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Keywords

Heterogenous agents, asset price, bubble, rational, regime switching

JEL Classification

C51, C53, D84, G12

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A Heterogenous Agent Foundation for Tests of Asset Price Bubbles*

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Abstract

We provide heterogenous agent foundations for regime-switching tests of asset price bubbles, and illustrate by applying the models to historical U.S. stock market data. While the tests remain unchanged, we show the specification of regimes can be based on the beliefs of investors that come from an underlying heterogenous agent model. This allows consideration of alternative specifications for investor beliefs, straightforward interpretation of extensions to more than three regimes, and added flexibility in determining the evolution of beliefs. Our empirical example shows that this can lead to results which differ from traditional regime-switching models.

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1 Introduction

Rational bubbles are a popular way to explain movements in asset prices which are difficult to understand with fundamentals. Particularly appealing to some is that rational bubbles do not depend on investor irrationality. This also allows tests of rational bubbles to be based on standard models of asset pricing in the spirit of Lucas (1978). The regime-switching tests of van Norden (1996) and Brooks and Katsaris (2005) are notable examples. These tests assume a rational bubble in asset prices always exists, but may be in a finite number of regimes. The model of van Norden (1996) has two regimes (bubble collapse and bubble survival), while that of Brooks and Katsaris (2005) has three regimes (bubble expansion, bubble collapse, and bubble survival). Each regime has an associated probability, based on exogenously specified factors, and the probabilities of being in a particular regime are the items of interest. While either two or three regimes seems to be a reasonable number, it is unclear where the regimes originate.

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In this paper we use an asset pricing model to provide heterogenous agent foundations for this regime-switching approach and illustrate by applying this model to historical U.S. stock market data. We also show that the models of van Norden (1996) and Brooks and Katsaris (2005) can be derived from the optimization problems of investors with heterogenous beliefs. Our primary contribution is to provide a simple and plausible heterogenous agent foundation for regime switching models of rational bubbles. We also seek to link direct tests of rational bubbles with the literature on asset pricing, particularly asset pricing with heterogenous agents.

The regime switching bubble tests of van Norden (1996) and Brooks and Katsaris (2005) have been applied to many different markets. Among others, van Norden and Schaller (1993), van Norden and Vigfusson (1998) and Brooks and Katsaris (2005) apply them to stock markets. van Norden (1996) and Maldonado et al. (2012) use these models to test for bubbles in exchange rate markets, while Anderson et al. (2011) do the same for real estate. And these two models have also been used to test for bubbles in oil prices by Shi and Arora (2012).

We begin by deriving the conditions required for a rational bubble under a representative agent, following the work of Lucas (1978). We then derive the optimality conditions and implications of an asset pricing model where investors have heterogenous beliefs following Brock and Hommes (1998). Next, we put forth the three-regime test of Brooks and Katsaris (2005). This test assumes the price of an asset is comprised of fundamental and bubble components, and that this bubble always exists and is in one of three regimes (with positive probability): bubble expansion, bubble survival, and bubble collapse. These assumptions lead to a model which can be estimated to assess the probability of being in any of the regimes at any given point in time. These probabilities are based on absolute bubble size, return spread, and abnormal trading volume.

The probabilities are the link between a heterogenous agent model and the regime switching test. In particular, we interpret the probability of being in a particular regime as the fraction of investors with a specific belief. The implication is that the number of regimes corresponds to the number of different beliefs. It is in this sense that the evolution of a bubble depends on the beliefs of individual investors. A two-regime model says that investors have two different types of beliefs about the expected value of excess returns, whereas a three-regime model posits three different types of beliefs.

The final step is to specify how the beliefs of investors change. In the regime switching models these are specified by the modeler, based on experience and personal judgement. We provide a general formulation that can accommodate such specifications, but also allows for different factors to impact belief evolution. Specifically, we assume that an investor selects or chooses their beliefs based upon a fitness or performance measure. The evolution of beliefs can evolve according to any performance measure investors think characterizes the expected value of a bubble. This could be absolute bubble size, return spread, and abnormal trading volume as in Brooks and Katsaris (2005), or it could be a different measure such as past realized profits.

We then apply the regime switching models (including one based on heterogenous agent foundations) to U.S. stock market data from January 1888 to November 2011. Our results highlight the importance of model specification. In particular, the model specification of Brooks and Katsaris (2005) and a three-regime extension of van Norden (1996) identify three periods of bubble survival (including the famous dot-com bubble episode), whereas a regime switching model based on heterogeneous agent foundations shows only two.

