf{V}^s \rightarrow \mathbf{V}^N$. Because equilibrium N Pareto-dominates all other symmetric Markov-perfect Stackelberg-Nash equilibria, the period-t policymaker's best response must switch from $\mathbf{F}_j$ to $\mathbf{F}_N$. On the contrary, if equilibrium N is not locally IE-stable under private sector learning, then although $\mathbf{H}^s$ may converge to $\widetilde{\mathbf{H}} \neq \mathbf{H}^N$ in the limit as $p_j \uparrow \infty$, because $\widetilde{\mathbf{H}} \neq \mathbf{H}^N$ the period-t policymaker's best response cannot be $\mathbf{F}_N$. ■

3.4 A comment on the Oudiz and Sachs (1985) solution method

As we noted earlier, the Oudiz and Sachs (1985) solution method identifies at most a single equilibrium. Uniqueness arises because a terminal period is imposed and the solution method is initialized from the solution to the terminal-period problem, which (because the problem is linear-quadratic) is unique under quite general regularity conditions. From the perspective of the agents that reside in the model, the terminal period operates very much like a coordination device—even if it resides in the far distant future—because its presence removes all ambiguity regarding the actions of other players. Interestingly, when a model has multiple symmetric Markov-perfect Stackelberg-Nash equilibria, our experience has been that the Oudiz and Sachs solution method identifies the Pareto-preferred equilibrium. To understand why the Oudiz and Sachs (1985) solution method tends to isolate the Pareto-preferred equilibrium, consider the finite horizon performance of the infinite-horizon policies. Over any finite interval, $s = t, ..., T$, the Oudiz and Sachs (1985) solution, which exploits the presence of a terminal period, invariably performs as well as or better than the best performing among the infinite-horizon policies.[9] Provided the Oudiz and Sachs (1985) solution does out-perform the infinite-horizon policies over any finite interval $s = t, ..., T$, then in the limit as $T \uparrow \infty$, the Oudiz and Sachs (1985) solution will converge to the equilibrium that is Pareto-preferred in the infinite horizon problem.

4 Examples of multiple equilibria

In this section we analyze two New Keynesian models that exhibit multiple symmetric Markov-perfect Stackelberg-Nash equilibria. The first is the sticky price model with government debt presented in Blake and Kirsanova (2007). The second is a sticky price New Keynesian model in the spirit of Woodford (2003, Chapter 5) and Sveen and Weinke (2007), but with partial inflation indexation. This second model is notable because it resides at the core of many New Keynesian models; the fact that it possesses multiple equilibria raises the prospect that other related New Keynesian models may also possess multiple equilibria.

4.1 A model with government debt

The economy is populated by a representative household, by a unit-continuum of monopolistically competitive firms, and by a single large government that conducts separately monetary policy and fiscal policy. Fiscal policy is conducted via a rule that relates government spending, g_t, inversely to the stock of real government debt, b_t. Monetary policy, in contrast, is conducted by choosing a setting for the nominal interest rate on a one-period nominal bond, r_t, optimally, but under discretion. Importantly, when formulating monetary policy the central bank takes the fiscal rule into account. Monopolistically competitive firms produce via a production function that depends only on labor and these goods are combined via a Dixit and Stiglitz (1977) technology to produce aggregate output, y_t, which is allocated to either private consumption, c_t, or government spending. Households choose their consumption and leisure, $1 - l_t$, and can transfer income through time through their holdings of government bonds.

[9] The Oudiz and Sachs (1985) solution method is not actually derived from decision problems for agents in which their planning horizon is finite. Instead, it is constructed by imposing a terminal condition on what are otherwise infinite-horizon planning problems. As a consequence, it is not unambiguously clear that it will necessarily out-perform the best performing infinite horizon policy over any finite interval $s = t, ..., T$.

The government issues debt period-by-period in order to pay the principle and interest on its existing debt and to fund any discrepancy between its spending and its tax revenues, τy_t, where $\tau \in (0,1)$ is the tax rate on income. Firms set prices subject to a Calvo (1983) nominal price rigidity and aggregation across prices leads to a New Keynesian Phillips curve relating inflation, π_t, to the expected future inflation, real marginal costs, and a serially correlated markup shock, v_t.

When log-linearized about a zero-inflation nonstochastic steady state the equations that constrain the monetary policy decision problem can be written as

$$\begin{aligned}
\pi_t &= \beta \mathrm{E}_t \pi_{t+1} + \frac{(1-\gamma\beta)(1-\gamma)}{\sigma\gamma(\varepsilon+\psi)}(\psi c_t + y_t) + v_t, \\
c_t &= \mathrm{E}_t c_{t+1} - \sigma(r_t - \mathrm{E}_t \pi_{t+1}), \\
y_t &= (1-\theta) g_t + \theta c_t, \\
b_{t+1} &= \chi r_t + \frac{1}{\beta}(b_t - \chi\pi_t + (1-\theta) g_t - \tau y_t), \\
g_t &= -\lambda b_t,
\end{aligned}$$

with the monetary policy objective function, a second-order accurate approximation[10] to household utility, taking the form (Blake and Kirsanova, 2007)

$$L_t = (1-\beta)\,\mathrm{E}_t \sum_{k=t}^{\infty} \beta^{(k-t)} \left[\pi_k^2 + \frac{\psi(1-\gamma\beta)(1-\gamma)}{(\epsilon+\psi)\gamma\epsilon}\left(\frac{1}{\psi} y_k^2 + \frac{\theta}{\sigma} c_k^2 + \frac{(1-\theta)}{\sigma} g_k^2\right)\right].$$

The parameter λ, which reflects the response of government spending to real debt, plays a crucial role in the analysis. If the fiscal response parameter is "relatively large" then fiscal policy bears the burden of stabilizing the stock of debt and there is unique symmetric Markov-perfect Stackelberg-Nash equilibrium. In this equilibrium, following an adverse markup shock, fiscal policy returns the real government debt quickly to its steady state level while monetary policy stabilizes inflation and output by raising the nominal interest rate in order to lower real marginal costs. If the fiscal control is "relatively small", then again there is a unique equilibrium in which monetary policy bears the burden of stabilizing the stock of debt. In the spirit of Leeper (1991), the former case can be thought of as one it which fiscal policy is passive and monetary policy is active and the latter case as one in which fiscal policy is active and monetary policy is passive.[11] For an "intermediate" strength of fiscal control, however,

[10] When deriving this approximation, Blake and Kirsanova (2007) assume the presence of an efficient production/employment subsidy, funded by a lump-sum tax, that offsets the output distortion caused by monopolistic competition.

