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Keywords

Vintage capital, technological innovation, dynamic programming.

JEL Classification

C61, E22, O33

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Using a discrete version of the Ramsey Vintage Capital Model we provide a characterization of the initial capital stocks compatible with a predefined scrapping time and a level of technical progress which generate feasible capital paths. From that characterization, it is proved that for a given level of technological progress there exists a minimum scrapping time for the machines which allows a sustainable growth path. Moreover, we reduce the infinite horizon dynamic programming problem to one of finite dimension and we show how this reduction allows us to find the optimal structure of the initial capital stock as well as the optimal scrapping time. A numerical example shows that, in accordance with the infinite horizon approach, the greater the technological progress, the lower the optimal scrapping time.

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1 Introduction

The full employment condition in the economic growth model with heterogeneous capital introduced by Solow ([6]) imposes restrictions on the feasible capital paths of the central planner problem. When the technology exhibits perfect complementarity between capital goods and labor force the condition is even more restrictive since there is not substitution between these factors. Despite that condition having been used in the literature to calculate the optimal capital path and the optimal vintage period length it has not been used to verify if the initial capital stocks are compatible with it (Boucekkine et al. ([3]), Fabbri and Gozzi ([4])).

The goal of this work is to analyse if the restriction of full employment exclude some compositions of initial capital stocks in the Ramsey Vintage Capital Model (RVCM). We show that, given a level of technological progress and a period length for the vintage capital, the initial capital stocks must remain in a compact and convex set determined by these values. We show some cases where such set is empty, thus the central planner problem will not have a solution. Furthermore, using the replacement echoes effect on the capital stocks, we transform the infinite horizon problem in a finite dimensional problem where the decision is to find the optimal scrapping time and the optimal composition of initial capital stocks.

The results have important theoretical and applied features. In practice, when solving the central planner problem of the RVCM, it is necessary to verify if the proposed initial capital stocks are feasible, otherwise the capital path will not be a solution. From the theoretical point of view, the results allow us to explain why some economies with vintage capital stocks cannot get a sustainable growth without generating unemployment (disequilibrium in the labor market) or even, using inefficiently the capital instaled . Thus, the model is capable to explain why economies with high level of human capital and a poor structure of initial capital stocks exhibit low growth with unemployment. Reciprocally, It also explains why economies with low levels of human capital and a good capital structure must invest in qualification or import labor force.

The manuscript is divided in four sections. In Section 2 we present the RVCM in a discrete version (which is more suitable for the kind of analysis that we develop) and state the main results. In Section 3 we show the transformation process of the central planner problem of the RVCM in a finite dimensional problem, where the goal is to find the optimal value for the scrapping time. In Section 4 we discuss some conclusions of the work and the proofs of the results of Section 2 are given in the appendix.

2 Discrete Version of the RVCM

In this sections we are going to present the discrete version of the RVCM in order to show how the technological progress parameter restricts the initial composition of capital stocks. As a consequence we will characterize the set of initial capital stocks compatible with the level of technical progress and the scrapping time for the machines. In particular, we will show that for each level of technological progress, there is a lower bound for the length period of capital usage that allows sustainable growth.

Time is discrete and the technology corresponds to that of the standard AK model with vintage capital. The length period of the vintage, or *scrapping time*, is denoted by T (constant for simplicity) and the parameter representing the Harrod-neutral technological progress is $\gamma \in (0,1)$. Thus, if $k_j \geq 0$ represent the capital stock (number of machines) in period $j = t-T, \cdots, t$ to be used in time t, then the amount of labor force needed to operate those machines is $\gamma^j k_j$. In this setting, the lower the parameter γ, the greater the technological progress. Labor has a totally inelastic supply which we normalize to one. There is one representative consumer with the instantaneous utility function $u : \mathbb{R}_+ \to \mathbb{R}$ and intertemporal discount factor $\beta \in (0,1)$. Therefore the central planner problem is to find a value for the scrapping time T and a path for the consumption, investment and production $(c_t, k_{t+1}, y_t)_{t\geq 0}$ such that:

$$\begin{array}{cc} \underset{\{T,(c_t,k_{t+1},y_t)_{t\geq 0}\}}{\text{Maximize}} & \displaystyle\sum_{t=0}^{+\infty} \beta^t u(c_t) \\ \text{subject to} & c_t + k_{t+1} = y_t \\ & y_t = \displaystyle\sum_{j=t-T}^{t} k_j \\ & \displaystyle\sum_{j=t-T}^{t} \gamma^j k_j = 1 \\ & k_0, k_{-1}, \cdots \text{ given.} \end{array} \tag{1}$$