2 Rational Bubbles and Regime Switching Tests

We begin by deriving the conditions required for a rational bubble under a representative agent, following the work of Lucas (1978). As is standard, we show that a rational bubble exists in this framework when the transversality condition fails. We then derive the optimality conditions and implications of an asset pricing model where investors have heterogenous beliefs following Brock and Hommes (1998). Specifically, we show that given a finite set of different beliefs about future prices, asset prices can exhibit behavior which differs from that under a representative agent. Next, we put forth the three-regime test of Brooks and Katsaris (2005). This test assumes that a bubble in asset prices can exist due to the failure of the transversality condition in the representative agent model.

Throughout, we use the terms rational bubble and bubble interchangeably. These are deviations of the price of an asset from their fundamental value, which is the expected stream of future dividends in the models below. The bubbles are rational in the models outlined below because investors know prices deviate from fundamental values, they also know the size of the deviations, and their expectations are set accordingly.

2.1 Rational Bubbles With a Representative Agent

Following Lucas (1978), a representative agent maximizes the discounted sum of expected utility at any time t. The agent receives some endowment each period, which can either be consumed or invested in an asset. Following the maximization, the price of this asset (ex dividend) at any time (P_t), assuming a constant risk-free rate (r_f), can be separated into a market fundamental component (P_t^*) and a bubble component (B_t):

$$P_t = P_t^* + B_t, \tag{1}$$

where $P_t^* \equiv \sum_{k=0}^{\infty} M^{-(k+1)} E_t\left(D_{t+k}\right)$ with $M = 1 + r_f$, and D_t is the dividend at period t.

The dividend process $\{D_t\}$ is assumed to be independent and identically distributed (IID) with $E_t\left[D_{t+1}\right] = D_t$. Based on the definition of P_t^*, we have:

$$MP_t^* = E_t\left(P_{t+1}^* + D_{t+1}\right). \tag{2}$$

In the presence of bubbles, the transversality condition fails and the bubble component is shown to be a submartingale process:

$$E_t\left(B_{t+1}\right) = \left(1 + r_f\right) B_t \equiv MB_t. \tag{3}$$

This condition says that the representative agents expects the bubble to grow at a constant rate when the transversality conditions fails. It will form the basis of the regime switching tests outlined below.

2.2 Rational Bubbles With Heterogenous Agents

The model generalizes the standard representative agent framework following Brock and Hommes (1998). There are now H traders, indexed by h in the market. The market has two assets, a risky asset, and a risk-free asset which is supplied at gross return M (the same as the risk-free rate above).

Traders are assumed to be mean-variance optimizers; we move away from utility maximization for tractability.[1] These mean-variance optimizers choose shares of risky assets to solve:

$$\max_{z_{h,t}} \left\{ E_{h,t}\left(W_{h,t+1}\right) - \frac{1}{2} a_h V_{h,t}\left(W_{h,t+1}\right) \right\}, \tag{4}$$

where $z_{h,t}$ denotes the number of shares of the risky asset purchased by trader h in period t, a_h is the agent-specific risk aversion parameter, and $V_{h,t}$ is the agent-specific conditional variance. Notice that a higher mean level of wealth increases utility and a higher variance of wealth decreases utility. The maximization is subject to the realized wealth, W_{t+1} denoted in units of the risky asset, of each trader,

$$W_{h,t+1} = M\left(W_{h,t} - z_{h,t}\right) + \frac{P_{t+1} + D_{t+1}}{P_t} z_{h,t}. \tag{5}$$

The realized wealth can be divided between risk-free and risky portions, and agents are choosing how many risky asset to purchase, with this risk characterized by the mean and variance of their realized wealth. Define the excess return, $R_{t+1} = \left(P_{t+1} + D_{t+1}\right)/P_t - M$, which is the difference between the gross return on a unit of the risky and risk-free assets. We assume that beliefs about the conditional variance of this excess return are constant over time and the same for all traders.[2] Further assume that all agents have the same degree of risk aversion, so that $a_h = a$ for all h. The optimality condition then yields:

$$z_{h,t} = \frac{1}{a\sigma^2} E_{ht}\left(R_{t+1}\right). \tag{6}$$

This condition says that for each trader, the number of risky shares purchased should equal the ratio of the expected excess return to the risk adjusted variance. That is, the number of risky shares should be the factor by which they believe the mean of the excess return will exceed the risk adjusted variance.