[11] Notice that here the discretionary central bank makes its policy decision taking the fiscal spending equation into consideration and can be viewed, as a consequence, a Stackelberg leader with respect to the fiscal authority (c.f. Leeper, 1991).

we find three equilibria.[12]

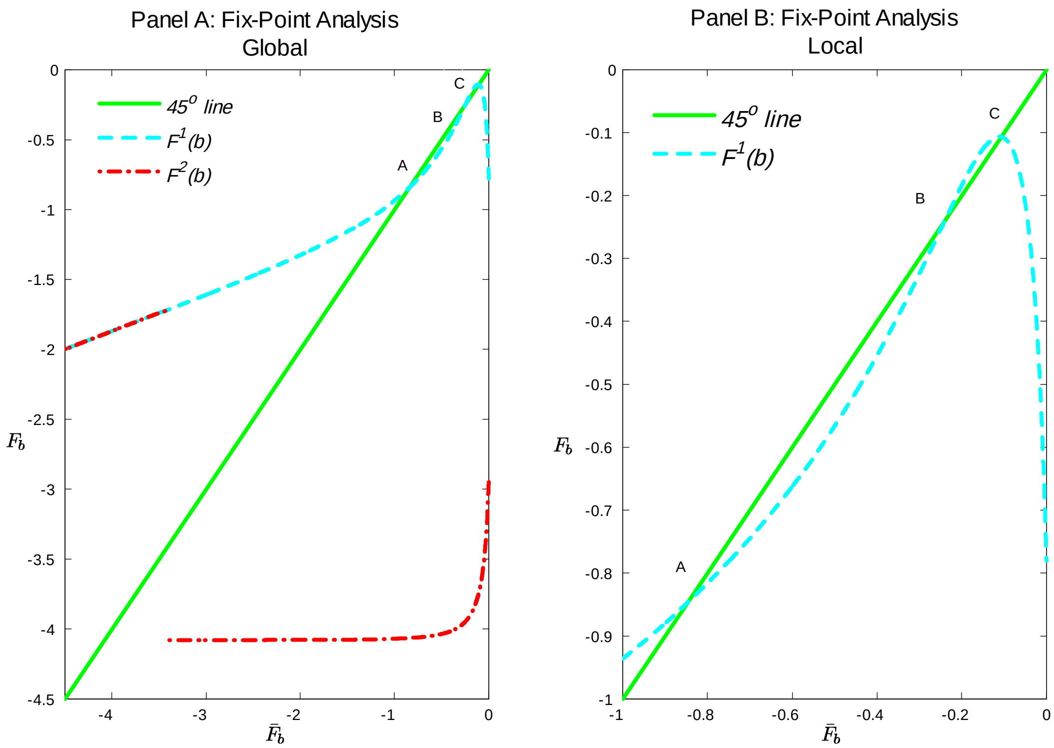

Figure 1: Government debt model

Figure 1 panel A shows the relationship between F_b and $\overline{F}_b$, varying $\overline{F}_b$ between -4.5 and 0. Over most of this space, each value of $\overline{F}_b$ is associated with two values of F_b with this multiplicity arising from the fact that the policy allows multiple private sector equilibria to exist at each point in time, which is to say that equation (20) has two solutions for $\mathbf{H}$ for which, given $\overline{F}_b$, the spectral radius of the state transition matrix, $|\overline{\mathbf{M}}|$, is less then one. The first solution (determined to be the one that minimizes $|\overline{\mathbf{M}}|$, given $\overline{F}_b$) corresponds to the solution branch depicted by the dashed line. The second solution corresponds to the solution branch depicted by the dash-dotted line. Where the dashed line and the dash-dotted lines overlap, $\overline{F}_b$ generates a unique private sector equilibrium. Although the dash-dotted line does not intersect the 45 degree line, the dashed line intersects three times and these points

[12] We parameterize the model as follows. We set the discount factor, β, to 0.99, the elasticity of intertemporal substitution, σ, to 0.5, the consumption-output ratio, θ, to 0.75, the steady-state debt-to-output ratio, χ, to 0.1, the elasticity of substitution between goods, ε, to 11, the Calvo price-rigidity, γ, to 0.75, the labor supply elasticity, ψ, to 2, the income tax rate, τ, to $(1-\beta)\chi + (1-\theta)$ and the fiscal policy parameter, λ, to 1.1. We set the AR(1) coefficient in the markup shock process to 0.3.

of intersection, denoted by A, B, and C, respectively, and shown in greater detail in panel B, represent the three equilibria in this model. By separating the two solution branches and tracking the relationship between F_b and $\overline{F}_b$, Figure 1 allows us to distinguish multiplicities arising through the interaction between private agents and the policymaker from multiplicities arising through the interaction between policymakers over time. In this example, only one solution arm crosses the 45 degree line, implying that the multiplicity is due to the interaction between policymakers over time.[13]

In Table 1, we report the policy rule, $\mathbf{F}$, the private-sector decision rules, $\mathbf{H}$, and the spectral radius of the transition matrix, $|\mathbf{M}|$, for all three equilibria.

Table 1: Policy rules in equilibrium

Eqm	$\mathbf{F} = \begin{bmatrix} F_v & F_b \end{bmatrix}$	$\mathbf{H} = \begin{bmatrix} H_{cv} & H_{cb} \\ H_{\pi v} & H_{\pi b} \end{bmatrix}$	$\lvert\mathbf{M}\rvert$
A	$\begin{bmatrix} -5.1657 & -0.8476 \end{bmatrix}$	$\begin{bmatrix} 2.4392 & 0.9401 \\ 1.4385 & 0.0585 \end{bmatrix}$	0.5326
B	$\begin{bmatrix} 7.7386 & -0.2414 \end{bmatrix}$	$\begin{bmatrix} -4.3044 & 0.4334 \\ 1.3073 & 0.0374 \end{bmatrix}$	0.6917
C	$\begin{bmatrix} 13.5114 & -0.1066 \end{bmatrix}$	$\begin{bmatrix} -8.3962 & 0.2392 \\ 1.1339 & 0.0216 \end{bmatrix}$	0.7437

The three policy rules presented in Table 1 are qualitatively and quantitatively quite different. Specifically, monetary policy can be thought of as being passive in equilibrium A and active in equilibria B and C.[14] Thus, characterizing equilibrium A as "passive", equilibrium B as "moderately active", and equilibrium C as "active", Table 1 reveals a trade-off between the response to government debt and the response to the markup shock: the more active the policy the more aggressively interest rates are raised in response to the markup shock. Table 1 also reveals a relationship between the response to government debt and the speed with

[13] Because Figure 1 has been constructed for a particular parameterization of the model, it is useful to consider how the set of equilibria is affected by parameter variation. Focusing on Panel B for ease of exposition, some parameter variations lower $F^1(b)$ causing equilibrium A to be the unique equilibrium; other parameter variations raise $F^1(b)$ causing equilibrium C to be the unique equilibrium. Although the model has a unique equilibrium for many parameterizations, it is not the case that most parameterizations lead to one particular equilibrium prevailing.