This is the analogous discrete version of the problem stated in Boucekkine et al. ([1]). In that article they used the replacement echoes property that arises in this problem and they proved that a constant scrapping time is the long run solution for their problem. For that reason, we fix the value of T and use the non-negativeness of the consumption path to define a *feasible capital path* for the problem (1) as a sequence $(k_t)_{t\geq -T}$ such that for all $t \geq 0$:

$$\begin{aligned} &(i) \quad k_{t+1} \leq \sum_{j=t-T}^{t} k_j; \\ &(ii) \quad \sum_{j=t-T}^{t} \gamma^j k_j = 1. \end{aligned} \tag{2}$$

Condition (ii) in (2), which is the equality between demand and labor supply, implies the replacement echoes effect in the model. Specifically, if we take the first difference of that equality we will obtain:

$$k_{t+1} = \gamma^{-T-1} k_{t-T}, \quad \text{for all } t \geq 0. \tag{3}$$

It means that the creative destruction process requires that the capital must grow at rate γ^{-T-1} in each vintage period in order to keep the full labor employment. Therefore, given the initial capital stocks $(k_{-T}, \cdots, k_0)$, the equation (3) will provide the complete capital path that equilibrates the demand and supply of labor. In this way, the problem (1) is reduced to find only the optimal value of T (the vintage period length).

Once we noticed that the equation (3) imposes a strong restriction on the whole capital path, a natural question arises: does the initial capital stock satisfy the market clear condition (ii) of (2)? The answer is no. Thus, we will say that the initial capital stocks is *compatible* with T and γ if and only if $\sum_{j=-T}^{0} \gamma^j k_j = 1$. If this condition is satisfied, a complete capital path is defined from (3). Such path will be feasible only if the sustainable condition (i) of (2) is satisfied. The following proposition provides a characterization of all compatible initial capital stocks which generates feasible capital paths satisfying (2).

Fixing the pair (T, γ), let us define the following matrix:

$$R_{T+1}(\gamma) = \begin{bmatrix} (\gamma^{-T-1} - 1) & -1 & -1 & \cdots & -1 \\ -\gamma^{-T-1} & (\gamma^{-T-1} - 1) & -1 & \cdots & -1 \\ -\gamma^{-T-1} & -\gamma^{-T-1} & (\gamma^{-T-1} - 1) & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\gamma^{-T-1} & -\gamma^{-T-1} & -\gamma^{-T-1} & \cdots & (\gamma^{-T-1} - 1) \end{bmatrix}, \tag{4}$$

and the set:

$$E_{T+1}(\gamma) = \left\{ \mathbf{x} = (x_1, \cdots, x_{T+1}) \in \mathbb{R}_+^{T+1} / \sum_{j=1}^{T+1} \gamma^{j-T-1} x_j = 1,\ R_{T+1}(\gamma)\mathbf{x} \leq 0 \right\} \tag{5}$$

Proposition 1 *(Feasible capital paths characterization) The sequence $(k_t)_{t \geq -T}$ is a feasible capital path for the problem (1) if and only if the initial capital stocks $(k_{-T}, \cdots, k_0) \in E_{T+1}(\gamma)$ and $k_{t+1} = \gamma^{-T-1} k_{t-T}$ for all $t \geq 0$.*

The consequences of Proposition (1) are important for obtaining a solution of the problem (1). If the set $E_{T+1}(\gamma)$ is empty, the central planner problem (1) has not an optimal capital path. The emptiness of the set $E_{T+1}(\gamma)$ means that the technological parameter γ is not compatible with the scrapping time T for producing sustainable capital paths for the economy; that might occur because either the labor market is not cleared or the consumption is not positive. On the other hand, is the set $E_{T+1}(\gamma)$ is not empty, the Proposition (1) asserts that in the initial period of vintage, the feasible capital stocks is in a compact and convex subset of $\mathbb{R}_+^{T+1}$ and in the subsequent periods it grows at rate γ^{-T-1}.