The key point to note is that this can be different for each trader h, depending on individual beliefs about expected excess returns. There may be as many different beliefs as individuals in the economy, and we assume that agents can choose from a finite set of different beliefs or predictors of the future price of a risky asset. Let $n_{h,t}$ be the fraction of investors in the economy with beliefs corresponding to h at time t. Using this, we can write the market equilibrium condition for risky assets as:

$$\sum_{h=1}^{H} n_{h,t} z_{h,t} = z_{s,t}. \tag{7}$$

This condition says that, over all H individuals in the market, the risky shares purchased by investor h with their specific beliefs must total z_{st}. As in Lucas (1978), we assume that the risky asset is in zero net supply, or $z_{s,t} = 0$. Substituting the optimality condition equation (6) into equation (7) and simplifying yields the equilibrium pricing equation for a share of the risky asset:

$$MP_t = \sum_{h=1}^{H} n_{h,t} E_{h,t}\left(P_{t+1} + D_{t+1}\right). \tag{8}$$

[1] Lengwiler (2004) discusses the special cases where the two are equivalent.

[2] Specifically, $V_{h,t}\left(R_{t+1}\right) = V_t\left(R_{t+1}\right) = \sigma^2$ for all h and t.

To ease comparison with the standard model, we also assume that the dividend process $\{D_t\}$ is independent and identically distributed (IID) with $E_t[D_{t+1}] = D_t$. Therefore, equation (2) still holds. The dividend process is common knowledge for all types of traders. Equation (8) can then be combined with equation (2) to yield:

$$\sum_{h=1}^{H} n_{h,t} E_{h,t}(B_{t+1}) = MB_t. \tag{9}$$

The bubble dynamic needs to satisfy equation (9) as opposed to $E_t(B_{t+1}) = MB_t$ in the representative agent case.

2.3 Regime Switching Tests of Rational Bubbles

The representative agent asset pricing model can be used to provide a means of testing for rational bubbles in asset prices. To show this, we now derive the three-regime model of Brooks and Katsaris (2005). The two regime model of van Norden (1996) is derived in a similar manner.

Brooks and Katsaris (2005) assume that the bubble component can be in one of three states (or regimes): a deterministic (D) regime, collapsing (C) regime, or a surviving (S) regime. In the two regime case, the assumption is that only two different states are possible. Following Blanchard and Watson (1982), equation (3) can be written conditional on one of these particular states (s_{t+1} is the regime next period):

$$\begin{aligned} E_t(B_{t+1}|s_{t+1} = D)\, d_t + E_t(B_{t+1}|s_{t+1} = C)\,(1-d_t)(1-q_t) \\ + E_t(B_{t+1}|s_{t+1} = S)\,(1-d_t)q_t = MB_t. \end{aligned} \tag{10}$$

where d_t is the probability of being in a deterministic regime, $(1-d_t)q_t$ is the probability of being in a survival regime, and $(1-d_t)(1-q_t)$ is the probability of being in a bubble collapsing regime. This equation first differentiates between deterministic and non-deterministic states, and then further restricts non-deterministic ones to be surviving or collapsing. There are many potential candidates to characterize each of the conditional expectations in this equation. Brooks and Katsaris (2005) assume that in the deterministic regime the bubble will grow at a constant rate:

$$E_t(B_{t+1}|s_{t+1} = D) = MB_t. \tag{11}$$

In the collapsing regime the conditional expectation is given by:[3]

$$E_t(B_{t+1}|s_{t+1} = C) = g(b_t)\, P_t, \tag{12}$$

where b_t is the relative size of the bubble in period t (i.e. $b_t = B_t/P_t$). This condition says that if the bubble is collapsing then the expected value of the bubble will be proportional to the price. Substituting equations (11) and (12) into equation (10) gives the conditional expected value in a surviving regime:

$$E_t(B_{t+1}|s_{t+1} = S) = \frac{M}{q_t}B_t - \frac{1-q_t}{q_t} g(b_t)\, P_t. \tag{13}$$

[3] Here, $g(b_t)$ is a continuous and everywhere differentiable function such that, $g(0) = 0$ and $0 \leq \partial g(b_t)/\partial b_t \leq Mb_t$.