[14] To this point, consider the monetary policy rule in equilibrium A. It is useful to express this policy rule as a relationship whereby the nominal interest rate responds to inflation and real debt. When written in this form the policy rule is

$$r_t = -3.59103\pi_t - 0.637409b_t.$$

In contrast, the policy rule in equilibrium C is

$$r_t = 11.9156\pi_t - 0.363488b_t.$$

In the spirit of Leeper (1991), the policy rule in equilibrium A can be thought of as being passive because it suggests that the interest rate be lowered in response to a rise in inflation.

which government debt is stabilized: the more passive is monetary policy the more quickly is government debt stabilized.

To understand why multiple equilibria arise in this model, recognize that following a markup shock the challenge facing the central bank is to bring inflation down without creating too large of a recession. According to the Phillips curve, in any stationary equilibrium inflation depends on the entire expected future path of real marginal costs,

$$\pi_t = \mathrm{E}_t \sum_{k=t}^{\infty} \beta^{(k-t)} \left[\frac{(1-\gamma\beta)(1-\gamma)}{\sigma\gamma(\varepsilon+\psi)} mc_k + v_k \right],$$

where real marginal costs are given by

$$mc_t = \psi c_t + y_t. \tag{46}$$

Notice that when the discount factor, β, is large mc_t and mc_{t+1} are highly substitutable in terms of their effect on period-t inflation. Clearly, if inflation is above target, then there are multiple paths for real marginal costs that will return inflation to target. Each of these paths for real marginal costs is associated with a different monetary policy and each has a different cost in terms of loss. The policymaker might choose a policy that involves lower future real marginal costs if that policy allows the costs of bringing inflation back to target to be deferred, even if that policy means tolerating slightly higher inflation today.

Consider the case where future policymakers are expected to employ a policy rule that responds only weakly to b_t. In this case a monetary policy that seeks to counter the markup shock by raising the real interest rate will be attractive. The higher real interest rate induces households to defer consumption, which, from equation (46), achieves the goal of lowering real marginal costs today and placing downward pressure on inflation. Of course, the higher interest rate also raises the cost of financing the government debt, which together with the fact that the decline in consumption lowers output and government tax revenues, leads to a rise in b_t. Where the success of this policy would be undone if future policymakers were to cut interest rates aggressively in response to the rise in b_t, because this would cause future real marginal costs to rise, it is sustained on the expectation that future policymakers will not attempt to solve the fiscal deficit problem by stimulating the economy. This line of reasoning gives rise to equilibrium C (or B).

In contrast, if future policymakers are expected to tighten monetary policy aggressively in response to a decline in government debt, then a monetary policy that stimulates the economy today can achieve lower inflation over time, even if it permits higher inflation today, provided real marginal costs decline in the future. By lowering the interest rate in response to the markup shock, monetary policy stimulates the economy and causes real marginal costs to rise, which is inflationary. However, because this policy raises government tax receipts and lowers the cost of financing debt, it causes the stock of government debt to decline. Since future policymakers are expected to tighten monetary policy aggressively in response to a decline in government debt, this policy achieves lower inflation over time because it induces tighter policy in the future. This line of reasoning gives rise to equilibrium A.

The economy's behavior in each of these equilibria can be seen clearly in Figure 2, which shows the responses of key variables to a unit markup shock. Focusing first on the active and moderately active equilibria (equilibria C and B, respectively), inflation rises following the markup shock (panel C) and the policy response is to raise the nominal interest rate (panel F). With the nominal interest rate rising by more than inflation, the real interest rate rises causing households to defer consumption (panel D). The decline in consumption lowers output (panel A) and government tax revenues (panel H), which leads to a rise in government debt (panel B). In subsequent periods, although interest rates are lowered to stimulate the economy and bring it out of recession, government debt is brought back to baseline predominantly through (primary) fiscal surpluses, rather than through a decline in the cost of financing government debt.

In the passive equilibrium (equilibrium A), monetary policy responds to the markup shock by lowering the interest rate, which stimulates consumption and output, raises real marginal costs, and causes inflation to rise by more than it otherwise would. This monetary policy causes tax revenues to rise and leads to a decline in government debt. To stabilize government debt, future policymakers raise the cost of financing government debt, which causes consumption, output, and real marginal costs to decline and places downward pressure on inflation.