We can illustrate both cases in the following example with $T = 1$. In this case, a vector $(x_1, x_2) \in E_2(\gamma)$ if and only if $\gamma^{-1}x_1 + x_2 = 1$; $(\gamma^{-2} - 1)x_1 - x_2 \leq 0$ and $-\gamma^{-2}x_1 + (\gamma^{-2} - 1)x_2 \leq 0$. The figure 1 shows the set $E_2(\gamma)$ in two different cases. In (a) we have that for $\gamma \in [(3-\sqrt{5})/2, 1]$ the set $E_2(\gamma) \neq \{\ \}$, and in (b) the set $E_2(\gamma) = \{\ \}$ for $\gamma \in (0, (3-\sqrt{5})/2)$.

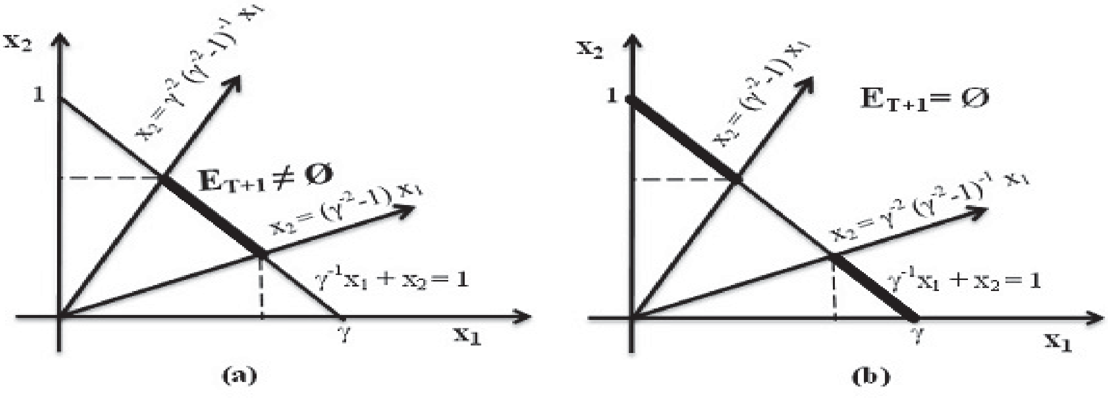

Figure 1: The set $E_2(\gamma)$

Therefore the following question arises: given the technological parameter $\gamma \in (0,1)$, what are the scrapping times T such that $E_{T+1}(\gamma) \neq \{\ \}$? The next theorem provides a complete answer for that question.

Theorem 1 *(Limits for the technological progress) For each* $T \geq 1$ *there exists a* $\gamma_{T+1} \in (0,1)$ *such that:*

(i) $R_{T+1}(\gamma)x \leq 0$ *has a solution* $x > 0$ *if and only if* $\gamma \geq \gamma_{T+1}$;

(ii) $\{\gamma_{T+1}\}_{T\geq 1}$ *is a strictly decreasing sequence;*

(iii) $\lim\limits_{T\to+\infty} \gamma_{T+1} = 1/2$.

The following conclusions come from the Theorem (1). First, given a lenght period of vintage T, high technological progress parameters (low values of γ) are not compatible with it. This results from the full labor employment condition (*ii*) of (2). In fact, for each pre-defined scrapping time, there is a maximum level of technological progress that allows it as feasible for the economy, without triggering unemployment. Second, for a given level of technological progress, the feasible values for the scrapping times has a lower bound, which increases as the technical progress augments. Finally, and less intuitive, there exists a global upper bound for the technological progress. In this simple formulation of the model, the upper bound is 0.5 (i.e. half of labor unit per machine). This number may change if we consider more complex production processes. However, this is a remarkable finding which establishes the following: given a production process with perfect complementarity of factors, there exists a maximum level of technological process that supports the sustained growth.

In figure 2 we depict the piecewise curve defined by γ_{T+1}. It relates the scrapping time with its maximum level of technological progress (minimum value of γ). This illustrates parts (*i*) and (*ii*) of Theorem (1).