This equation implies that the rate of bubble expansion in the surviving regime moves in tandem with the magnitude of bubble collapse. Big collapses (i.e. when $g(b_t)P_t$ is small) are associated with fast expansions. If $g(b_t)P_t$ goes to zero during collapse, the rate of bubble expansion in the surviving regime is faster than in the deterministic regime. This means that $M/q_t \geq M$. Moreover, when the value of q_t is small, bubbles expand at a faster rate and the probability of bubble collapse is high.[4]

For estimation purpose, the Brooks and Katsaris (2005) model is written in terms of gross return (R_t^*). The expected gross return for each of the conditional expectations derived above are:

$$E_t\left(R_{t+1}^*|s_{t+1}=D\right) = M, \tag{14}$$

$$E_t\left(R_{t+1}^*|s_{t+1}=S\right) = M + \frac{1-q_t}{q_t}\left[Mb_t - g(b_t)\right], \tag{15}$$

$$E_t\left(R_{t+1}^*|s_{t+1}=C\right) = M + g(b_t) - Mb_t. \tag{16}$$

The final step in setting up the model is to outline how the probabilities of being in any of the regimes is determined (d_t and q_t). The probability of being in a deterministic regime is assumed to be related to the absolute bubble size $|b_t|$ and the return spread S_t^{fa}:

$$d_t = \Omega\left(\beta_{d0} + \beta_{db}|b_t| + \beta_{ds}S_t^{fa}\right), \tag{17}$$

where Ω is the standard normal cumulative density function and the β's are coefficients. The interpretation is that a higher return spread and/or deviation from fundamental values raises the likelihood of being in the non-deterministic regime.[5] The probability q_t is assumed to be a function of the absolute bubble size and abnormal trading volume:

$$q_t = \Omega\left(\beta_{q0} + \beta_{qb}|b_t| + \beta_{qv}J_t\right), \tag{18}$$

where J_t is the percentage deviation of last month's volume from the 12 month moving average. Notice that both probabilities are based on measures which are believed to be important for bubble expansion and collapse, but which do not come from the underlying asset pricing model. Equations (14)-(18) constitute the three-regime BK model.

The two-regime model of van Norden (1996) consists only of the bubble survival and collapsing regimes. It does not utilize information from the return spread or abnormal trading volume, and is formalized by equations (12), (13), (15) and (16), with $q_t = \Omega\left(\beta_{q0} + \beta_{qb}|b_t|\right)$.

3 A Heterogenous Foundation for Regime Switching Tests

In this section we provide heterogenous agent foundations for the regime switching tests outlined above. The heterogenous agent asset pricing model yields an interpretation which allows the different regimes to be interpreted as varying beliefs among investors. While these varying beliefs lead to potentially many different types of regimes, we can specify a general formulation of how those beliefs evolve. Thus there are as many regimes as there are investor beliefs, and the price of an asset may change regimes based on changing beliefs, which can be specified in general terms. We then estimate the models and illustrate their use through an empirical application.

[4]This is due to the fact that $\frac{\partial E_t\left(B_{t+1}|s_{t+1}=S\right)}{\partial q_t} = -\frac{1}{q_t^2}\left[MB_t + g(b_t)P_t\right] < 0$.

[5]Parameters β_{db}, β_{ds} are restricted to be negative.

3.1 The General Case

In the heterogenous agent setting of Brock and Hommes (1998), a rational bubble will grow at a constant rate which is the weighted average over the expected value of the bubble, as in equation (9). It is in this sense that the evolution of a bubble depends upon the beliefs of individual investors. These beliefs are the key variables which link the heterogenous agent model to regime switching tests of rational bubbles.

We can interpret individual beliefs ($n_{h,t}$) as the fraction of investors with a certain belief. For example, $n_{1,t}$ might represent the fraction who believe that the bubble will rise at a constant rate, while $n_{2,t}$ the fraction who believe it will collapse. If there are large number of investors, these fractions will also correspond to the probability that investors have certain beliefs. This linkage has previously been documented in discrete choice literature by Anderson et al. (1992) and Brock and Durlauf (2001).[6] In this case, equation (9) states that the bubble grows at a constant rate which is a probability weighted average over the expected value of the bubble. And these probabilities are the probabilities that investors have certain beliefs about the expected value of the bubble.

To see explicitly how the heterogenous agent model provides foundations for the regime switching tests, consider again the three-regime model of Brooks and Katsaris (2005). This test is derived from equation (3) by assuming three states, yielding equation (10):

$$\begin{aligned} &E_t\left(B_{t+1}|s_{t+1}=D\right)d_t + E_t\left(B_{t+1}|s_{t+1}=C\right)(1-d_t)(1-q_t) \\ &\quad + E_t\left(B_{t+1}|s_{t+1}=S\right)(1-d_t)q_t = MB_t. \end{aligned} \tag{19}$$