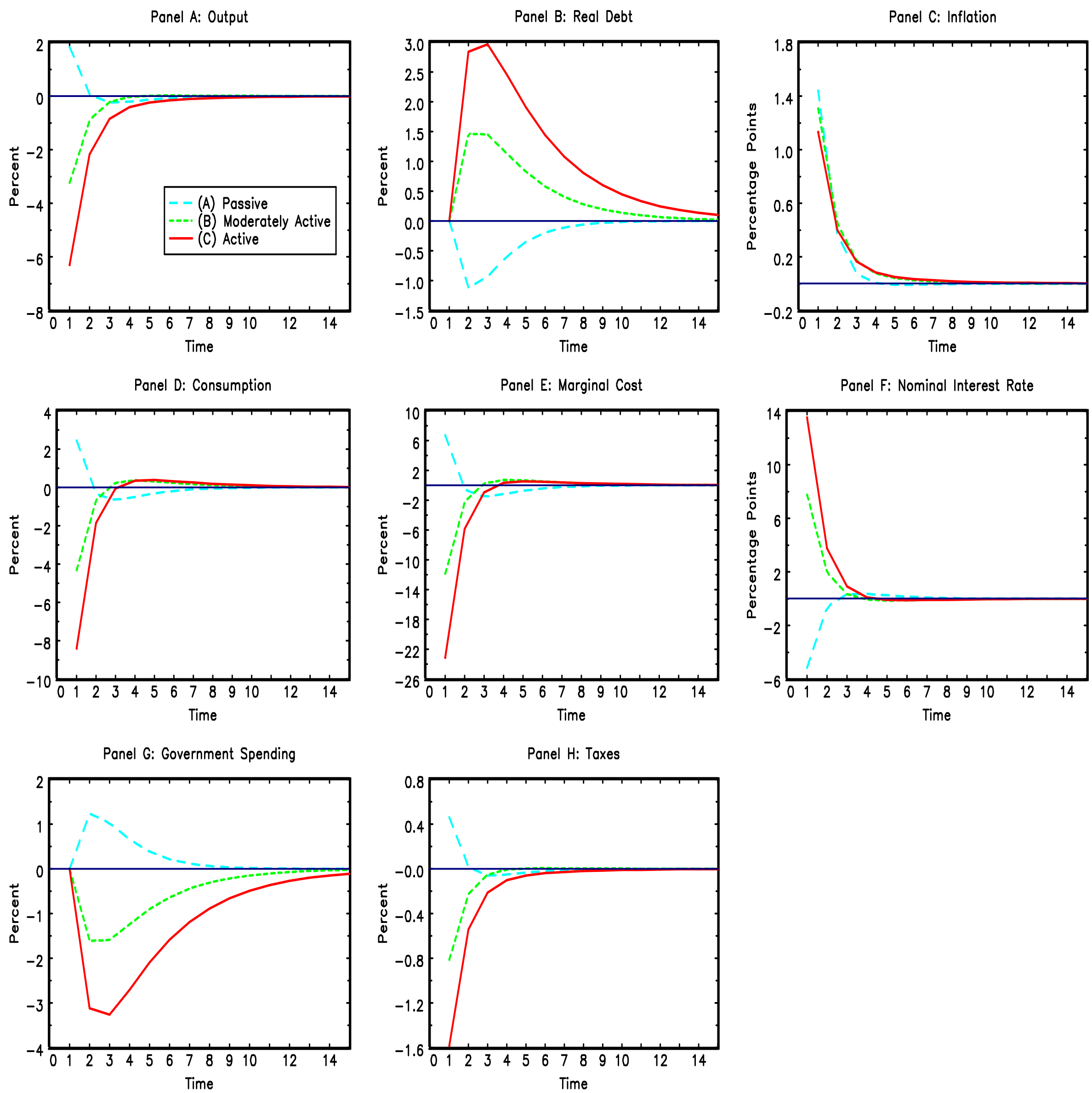

Figure 2: Responses to unit markup shock

It is clear from Figure 2 that monetary policy and the economy's behavior more generally is very different in equilibrium A than it is in either equilibrium B or equilibrium C. With these differences in mind, we now apply the equilibrium selection methods described in Section 4 and report the results in Table 2. Table 2 also identifies the equilibria that can be obtained via the iterative Backus and Driffill (1986) and Oudiz and Sachs (1985) solution procedures and shows the average loss associated with each equilibrium.

Table 2: Equilibrium characteristics

	Equilibrium		
Criterion	A	B	C
(1) Average loss	2.2252	1.9935	1.6700
(2) Minimax loss rank	1	2	3
(3) Minimax regret rank	3	1	2
(4) IE-stable (Joint)	yes	no	yes
(5) IE-stable (Private sector)	yes	yes	yes
(6) Coalition size (p_j)	5	3	—
(7) Backus-Driffill	yes	no	yes
(8) Oudiz-Sachs	no	no	yes

Row (1) of Table 2 reports the average loss associated with each equilibrium. This row shows that the three equilibria can be welfare ranked and that equilibrium C, the equilibrium in which monetary policy is most active in stabilizing output and inflation, is the Pareto-preferred equilibrium. In contrast, equilibrium A, in which monetary policy seeks to stabilize output and inflation by manipulating government debt, performs worst. Clearly, if policymakers and private agents could choose, they would choose to coordinate on equilibrium C. Rows (2) and (3) of Table 2 show that if policymakers were to choose their policy action using minimax loss or minimax regret, then they would coordinate on equilibrium A and equilibrium B, respectively. Thus, these robustness-based criteria suggest that the Pareto-preferred equilibrium C might be unattainable. However, row (4) reveals that equilibrium B is not locally IE-stable under joint learning. If we require the equilibrium to be both jointly learnable and private-sector learnable (locally IE-stable under private sector learning), then rows (4) and (5) preclude equilibrium B, and the minimax regret criterion identifies equilibrium C. As expected, equilibrium C is also the unique equilibrium located by the Oudiz and Sachs (1985) solution procedure and, since it is jointly learnable, it is also an equilibrium located by the Backus and Driffill (1986) solution procedure.

Although either equilibrium A or equilibrium C might prevail, we now ask whether policymakers and private agents could reasonably coordinate on the Pareto-preferred equilibrium C. Equilibrium C is jointly learnable and it is locally IE-stable under private sector learning. In addition, row (6) of Table 2 shows that the largest coalition of policymakers needed to support equilibrium C is 5, which seems "relatively small", and is certainly smaller than the typically central banker's tenure, suggesting that coordination on equilibrium C could be possible.

4.2 A model with capital accumulation and inflation indexation

Following Woodford (2003, Chapter 5), the economy is populated by households, intermediate-good producing firms, final-good producing firms, and a central bank. Households are identical and infinitely lived, choosing consumption, c_t, labor, l_t, and nominal holdings of next period bonds, b_{t+1}, to maximize expected discounted utility subject to a budget constraint. On the production side, a unit-continuum of monopolistically competitive intermediate-good producing firms, indexed by $\omega \in [0,1]$, produce by combining labor services hired in a perfectly competitive market with their firm-specific capital. These intermediate-good producing firms make labor and investment decisions, seeking to maximize their value subject to their production technology

$$Y_t(\omega) = e^{u_t} K_t(\omega)^{\alpha} L_t(\omega)^{(1-\alpha)},$$

their capital accumulation equation

$$I_t(\omega) = I\left(\frac{K_{t+1}(\omega)}{K_t(\omega)}\right) K_t(\omega),$$

where $I(1) = \delta$, $I^{'}(1) = 1$, and $I^{''}(1) = \eta$, and a Calvo (1983) price rigidity, where firms that cannot optimally set their price in a given period are assumed to index their price to lagged aggregate inflation (Smets and Wouters, 2003). Profits are aggregated and returned to households (shareholders) in the form of a lump-sum dividend. The final-good producing firms purchase intermediate goods, aggregate them into a final good according to a Dixit and Stiglitz (1977) production technology, and sell these final goods in a perfectly competitive market to households and firms to consume and invest, respectively.