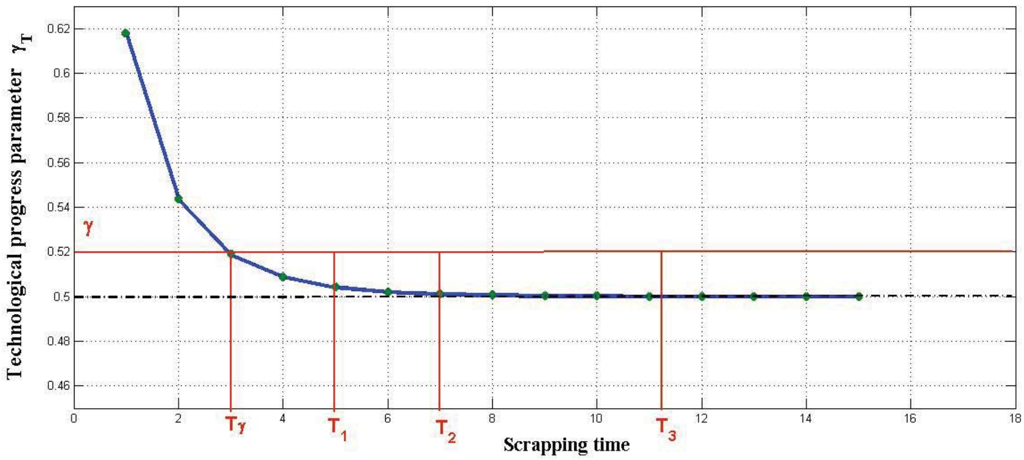

Figure 2: Maximum level of technical progress for each scrapping time

3 Optimal Capital Stock Composition and Scrapping Time

In Proposition (1) we saw that the set of feasible capital paths given by (2) is closely related with the set $E_{T+1}(\gamma)$. In this section we will explore this singular property of the RVCM in order to characterize the solution of the problem (1) and to perform a sensitivity analysis of the technological progress parameter on that solution.

Let us introduce the following notation. If $(x_1, \cdots, x_{T+1}) \in E_{T+1}(\gamma)$, the unique feasible capital path from this is $k_{-T} = x_1, \cdots, k_0 = x_{T+1}$ and $k_{t+1} = \gamma^{-T-1} k_{t-T}$, for all $t \geq 0$. From that path $(k_t)_{t \geq -T}$, let us denote $\mathbf{k}_0 = (k_{-T}, \cdots, k_0) \in E_{T+1}(\gamma)$. We may notice that the first component of $R_{T+1}(\gamma)\mathbf{k}_0$ is equal to $(\gamma^{-T-1} - 1)k_{-T} - k_{-T+1} - \cdots - k_0$ and using (3) it results $k_1 - k_{-T} - k_{-T+1} - \cdots - k_0$ which is equal to $-c_0$. Analogously, the following components will result $-c_1, -c_2, \cdots, -c_T$, the negative of the consumption plan in the first vintage period. Denoting by $\mathbf{c}_0 = (c_0, \cdots, c_T)$, the condition $R_{T+1}(\gamma)\mathbf{k}_0 \leq 0$ is equivalent to $\mathbf{c}_0 = -R_{T+1}(\gamma)\mathbf{k}_0 \geq 0$. The same procedure may be followed for each subsequent period of vintage, namely, for $t \geq 0$, we denote $\mathbf{k}_t = (k_{(T+1)t-T}, \cdots, k_{(T+1)t})$, this vector belongs to $E_{T+1}(\gamma)$ and $\mathbf{c}_t = -R_{T+1}(\gamma)\mathbf{k}_t \geq 0$, where $\mathbf{c}_t = (c_{(T+1)t}, \cdots, c_{(T+1)t+T})$ is the consumption plan in the $t+1$ period of vintage.

Using the notation given above and (3) we can conclude that $\mathbf{k}_{t+1} = \gamma^{-T-1}\mathbf{k}_t$ for all $t \geq 0$

and $\mathbf{c}_{t+1} = \gamma^{-T-1}\mathbf{c}_t$. In this way, the objective function of (1) can be expressed in terms of the consumption in the first vintage period:

$$G(\mathbf{c}_0) = \sum_{k=0}^{+\infty}\sum_{j=0}^{T} \beta^{k(T+1)+j} u(\gamma^{-k(T+1)} c_j). \qquad (6)$$

Thus, we are able to define the welfare of the economy as a function of the scrapping time T :

$$W_T = \begin{array}{cc} \underset{\mathbf{x}\in E_{T+1}(\gamma)}{\text{Maximize}} & G(\mathbf{c}_0(\mathbf{x})) \\ \text{subject to} & \mathbf{c}_0(\mathbf{x}) = -R_{T+1}(\gamma)\mathbf{x} \end{array} \qquad (7)$$

Finally, the optimal scrapping time is defined by $T^* = \underset{T\geq 0}{\text{ArgMax}}\ W_T$. Therefore, the optimal capital stock for the vintage capital problem (1) is the solution of (7) when $T = T^*$.