To ease comparison we assume that there are three groups of investors, each with differing beliefs. To transform this equation to its heterogenous agent equivalent, first replace the expected value of the bubble in each state by the beliefs of particular investors about the bubble, i.e. $E_{ht}\left(B_{t+1}\right)$. Second, d_t and q_t (the probabilities associated with each state) are replaced by $n_{1,t-1}$, $n_{2,t-1}$, and $n_{3,t-1}$ (the fraction of investors that have certain types of beliefs about the state of the bubble). The equation above becomes:

$$E_{1t}\left(B_{t+1}\right)n_{1,t} + E_{2t}\left(B_{t+1}\right)n_{2,t} + E_{3t}\left(B_{t+1}\right)n_{3,t} = MB_t, \tag{20}$$

which is the bubble dynamic equation in the heterogenous agent framework. The rate of growth of the bubble is a weighted average over expected beliefs, where the weights are the fraction of investors with a particular belief. We further assume that the specifications of investor beliefs are the same as the conditional expected values of bubbles, namely:

$$\begin{aligned} E_{1t}\left(B_{t+1}\right) &= MB_t, \\ E_{2t}\left(B_{t+1}\right) &= g\left(b_t\right)P_t, \\ E_{3t}\left(B_{t+1}\right) &= MB_t + \frac{n_{2,t}}{n_{3,t}}\left[MB_t - g\left(b_t\right)P_t\right]. \end{aligned}$$

The last equation was obtained by replacing q_t with $n_{3,t}/\left(n_{2,t}+n_{3,t}\right)$. These three equations yield an intuitive interpretation based on the beliefs of investors, derived from the underlying optimization problem. For example, type 2 investors (those with expectationd denoted by E_{2t}) are

[6] It is based on the assumption of identical and independent investors and law of large numbers.

similar to agents who are called *contrarians* in Brock and Hommes (1998), and type 1 and type 3 investors are *trend chasers*.

Type 2 investors believe that the bubble will collapse next period and the remaining bubble size is proportional to the current prices. Type 1 investors are pure trend chasers who believe the bubble will grow at a constant rate. Type 3 investors are more sophisticated than type 1 investors, and their belief on the rate of extrapolation at period t depends on the ratio of the fractions of type 2 and type 3 investors at the beginning of period t, i.e. $n_{2,t}/n_{3,t}$.

From this point, one could follow the steps in previous section to derive the equations for excess returns of each investor type, which can then be estimated. In this case the evolution of the $n_{h,t-1}$ are assumed to follow those of d_t and q_t. These can follow many different processes, with equations (17) and (18) one specific example.

As another example, which will be estimated below, Brock and Hommes (1998) use realized profits. To see this in the context of the model, first the excess return can be rewritten as $R_{t+1} = (B_{t+1} - MB_t + \delta_{t+1})/P_t$, where $\delta_{t+1} = P^*_{t+1} + D_{t+1} - E_t\left(P^*_{t+1} + D_{t+1}\right)$ is a martingale difference sequence. The realized profits for trader type h is:[7]

$$\pi_{h,t} = R_t.z_{h,t-1} = \frac{1}{a\sigma^2}\frac{1}{P^2_{t-1}}\left[E_{h,t-1}\left(B_t\right) - MB_{t-1}\right]\left(B_t - MB_{t-1}\right). \tag{21}$$

We assume the fraction of investors with belief h evolves according to the following equation:[8]

$$d_t = \Omega\left[\gamma_{d0} + \gamma_d\left(\Delta\pi_t^{1,2} + \Delta\pi_t^{1,3}\right)\right] \tag{22}$$

$$q_t = \Omega\left[\gamma_{q0} + \gamma_q\left(\Delta\pi_t^{3,1} + \Delta\pi_t^{3,2}\right)\right] \tag{23}$$

where the parameters $\gamma_d, \gamma_q > 0$ are called the intensity of choice and $\Delta\pi_t^{h,k} = \pi_{h,t} - \pi_{k,t}$ denotes the difference in realized profits of belief type h compared to type k. The key feature of this equation is that strategies with higher fitness (realized profits) in the recent past attract more followers. Here, when $\gamma_i = \infty$ with $i = \{d, q\}$, all type h investors will change to the more accurate strategy in predicting the actual price. This means that agents are infinitely sensitive to differences in forecasting accuracy. With $\gamma_i = 0$, agents will not change their beliefs at all. Agents are boundedly rational in the sense that they abandon beliefs that performed poorly in the recent past.