After aggregating and log-linearizing about a zero-inflation nonstochastic steady state, the model's constraints and first-order conditions are

$$\begin{aligned}
\pi_t &= \frac{\theta}{1+\theta\beta}\pi_{t-1} + \frac{\beta}{1+\theta\beta}\mathrm{E}_t\pi_{t+1} + \frac{(1-\xi)(1-\beta\xi)}{(1+\theta\beta)\xi}mc_t + v_t, \\
c_t &= \mathrm{E}_t c_{t+1} - \frac{1}{\sigma}\left(r_t - \mathrm{E}_t\pi_{t+1} - g_t + \mathrm{E}_t g_{t+1}\right), \\
k_{t+1} &= \frac{1}{1+\beta}k_t + \frac{\beta}{1+\beta}\mathrm{E}_t k_{t+2} + \frac{1-\beta(1-\delta)}{(1+\beta)\eta}\mathrm{E}_t ms_{t+1} - \frac{1}{(1+\beta)\eta}\left(r_t - \mathrm{E}_t\pi_{t+1}\right) \\
mc_t &= w_t - y_t + l_t, \\
w_t &= \chi l_t + \sigma c_t - g_t, \\
y_t &= (1-\gamma)c_t + \frac{\gamma}{\delta}\left[k_{t+1} - (1-\delta)k_t\right], \\
y_t &= u_t + \alpha k_t + (1-\alpha)l_t, \\
ms_t &= w_t - k_t + l_t
\end{aligned}$$

where $\beta \in (0,1)$ is the discount factor, $\rho \equiv \frac{1-\beta}{\beta}$ is the discount rate, $\gamma \equiv \frac{\alpha\delta}{\rho+\delta}\frac{\varepsilon-1}{\varepsilon}$ is the steady-state share of investment in output, $\varepsilon > 1$ is the steady-state elasticity of substitution between intermediate goods, $\delta \in (0,1)$ is the depreciation rate, and $\eta > 0$ is the elasticity of the investment-to-capital ratio with respect to Tobin's q evaluated at steady state (Eichenbaum and Fisher, 2007).

Although the model allows for three stochastic elements: an aggregate consumption-preference shock, g_t; an aggregate markup shock, v_t; and an aggregate technology shock, u_t, we zero-out g_t and u_t in order to focus on the policy trade-offs associated with the markup shock, v_t, which is assumed to evolve over time according to

$$v_{t+1} = \rho_v v_t + \epsilon_{vt+1},$$

where $\rho_v \in (-1,1)$ and $\epsilon_{v_{t+1}}$ is *i.i.d.* distributed with zero mean and finite variance.[15]

The central bank's loss function is assumed to have the form

$$L_t = (1-\beta)\,\mathrm{E}_t \sum_{k=t}^{\infty} \beta^{(k-t)} \left[\pi_k^2 + \frac{(1-\xi)(1-\beta\xi)}{(1+\theta\beta)\,\xi\varepsilon} y_k^2\right].$$

With monetary policy conducted under discretion this model has three symmetric Markov-perfect Stackelberg-Nash equilibria. The policy rule and the private-sector decision rules for each of these equilibria are reported in Table 3.

Table 3: Policy rules in equilibrium

Eqm	$\mathbf{F} = \begin{bmatrix} F_v & F_k & F_\pi \end{bmatrix}$	$\mathbf{H} = \begin{bmatrix} H_{\pi v} & H_{\pi k} & H_{\pi\pi} \\ H_{cv} & H_{ck} & H_{c\pi} \\ H_{kv} & H_{kk} & H_{k\pi} \end{bmatrix}$	$\lvert\mathbf{M}\rvert$
A	$\begin{bmatrix} -4.1623 & 4.8258 & -0.7321 \end{bmatrix}$	$\begin{bmatrix} 2.4376 & -2.3107 & 0.6257 \\ 1.6494 & -2.3572 & 0.2771 \\ 0.7774 & -0.4040 & 0.1304 \end{bmatrix}$	0.3000
B	$\begin{bmatrix} 1.0633 & -0.0914 & 0.6072 \end{bmatrix}$	$\begin{bmatrix} 0.3707 & -0.1749 & 0.1201 \\ -0.9957 & 0.2727 & -0.3789 \\ -0.5546 & 0.9242 & -0.1996 \end{bmatrix}$	0.9655
C	$\begin{bmatrix} 1.0163 & -0.0247 & 0.5994 \end{bmatrix}$	$\begin{bmatrix} 0.1934 & -0.0050 & 0.0768 \\ -1.0822 & 0.3487 & -0.4007 \\ -0.6019 & 0.9663 & -0.2115 \end{bmatrix}$	0.9674

Clearly, equilibria B and C are "close", sharing policy rules that are qualitatively and quantitatively similar. In these two equilibria, the coefficients in the policy rule indicate that

[15] To parameterize the model, we set the discount factor (β) to 0.99, the Calvo price rigidity (ξ) to 0.75, the inflation indexation parameter (θ) to 0.60, the Cobb-Douglas production function parameter (α) to 0.36, the labor supply elasticity (χ) to 1, the elasticity of intertemporal substitution (σ) to 2, the depreciation rate (δ) to 0.025, the elasticity of subititution between goods (ε) to 11, and the shock persistece (ρ_v) to 0.3.

the central bank will raise the nominal interest rate in response to a positive markup shock and in response to higher (lagged) inflation, but lower the nominal interest rate in response to a higher capital stock. In contrast, the policy responses in equilibria A are qualitatively the very opposite of those in equilibria B and C.

To understand why this model has multiple equilibria we again turn to the Phillips curve and to the problem of stabilizing inflation. Adapting a result from Dennis and Söderström (2006), the forward representation of the inflation equation is given by

$$\pi_t = \theta\pi_{t-1} + \frac{(1-\xi)(1-\beta\xi)}{\xi}\mathrm{E}_t\sum_{k=t}^{\infty}\beta^{(k-t)}mc_k + \frac{1+\theta\beta}{1-\rho_v\beta}v_t. \tag{47}$$

Moreover, real marginal costs can be expressed as

$$\begin{aligned} mc_t &= \alpha ms_t + (1-\alpha)w_t \\ &= \left(\frac{\alpha+\chi}{1-\alpha} + \frac{\sigma}{1-\gamma}\right)y_t + \left[\frac{\sigma\gamma(1-\delta)}{(1-\gamma)\delta} - \frac{\alpha(\alpha+\chi)}{1-\alpha}\right]k_t - \frac{\sigma\gamma}{(1-\gamma)\delta}k_{t+1}. \end{aligned} \tag{48}$$

Analogous to the model with government debt, equation (47) shows that movements in mc_t and mc_{t+1} are highly substitutable in terms of their effect on π_t and that, for any initial value of inflation, there are multiple paths for mc_t that will return inflation to target. As earlier, these different paths for real marginal costs are associated with different monetary policies and with different performance in terms of loss. Equation (48) shows that monetary policy can affect mc_t through two distinct channels. To lower real marginal costs, the central bank can either raise the real interest rate, weakening aggregate demand and thereby causing y_t to decline or it can lower the real interest rate to stimulate investment and thereby boost the future capital stock. Notice that raising (lowering) the real interest rate causes both y_t and k_{t+1} to decline (rise) and that y_t and k_{t+1} have countervailing effects on mc_t. As a consequence, the desirability of each policy from the perspective of the period-t policymaker turns on how future policymakers are expected to respond to the capital stock.