3.1 Example

Let us consider the following example to illustrate the method and to analyse the sensitivity of the results to changes of the technological progress parameter.

The instantaneous utility is $u(c) = (c^{1-\theta} - 1)/(1-\theta)$. It implies that the total utility of a feasible consumption path given by (6) is:

$G(\mathbf{c}_0) = A\left(c_0^{1-\theta} + \beta c_1^{1-\theta} + \cdots + \beta^T c_T^{1-\theta}\right) + B,$

where the constants A and B depend on γ, T, β and θ. The solution of the maximization (7) is:

$c_0 = d_0^{-1/\theta}\left[d_0^{1-1/\theta} + \beta^{1/\theta} d_1^{1-1/\theta} + \cdots + \beta^{T/\theta} d_{T+1}^{1-1/\theta}\right]^{-1},$

$c_k = \beta^{(k-1)/\theta} d_0^{1/\theta} d_k^{-1/\theta} c_0;\ k = 1, \cdots, T;$

where $d = [d_0\ d_1\ \cdots d_T]' = -\left[(R_{T+1}(\gamma))'\right]^{-1}\left[\gamma^{-T}\gamma^{-T+1}\cdots\gamma^{-1}1\right]'$. From that consumptiom plan we have that the optimal capital stock is $\mathbf{k}_0 = -\left[R_{T+1}(\gamma)\right]^{-1}\mathbf{c}_0$. Making these same calculations for $T = 1, \cdots$ we will obtain the function W_T.

Using the parameter values of Boucekkine et al. ([1]; [2]) $\theta = 0.15$; $\gamma = 0.97$; and $\beta = 0.95123$ we obtain $T^* = 9$. In addition we are able to describe the shape of the welfare function W_T and see how it varies as the technological progress parameter decreases. Figure 3 shows the optimal scrapping time when the parameter γ decreases.

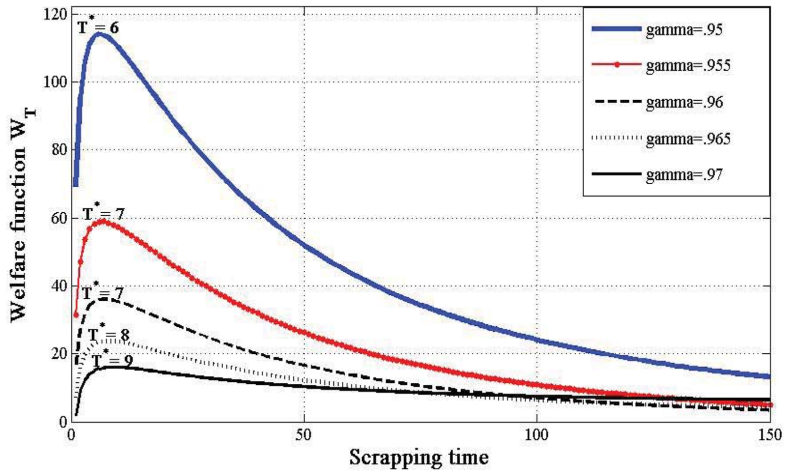

Figure 3: Total welfare and the optimal scrapping time

Therefore, we can conclude that economies with higher levels of technological progress exhibit shorter periods of vintage, thus the capital reposition is more frequent.

4 Conclusions

The consequences of technological advances on the involuntary unemployment have been a central issue for policy makers in the labor market. The excess in the human capital accumulation in economies with a defficient infrastructure unable to absorve the labor force may lead the economy to low wages, unemployment and slow development. On the other side of the coin, economies with a labor force that does not follow the technological progress may be forced to import qualified labor to meet such advances.

The Ramsey vintage capital model provides a clear framework to analyse the effect of technological progress in heterogenous capital. In this work we use that model to explain that certain capital structures may not be compativel with the technological progress that the economy is receiving.

Two main results are provided. The first asserts that, depending on the level of technological progress, some scrapping times for the capital are not feasible (in the sense that they could generate unemployment). In addition we prove that there exists an upper bound for the technological progress that allows for sustainable growth with full employment. The second result is that the central planner problem of the model may be reduced to a simple finite dimensional problem of serching for an optimal scrapping time for the model. Once that scrapping time is found, the optimal initial capital structure is provided and the whole capital path results from the replacement echoes effect present in this model.