One can then derive that the realized profit differences are nonlinear functions of q_{t-1}, b_t and b_{t-1}, namely:

$$\Delta\pi_t^{1,2} + \Delta\pi_t^{1,3} = -\frac{1}{a\sigma^2}\frac{1}{q_{t-1}}\Psi_t,$$

$$\Delta\pi_t^{3,1} + \Delta\pi_t^{3,1} = \frac{1}{a\sigma^2}\left(1 - \frac{2}{q_{t-1}}\right)\Psi_t,$$

where $\Psi_t = \left[g\left(b_{t-1}\right) - Mb_{t-1}\right]\left(b_t - Mb_{t-1}\right)$.

[7] For simplicity, we assume $\delta_t = 0$.

[8] Brock and Hommes (1998) use a discrete choice model with multi-nominal logit probabilities.

3.2 Model Estimation

To estimate the model we assume in equations (14) - (16) that $g(b_t) = cb_t$, with $0 < c \leq M$ capturing the magnitude of bubble collapse and $\kappa = M - c$.[9] The model is then:

$$R^{*D}_{t+1} = M + \varepsilon^D_{t+1}, \ \varepsilon^D_{t+1} \sim N\left(0, \sigma^2_D\right) \tag{24}$$

$$R^{*S}_{t+1} = M + \kappa \frac{1-q_t}{q_t} b_t + \varepsilon^S_{t+1}, \ \varepsilon^S_{t+1} \sim N\left(0, \sigma^2_S\right) \tag{25}$$

$$R^{*C}_{t+1} = M - \kappa b_t + \varepsilon^C_{t+1}, \ \varepsilon^C_{t+1} \sim N\left(0, \sigma^2_C\right) \tag{26}$$

The residuals of each equation are assumed to be normally distributed with zero mean and a given variance. The transition probabilities of the heterogeneous agent model are:

$$d_t = \Omega\left[\gamma_{d0} + \gamma_{d1} \frac{1}{q_{t-1}} \Psi'_t\right], \tag{27}$$

$$q_t = \Omega\left[\gamma_{q0} + \gamma_{q1}\left(1 - \frac{2}{q_{t-1}}\right)\Psi'_t\right], \tag{28}$$

where $\Psi'_t = b_{t-1}\left(b_t - Mb_{t-1}\right)$.

The log likelihood function of this system is:

$$l\left(R_1, R_2, \cdots, R_T; \Psi\right) = \sum_{t=1}^{T-1} \ln\left[d_t \frac{\phi_D}{\sigma_D} + (1-d_t)\, q_t \frac{\phi_S}{\sigma_S} + (1-d_t)(1-q_t)\frac{\phi_C}{\sigma_C}\right],$$

where Ψ contains all of the unknown parameters, the σ's are standard deviations, and ϕ_D, ϕ_S and ϕ_C are the probability density functions of $N\left(R^D_{t+1} - M, \sigma^2_D\right)$, $N\left(R^S_{t+1} - M - \kappa\frac{1-q_t}{q_t}b_t, \sigma^2_S\right)$ and $N\left(R^C_{t+1} - M + \kappa b_t, \sigma^2_C\right)$ respectively.

This likelihood function is unbounded.[10] To avoid this problem we use the Quasi-Bayesian approach of Hamilton (1991), where one adjusts the log likelihood function to be:

$$l^*\left(R_1, \cdots, R_T; \Psi\right) = l\left(R_1, R_2, \cdots, R_T; \Psi\right) - \sum_{k \in \{D,S,C\}} \frac{a_k}{2}\log\sigma^2_k - \sum_{k \in \{D,S,C\}} \frac{b_k}{2\sigma^2_k},$$

where the ratio b_k/a_k corresponds to our prior for σ^2_k, and a_k characterizes the weight of the prior. The model is estimated using the Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm with 100 sets of randomly generated start-up values and we take estimates associated with the largest likelihood value.

3.3 Empirical Application

We now revisit the empirical application of Brooks and Katsaris (2005) with the sample period extended to November 2011, using the same data sources. The monthly observations on the S&P

[9] Unlike Brooks and Katsaris (2005), we do not apply first order linear approximation to these equations because important information may be lost.

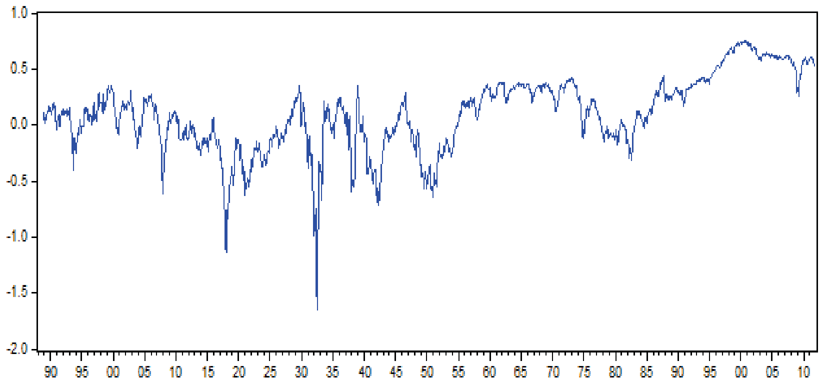
[10] The unbounded likelihood function problem associated with the mixture normal model has been well documented in the literature. See Fruhwirth-Schnatter (2006).