Consider the case where future policymakers are expected to lower the interest rate in response to a rise in the capital stock. Following a positive markup shock, the policy of raising the real interest rate and causing y_t and k_{t+1} to decline will successfully deliver lower real marginal costs and inflation because the boost in future real marginal costs caused by the decline in the capital stock is offset by higher interest rates in the future. Under this approach, monetary policy responds to the positive markup shock by contracting demand, lowering real marginal costs and inflation, and by then lowering interest rates as inflation declines allowing the economy to recover, producing equilibrium C (or B). Alternatively, if future policymakers are expected to raise the interest rate in response to a higher capital stock, then a policy that lowers the real interest rate and stimulates investment can bring about a decline in inflation, despite the boost to y_t and mc_t today, because future policymakers respond to the higher capital stock by tightening monetary policy, producing equilibrium A.

The economy's behavior in the different equilibria are shown in Figure 3 which displays the responses of key variables to a unit markup shock.

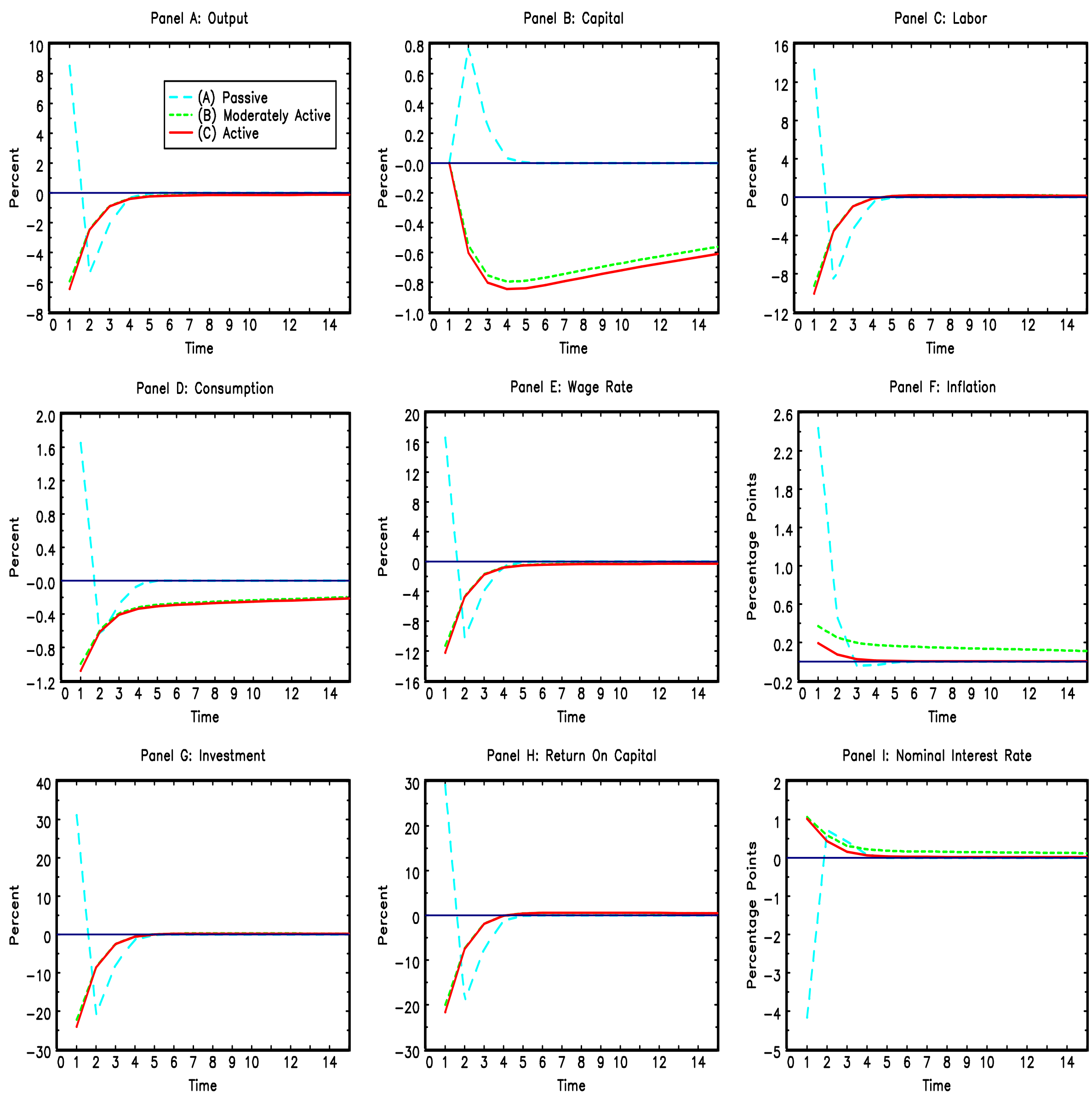

Figure 3: Responses to unit markup shock

Focusing first on equilibria B and C, following the markup shock the interest rate is raised (Panel I) by more than the increase in inflation (Panel F), causing the real interest rate to rise. The higher real interest rate generates a decline in consumption (Panel D) and investment (Panel G), which lowers output (Panel A) and real marginal costs (Panels E and H). Further, the fall in investment leads to a decline in the capital stock (Panel B). In subsequent periods, the decline in real marginal costs causes inflation to moderate. With inflation declining back to baseline, monetary policy responds by lowering the interest rate and stimulating demand. In these two equilibria, monetary policy stabilizes the economy in the traditional way, contracting output and hence real marginal costs in order to keep inflationary pressures contained. Although equilibria B and C are similar in many respects, it is notable that the economy returns to steady state somewhat more slowly in equilibrium B than it does in equilibrium C.

In constrast, in equilibrium A the interest rate is lowered in response to the positive markup shock, generating a big decline in the real interest rate. The lower real interest rate stimulates consumption and investment, which pushes up output and real marginal costs and further boosts inflation. However, the rise in investment causes the capital stock to increase and the capital build up eventually lowers real marginal costs while inducing tighter monetary policy. Although the policy tightening is aimed primarily at lowering investment, it also serves to lower output, which causes a further decline in real marginal costs. In this equilibrium, monetary policy responds to the markup shock by stimulating the economy in order to boost capital spending. This policy succeeds in stabilizing the economy because the higher capital stock causes future real marginal costs to decline and future monetary policy to tighten.