Those results may open research lines where the productive factors have some degree of substitution or where the labor has not a totally inelastic supply.

A Appendix

Proof. (Proposition 1 (Feasible capital paths characterization))

Let $(k_t)_{t\geq -T}$ be a feasible capital path, namely, satisfying condition (i) and (ii) of (2). Let us prove that $\mathbf{k}_0 = (k_{-T}, \cdots, k_0) \in E_{T+1}(\gamma)$. Condition (ii) corresponds to the first restriction in $E_{T+1}(\gamma)$. For $t = 0$, condition (i) is $k_1 \leq k_{-T} + k_{-T+1} + \cdots + k_0$; substituting $k_1 = \gamma^{-T-1}k_{-T}$ (equation (3)) it results $(\gamma^{-T-1}-1)k_{-T}-k_{-T+1}-\cdots-k_0 \leq 0$, which is the first element of $R_{T+1}(\gamma)\mathbf{k}_0$. Analogously, $k_2 \leq k_{-T+1}+k_{-T+2}+\cdots+k_1$; substituting $k_1 = \gamma^{-T-1}k_{-T}$ and $k_2 = \gamma^{-T-1}k_{-T+1}$ it results $-\gamma^{-T-1}k_{-T} + (\gamma^{-T-1} - 1)k_{-T+1} - k_{-T+2} - \cdots - k_0 \leq 0$,which is the second element of $R_{T+1}(\gamma)\mathbf{k}_0$. Following the same procedure we conclude that $R_{T+1}(\gamma)\mathbf{k}_0 \leq 0$.

Now, consider a capital path $(k_t)_{t\geq -T}$ such that $\mathbf{k}_0 = (k_{-T}, \cdots, k_0) \in E_{T+1}(\gamma)$ and $k_{t+1} = \gamma^{-T-1}k_{t-T}$ for $t \geq 0$. Part (ii) of 2) will be proved by induction. For $t = 0$, it is the first condition of $E_{T+1}(\gamma)$. If (ii) is valid for t, then $\sum\limits_{j=t+1-T}^{t+1} \gamma^j k_j = \sum\limits_{j=t+1-T}^{t} \gamma^j k_j + \gamma^{t+1}k_{t+1}$. Using (3) it will be equal to $\sum\limits_{j=t+1-T}^{t} \gamma^j k_j + \gamma^{t+1}\gamma^{-T-1}k_{t-T} = \sum\limits_{j=t-T}^{t} \gamma^j k_j = 1$. Now, let us prove (i). Denoting by $\mathbf{k}_t = (k_{(T+1)t-T}, \cdots, k_{(T+1)t})$ the capital path corresponding to the $t-$period of vintage, we will have that for all $t \geq 0$, $\mathbf{k}_{t+1} = \gamma^{-T-1}\mathbf{k}_t$ or $\mathbf{k}_0 = \gamma^{t(T+1)}\mathbf{k}_t$. Thus, $R_{T+1}(\gamma)\mathbf{k}_0 \leq 0$ implies $R_{T+1}(\gamma)\mathbf{k}_t \leq 0$ for all $t \geq 0$. Using (3) in each component of these inequalities it results (i) for all $t \geq$. ■

Proof. (Theorem 1 (Limits for the technological progress))

For the sake of simplicity, let $N = T+1$ and $R = R_{T+1}(\gamma)$. Since the set $\left\{x \in \mathbb{R}^N_+ / Rx \leq 0\right\}$ is a cone, let us consider $x_1 + \cdots + x_N = 1$.