price index, dividends, and the US Consumer Price Index (All Items Seasonally Adjusted) are taken from Robert Shiller's website. The monthly share volume is the monthly average of the daily share volume from the NYSE. We calculate the monthly abnormal volume as the percentage deviation of last month's volume from the 6-month moving average.

The market fundamental values are obtained using the method of Campbell and Shiller (1987). The bubble deviations from the market fundamental values are displayed in Figure 1, where we observe significant drops around 1917 and 1932 and a fast increase around 1995. We estimate three different models: the nonlinear Brooks and Katsaris (2005) model, the nonlinear three regime extension of the van Norden (1996) model, and our extended heterogeneous agent model. The prior coefficients in all models are set to be: $a_k = 0.1$ for all $k \in \{D, S, C\}$ and $b_D = 0.01, b_S = 0.01, b_C = 0.1$.

FIGURE 1

Bubble Deviations of Actual Prices from Campbell and Shiller Fundamental Values.



From Table 1, we can see that coefficients β_{dS} and β_{qV} are insignificant in both models. Estimates from the Brooks and Katsaris (2005) model and the van Norden (1996) model for other coefficients are almost identical. In addition, we fail to reject the null hypothesis of the likelihood ratio test at the 10% significance level. This suggests that the return spread and abnormal trading volume do not provide additional information for parameter estimation of either model over the sample period. Transition probabilities obtained from these two models are the same (see panels (a) and (b) of Figure 2). This is in contrast to Brooks and Katsaris (2005), where they conclude that these two exogenous variables provide richer dynamics to the transition probabilities. Panel (a) of Figure 2a shows that there are several episodes of dramatic increase in the bubble surviving probability around 1917M07 (post panic of 1917), 1932M01 (post great depression), and 1995M04 (beginning of the dot-com bubble).

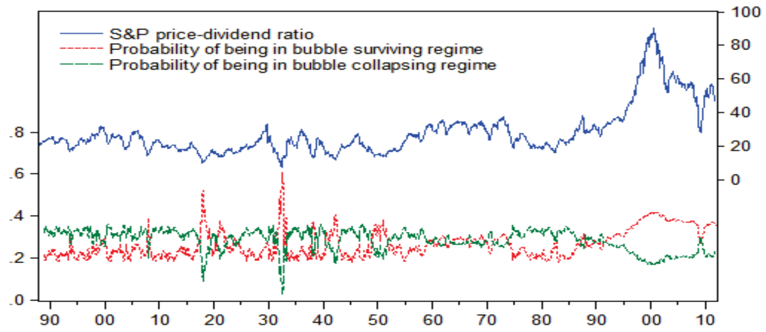
Estimates from the heterogeneous agent model are displayed in the last column of Table 1.[11] The estimated value of M is 1.05 (implying a 5% risk-free rate), similar to the regime-switching models. The estimates of κ and σ_C, however, are much larger than the other two, implying a more dramatic and volatile collapse regime. Panel (c) of Figure 2 shows that the heterogenous agent

[11] We have also considered using the accumulated past realized profit (of previous two and three periods) as fitness measures. Results are similar to the reported one.