Table 4: Equilibrium characteristics

	Equilibrium		
Criterion	A	B	C
(1) Average loss	6.6737	0.8535	0.2849
(2) Minimax loss rank	1	3	2
(3) Minimax regret rank	3	2	1
(4) IE-stable (Joint)	yes	no	yes
(5) IE-stable (Private sector)	yes	no	no
(6) Coalition size (p_j)	$\nexists$	$\nexists$	—
(7) Backus-Driffill	yes	no	yes
(8) Oudiz-Sachs	no	no	yes

Clearly the economy behaves very differently in equilibrium A than it does in equilibrium C. But are the conventional policies (equilibria B and C) superior to the unconventional policy (equilibrium A)? Table 4 (row 1) shows that the conventional policies are superior to the unconventional policy and that equilibrium C is superior to equilibrium B. Rows (2) and (3) show that the minimax loss and minimax regret criteria identify equilibrium A and equilibrium C, respectively. In addition, equilibrium B is neither jointly learnable (row 4) nor learnable by private agents (row 5). Interestingly, the Pareto-preferred equilibrium C is also not learnable by private agents. As a consequence, only equilibrium A is both jointly learnable and learnable by private agents, and no sequence of trembling hand errors could occur to induce coordination on equilibrium C. In this model, unless the policymaker is a minimax-regret decisionmaker, the equilibrium of interest appears to be equilibrium A in which monetary policy seeks to stabilize output and inflation by manipulating investment and the capital stock.

5 Conclusion

Discretionary policymakers cannot manage either the expectations of private agents or the actions of future policymakers. As a consequence, discretionary policymakers are susceptible to expectations traps and coordination failures and discretionary control problems can have multiple equilibria. Recognizing this potential for multiple equilibria, this paper addresses the important issue of equilibrium selection, an issue related intrinsically to the capacity for agents to coordinate. One contribution of this paper is to cast the discretionary control problem as a dynamic game, allowing us to explain clearly the strategic interactions that give rise to multiple equilibria. However, the paper's main contribution is to develop a range of equilibrium selection criteria, where these criteria are motivated by robustness, learning, and the possibility of systematic policy errors. Among other results, the paper finds coordination on an equilibrium that is not learnable by private agents and not learnable jointly by private agents and by policymakers to be unlikely. Where more than one equilibrium satisfies these learnability criteria, a policymaker's desire for robustness can be invoked to achieve uniqueness.

We illustrate the equilibrium selection criteria by applying them to two New Keynesian models. In the first model, the Pareto-preferred equilibrium is one of two equilibria that is both jointly learnable and learnable by private agents, and, between these two equilibria, it is the equilibrium that would be coordinated on by minimax-regret policymakers. Since, in addition, the Pareto-preferred equilibrium can be supported by a relatively short sequence of trembling hand errors, is seems reasonable to expect that agents residing in this model might plausibly coordinate upon the Pareto-preferred equilibrium. In the second model, however, although the Pareto-preferred equilibrium is jointly learnable it is not learnable by private agents and it cannot be supported by any sequence of trembling-hand errors. Accordingly, to the extent that private sector learnability is critical for coordination, it appears likely that the Pareto-worst equilibrium and not the Pareto-best equilibrium will be the equilibrium outcome. At the same time, we do not rule out entirely the possibility that agents might coordinate on the Pareto-preferred equilibrium in this model. The Pareto-preferred equilibrium is jointly learnable and it is the equilibrium identified by the minimax regret selection criterion. Finally, where model uncertainty influences behavior, the question arises as to whether policymakers can facilitate coordination by making their forecasts public. We leave this question to further research.

References

[1] Albanesi, S., Chari, V., and L. Christiano, (2003), "Expectation Traps and Monetary Policy," *Review of Economic Studies*, 70, pp. 715-741.

[2] Anderson, E., Hansen, L., McGrattan, E., and T. Sargent, (1996), "Mechanics of Forming and Estimating Dynamic Linear Economies" in Amman, H., Kendrick, D., and J. Rust (eds) *Handbook of Computational Economics*, Volume 1, Chapter 4, North Holland, New York.

[3] Backus, D., and J. Driffill, (1986), "The Consistency of Optimal Policy in Stochastic Rational Expectations Models," Centre for Economic Policy Research Discussion Paper #124.

[4] Barro, R., and D. Gordon, (1983), "Rules, Discretion and Reputation in a Model of Monetary Policy," *Journal of Monetary Economics*, 12, pp. 101-121.

[5] Blake, A., (2004), "Open Loop Time Consistency for Linear Rational Expectations Models," *Economics Letters*, 82, pp. 21-27.

[6] Blake, A., and T. Kirsanova, (2007), "Discretionary Policy and Multiple Equilibria in LQ RE Models," University of Exeter Working Paper.

[7] Blake, A., and T. Kirsanova, (2008), "'Wet' or 'Impatient'? New Perspective on Discretionary Monetary Policymaking," University of Exeter Working Paper.

[8] Calvo, G., (1983), "Staggered Contracts in a Utility-Maximising Framework," *Journal of Monetary Economics*, 12, pp. 383-398.

[9] Chari, V., and P. Kehoe, (1990), "Sustainable Plans," *Journal of Political Economy*, 98, 4, pp. 783-802.

[10] Chari, V., Christiano, L., and M. Eichenbaum, (1998), "Expectations Traps and Discretion," *Journal of Economic Theory*, 81, pp. 462-92.

[11] Christiano, L., Eichenbaum, M., and C. Evans, (2005), "Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy," *Journal of Political Economy*, 113, pp. 1-45.

[12] Chow, G., (1997), *Dynamic Economics: Optimization by the Lagrange Method*, Oxford University Press, Oxford.

[13] Clements, D., and H. Wimmer, (2003), "Existence and Uniqueness of Unmixed Solutions of the Discrete-Time Algebraic Riccati Equation," *Systems Control Letters*, 50, pp. 343-346.

[14] Cohen, D., and P. Michel, (1988), "How Should Control Theory be Used to Calculate a Time-Consistent Government Policy?" *Review of Economic Studies*, 55, pp. 263-274.

[15] Cooper, R., and A. John, (1988), "Coordination Failures in New Keynesian Models," *The Quarterly Journal of Economics*, 103, 3, pp. 441-463.

[16] Currie, D. and P. Levine, (1985), "Optimal Feedback Rules in an Open Economy Macromodel with Rational Expectations," *European Economic Review*, 27, pp. 141-163.

[17] Currie, D. and P. Levine, (1993), "*Rules, Reputation and Macroeconomic Policy Coordination*," Cambridge University Press, Cambridge, UK.

[18] Davig, T., and E. Leeper, (2006), "Fluctuating Macro Policies and the Fiscal Theory," in *NBER Macroeconomics Annual*, MIT Press.

[19] Dennis, R., (2007), "Optimal Policy in Rational Expectations Models: New Solution Algorithms," *Macroeconomic Dynamics*, 11, pp. 31-55.

[20] Dennis, R., and U. Söderström, (2006), "How Important is Precommitment for Monetary Policy?" *Journal of Money, Credit, and Banking*, 38, 4, pp. 847-872.