Let us notice that $x \in \mathbb{R}^N_+$ is a non-zero solution of $Rx \leq 0$ if and only if the components of x satisfy:

$$x_n \leq \gamma^N + (1-\gamma^N)\sum_{i=1}^{n-1} x_i; \quad n = 1, \cdots, N \tag{8}$$

Inequalities in (8) imply:
$x_1 \leq \gamma^N$,
$x_2 \leq \gamma^N + (1-\gamma^N)x_1 \leq \gamma^N(2-\gamma^N)$,
$x_n \leq \gamma^N(2-\gamma^N)^{n-1}$; $n = 1, \cdots, N$
Summing up in n it results:

$$1 \leq \gamma^N \sum_{n=1}^{N} (2-\gamma^N)^{n-1} \;\text{ or }\; 0 \leq P_N(\gamma^N) \tag{9}$$

where P_N is the polynomial $P_N(m) = m\sum\limits_{n=1}^{N}(2-m)^{n-1} - 1$. Therefore (9) is a necessary condition for the existence of non-zero solution for $Rx \leq 0$. Furthermore, if (9) is satisfied, we can define:

$$x_n = \frac{(2-\gamma^N)^{n-1}}{\sum\limits_{k=1}^{N}(2-\gamma^N)^{k-1}}; n = 1, \cdots, N,$$

and it is not difficult to check that it is a solution of (8) which is equivalent to $Rx \leq 0$ as noticed before. Therefore (9) is also a sufficient condition for the existence of solution of $Rx \leq 0$.

Now we will prove that for each $N \geq 2$ there exists a $\gamma_N \in (0,1)$ such that $P_N(\gamma_N^N) = 0$, $P_N(\gamma^N) > 0$ if $\gamma \in (\gamma_N, 1)$ and $P_N(\gamma^N) < 0$ if $\gamma \in (0, \gamma_N)$, so part (i) will be proved.

The polynomial P_N may be written as $P_N(m) = m(1-m)^{-1}\left[(2-m)^N - 1\right] - 1$. Thus: $P_N(m) \geq 0 \Leftrightarrow 2 - m \geq \frac{1}{m^{1/N}}$.

Let us notice that there exists a unique $\bar{m}_N \in (0,1)$ such that:

$$2 - \bar{m}_N \geq \frac{1}{\bar{m}_N^{1/N}}; \text{ i.e. } P_N(\bar{m}_N) = 0, \tag{10}$$

$$2 - m \geq \frac{1}{m^{1/N}}; \text{ i.e. } P_N(m) > 0 \text{ if } m \in (\bar{m}_N, 1), \text{ and}$$

$$2 - m \geq \frac{1}{m^{1/N}}; \text{ i.e. } P_N(m) < 0 \text{ if } m \in (0, \bar{m}_N).$$

Now, let us define $\gamma_N = (\bar{m}_N)^{1/N}$. If $\gamma > \gamma_N$ then $\gamma^N > \bar{m}_N$ and $P_N(\gamma^N) > 0$. If $\gamma < \gamma_N$ then $\gamma^N < \bar{m}_N$ and $P_N(\gamma^N) < 0$. Therefore we conclude the part (i).

Claim: $2^{-N} < \bar{m}_N < 2^{-N+1}$.

To prove this, it is sufficient to verify that $P_N(2^{-N}) < 0$ and $P_N(2^{-N+1}) > 0$. Indeed: $2 - 2^{-N} < \frac{1}{(2^{-N})^{1/N}} = 2$, so $P_N(2^{-N}) < 0$.

To prove $P_N(2^{-N+1}) > 0$, notice that it is equivalent to $2 - 2^{-N+1} > \frac{2}{(2^{-N+1})^{1/N}}$ or $\frac{1}{2} < \left[1 - \frac{1}{2^N}\right]^N$. The sequence $a_N = \left[1 - \frac{1}{2^N}\right]^N$ is strictly decreasing, because:

$$\frac{a_{N+1}}{a_N} = \left(\frac{2^{2N+2} - 2^{N+2} + 1}{2^{2N+2} - 2^{N+2}}\right)\left(\frac{2^{N+1} - 1}{2^{N+1} - 2}\right)^{N-1} > 1.$$

Since $a_2 = 0.5625$, we conclude that $a_N > 1/2$ for all $N \geq 2$. Then the Claim is correct.

From the Claim above we have that $\bar{m}_{N+1} < \bar{m}_N$ then $2 - \bar{m}_{N+1} > 2 - \bar{m}_N$. Using (10) it results $\bar{m}_N^{1/N} > \bar{m}_{N+1}^{1/(N+1)}$ then $\gamma_N > \gamma_{N+1}$, therefore the part (ii) is proved.

Finally, using again the Claim above, $2^{-N} < \gamma_N^N < 2^{-N+1}$, then $2^{-1} < \gamma_N < 2^{-1+(1/N)}$. Taking $N \to +\infty$ it results $\gamma_N \to 1/2$, and the part (iii) is proved. ■

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