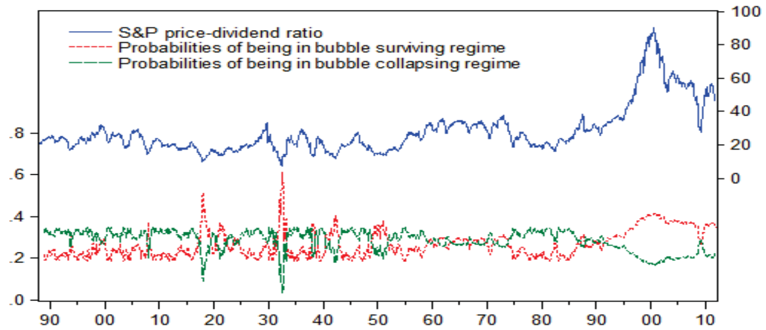
FIGURE 2

Probabilities of being in a surviving or collapsing regime

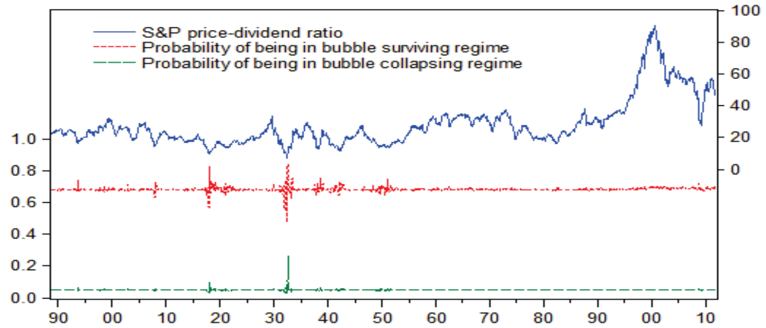
(a) *The Brooks and Katsaris (2005) model*



(b) *The three regime extension of van Norden (1996) model*



(c) *The heterogeneous agent model*



model has similar rises in the bubble surviving probability as the other two models in 1917 and 1932, but this is not the case in the mid-1990s. In general, however, there is less volatility in the estimates from the heterogenous agent model. This may reflect the fact that past realized profits have become less informative of investor beliefs, or that calculated deviations from fundamental

TABLE 1

The S&P stock market: regime switching bubble models and the heterogeneous agent bubble model. Figures in parentheses are p-values.

	BK	Extension of VNS		HAM
M	1.05 (0.00)	1.05 (0.00)	M	1.05 (0.00)
κ	0.14 (0.00)	0.14 (0.00)	κ	0.82 (0.00)
σ_D	0.01 (0.00)	0.01 (0.00)	σ_D	0.01 (0.00)
σ_S	0.01 (0.00)	0.01 (0.00)	σ_S	0.03 (0.00)
σ_C	0.06 (0.00)	0.06 (0.00)	σ_C	0.29 (0.00)
β_{d0}	-0.08 (0.37)	-0.08 (0.38)	$\gamma_{d,0}$	-0.61 (0.00)
β_{db}	-0.16 (0.44)	-0.17 (0.41)	$\gamma_{d,1}$	1.84 (0.18)
β_{dS}	0.73 (0.60)		$\gamma_{q,0}$	1.51 (0.00)
β_{q0}	-0.39 (0.00)	-0.39 (0.00)	$\gamma_{q,0}$	-0.55 (0.00)
β_{qb}	1.27 (0.00)	1.26 (0.00)		
β_{qV}	0.05 (0.42)			
lld	8298.18	8297.73		8105.22
Test for Restrictions				
$M \geq \kappa$	6085.97 (0.00)	6026.04 (0.00)	$M \geq \kappa$	2.65 (0.05)
$\beta_{q,V} < 0$	0.65 (0.21)	-		
$\beta_{q,b} < 0$	267.23 (0.00)	270.82 (0.00)		
$\beta_{d,S} < 0$	0.27 (0.30)	-		
$\beta_{d,b} < 0$	0.61 (0.22)	0.68 (0.21)		
LR Test	0.88 (0.64)			

values are too volatile.

4 Conclusion

Providing heterogenous agent foundations for regime switching tests of asset price bubbles does not change in any way their mechanics or computation. But it does allow consideration of alternative specifications for investor beliefs, straightforward interpretation of extensions to more than three regimes, and added flexibility in determining the evolution of beliefs. When the regime switching tests are derived from a representative agent model, the bubble grows at a constant rate which is the probability weighted average of assumed states. These states are not represented in the underlying asset pricing model, nor do they play a part in a solution where the transversality condition holds.

In contrast, the weights in the heterogenous agent model are fundamental to the model irrespective of whether a bubble exists or not. These beliefs can correspond to the regimes in the models above, but can also be more general. This fits in well with the heterogenous agent literature where many different types of investor beliefs have been proposed. Examples of such beliefs include chartist, trend chaser, agents with perfect foresight, and fundamentalist [see Frankel and Froot (1990), Brock and Hommes (1998), Boswijk et al. (2007) among others].

Our empirical application to the U.S. stock market shows the importance of the regime switching model specification. The specification of transition probabilities, whether assumed or based on a fitness measure, is particularly important. This is highlighted by the fact that the inferred transition probabilities of the Brooks and Katsaris (2005) specification show the expansion phase of the dot-com bubble along with the recovery phases of the 1917 crash and the great depression, whereas the transition probabilities calculated from the Brock and Hommes (1998) specification do not reflect the dot-com episode.

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