[21] de Zeeuw, A., and F. van der Ploeg, (1991), "Difference Games and Policy Evaluation: A Conceptual Framework," *Oxford Economic Papers*, 43, 4, pp. 612-636.

[22] Dixit, A., and J. Stiglitz, (1977), "Monopolistic Competition and Optimum Product Diversity," *American Economic Review Papers and Proceedings*, 67, pp. 297-308.

[23] Eichenbaum, M., and J. Fisher, (2007), "Estimating the Frequency of Price Reoptimization in Calvo-Style Models," *Journal of Monetary Economics*, 54, pp. 2032-2047.

[24] Evans, G., (1986), "Selection Criteria for Models with Non-Uniqueness," *Journal of Monetary Economics*, 18, pp. 147-157.

[25] Evans, G., (2001), "Expectations in Macroeconomics: Adaptive versus Eductive Learning," *Revue Economique*, 52, 3, pp. 573-582.

[26] Evans, G., R. Guesnerie, (1993), "Rationalizability, Strong Rationality, and Expectational Stability," *Games and Economic Behavior*, 5, pp. 632–646.

[27] Evans, G., R. Guesnerie, (2003), "Coordination on Saddle-Path Solutions, The Eductive Viewpoint: Linear Univariate Models," *Macroeconomic Dynamics*, 7, pp. 42–62.

[28] Evans, G., and R. Guesnerie, (2005), "Coordination on Saddle-Path Solutions: The Eductive Viewpoint—Linear Multivariate Models," *Journal of Economic Theory*, 124, pp. 202-229.

[29] Evans, G., and S. Honkapohja, (2001), *Learning and Expectations in Macroeconomics*, Princeton University Press, Princeton.

[30] Guesnerie, R., and M. Woodford, (1992), "Endogenous Fluctuations," in Laffont, J-J. (ed), *Advances in Economic Theory*, Cambridge University Press, Cambridge.

[31] Hansen, L., and T. Sargent, (2008), *Robustness*, Princeton University Press, Princeton, New Jersey.

[32] Ireland, P., (1997), "Sustainable Monetary Policies," *Journal of Economic Dynamics and Control*, 22, pp. 87-108.

[33] Jensen, H., (2002), "Targeting Nominal Income Growth or Inflation?" *American Economic Review*, 92, 4, pp. 928-956.

[34] Judd, K., (1998), *Numerical Methods in Economics*, MIT Press, Cambridge, Massachusetts.

[35] Kurozumi, T., (2008), "Optimal Sustainable Monetary Policy," *Journal of Monetary Economics*, 55, pp. 1277-1289.

[36] King, R., and A. Wolman, (2004), "Monetary Discretion, Pricing Complementarity, and Dynamic Multiple Equilibria," *The Quarterly Journal of Economics*, 199, 4, pp. 1513-1553.

[37] Kydland, F., and E. Prescott, (1977), "Rules Rather Than Discretion: The Inconsistency of Optimal Plans," *Journal of Political Economy*, 87, pp. 473-492.

[38] Lancaster, P., and L. Rodman, (1995), *The Algebraic Riccati Equation*, Oxford University Press, Oxford.

[39] Laub, A., (1979), "A Schur Method for Solving Algebraic Riccati Equations," *IEEE Transactions on Automatic Control*, 24, pp. 913-921.

[40] Leeper, E., (1991), "Equilibria under 'Active' and 'Passive' Monetary and Fiscal Policies," *Journal of Monetary Economics*, 27, pp. 127-147.

[41] Lucas, R., and N. Stokey, (1983), "Optimal Fiscal and Monetary Policy in an Economy Without Capital," *Journal of Monetary Economics*, 12, pp. 55-93.

[42] McCallum, B., (1983), "On Non-Uniqueness in Rational Expectations Models: An Attempt at Perspective," *Journal of Monetary Economics*, 11, pp. 139-168.

[43] Magnus, J., and H. Neudecker, (1988), *Matrix Differential Calculus with Applications to Statistics and Econometrics*, John Wiley and Sons, New York.

[44] Marcet, A., and T. Sargent, (1989), "Convergence of Least Squares Learning Mechanisms in Self-Referential Linear Stochastic Models," *Journal of Economic Theory*, 48, pp. 337-368.

[45] Ortigueira, S., (2006), "Markov-Perfect Optimal Taxation," *Review of Economic Dynamics*, 9, pp. 153-78.

[46] Ortigueira, S., and J. Pereira, (2009), "Markov-perfect Optimal Fiscal Policy: The Case of Unbalanced Budgets," manuscript (version dated November 4, 2009).

[47] Oudiz, G., and J. Sachs, (1985), "International Policy Coordination in Dynamic Macroeconomic Models," in Buiter, W., and R. Marston, (eds), *International Economic Policy Coordination*, Cambridge University Press, Cambridge, pp. 275-319.

[48] Rogoff, K., (1985), "The Optimal Degree of Commitment to an Intermediate Monetary Target," *The Quarterly Journal of Economics*, 100, 4, pp. 1169-1189.

[49] Savage, L., (1951), "The Theory of Statistical Decision, Journal of American Statistical Association," 46, pp. 55-67.

[50] Selten, R., (1975), "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games," *International Journal of Game Theory*, 4, 1, pp. 25-55.

[51] Smets, F., and R. Wouters, (2003), "An Estimated Stochastic Dynamic General Equilibrium Model of the Euro Area," *Journal of the European Economic Association*, 1, 5, pp. 1123-1175.

[52] Smets, F., and R. Wouters, (2007), "Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach," *American Economic Review*, 97, 3, pp. 586-606.

[53] Söderlind, P., (1999), "Solution and Estimation of RE Macromodels with Optimal Policy," *European Economic Review*, 43, pp. 813-823.

[54] Sveen, T., and L. Weinke, (2007), "Firm-Specific Capital, Nominal Rigidities, and the Taylor Principle," *Journal of Economic Theory*, 136, pp. 729-737.

[55] Svensson, L., (2000), "Open-Economy Inflation Targeting," *Journal of International Economics*, 50, pp. 155-183.

[56] Taylor, J., (1977), "Conditions for Unique Solutions in Stochastic Macroeconomic Models with Rational Expectations," *Econometrica*, 45, pp. 1377–l 385.

[57] Walsh, C., (1995), "Optimal Contracts for Central Bankers," *American Economic Review*, 85, 1, pp. 150–167.

[58] Walsh, C., (2003), "Speed Limit Policies: The Output Gap and Optimal Monetary Policy," *American Economic Review*, 93, 1, pp. 265-278.

[59] Woodford, M., (2003), *Interest and Prices: Foundations of a Theory of Monetary Policy*, Princeton University Press, Princeton, New Jersey.(1988)