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Tractable Latent State Filtering for Non-Linear DSGE Models Using a Second-Order Approximation

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Abstract

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Keywords

Latent state filtering; estimation of DSGE models; second-order approximation; pruning; Kalman filter; particle filter; quasi-maximum likelihood.

JEL Classification

C63, C68, E37

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Tractable Latent State Filtering for Non-Linear DSGE Models Using a Second-Order Approximation

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May 6, 2013

This paper develops a novel approach for estimating latent state variables of Dynamic Stochastic General Equilibrium (DSGE) models that are solved using a second-order accurate approximation. I apply the Kalman filter to a state-space representation of the second-order solution based on the 'pruning' scheme of Kim, Kim, Schaumburg and Sims (2008). By contrast to particle filters, no stochastic simulations are needed for the filter here--the present method is thus much faster. In Monte Carlo experiments, the filter here generates more accurate estimates of latent state variables than the standard particle filter. The present filter is also more accurate than a conventional Kalman filter that treats the linearized model as the true data generating process. Due to its high speed, the filter presented here is suited for the estimation of model *parameters*; a quasi-maximum likelihood procedure can be used for that purpose.

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1. Introduction

Dynamic Stochastic General Equilibrium (DSGE) models typically feature state variables that cannot directly be measured empirically (such as preference shocks), or for which data include measurement error. A vast literature during the past two decades has taken *linearized* DSGE models to the data, using likelihood-based methods (e.g., Smets and Wouters (2007), Del Negro and Schorfheide (2011)). Linearity (in state variables) greatly facilitates model estimation, as it allows to use the standard Kalman filter to infer latent variables and to compute sample likelihood functions based on prediction error decompositions. Recent research has begun to estimate *non-linear* DSGE models using particle filters;[1] these filters infer latent states using *Monte Carlo* methods, and are thus slow computationally, which limits their use to small models.

This paper develops a novel *deterministic* filter for estimating latent state variables of DSGE models that are solved using a *second-order* accurate approximation (as derived by Jin and Judd (2000), Sims (2000), Collard and Juillard (2001), Schmitt-Grohé and Uribe (2004), Kollmann (2005) and Lombardo and Sutherland (2007)). That approximation provides the most tractable *non-linear* solution technique for medium-scale models, and has thus widely been used in macroeconomics (see Kollmann (2002) and Kollmann, Kim and Kim (2011) for detailed references).

When simulating second-order accurate model solutions, it is common to use the 'pruning' scheme of Kim, Kim, Schaumburg and Sims (2008), under which second-order terms are replaced by products of the linearized solution. This paper assumes that the 'pruned' second-order model is the *true* data generating process (DGP). The method presented here exploits the fact that the state equation of the pruned system is *linear* in a state vector that consists of variables solved to second- and first-order accuracy, and of products of first-order accurate variables. I apply the Kalman filter to that state equation. In Monte Carlo experiments, the filter here generates more accurate estimates of latent state variables than the standard particle filter, especially when the model has strong curvature or when shocks are large. Importantly, the filter here is much faster than particle filters, as it is not based on stochastic simulations. The present filter is also more accurate than a conventional Kalman filter that treats the linearized model as

[1] See Fernández-Villaverde and Rubio-Ramírez (2007) and An and Schorfheide (2007) for early applications.

the true DGP.[2] Due to its high speed, the filter presented here is suited for the estimation of model *parameters*; a quasi-maximum likelihood procedure can be used for that purpose.

2. Model format and filter

Model format and second-order solution

Many widely-used DSGE models can be expressed as:

$$E_t G(\Omega_{t+1}, \Omega_t, \varepsilon_{t+1}) = 0, \tag{1}$$

where E_t is the mathematical expectation conditional on date *t* information; $G: R^{2n+m} \to R^n$ is a function, and Ω_t is an *n*x1 vector of endogenous and exogenous variables known at *t*; ε_{t+1} is an *m*x1 vector of innovations to exogenous variables. In what follows, ε_t is Gaussian: $\varepsilon_t \sim N(0, \xi^2 \Sigma_\varepsilon)$, where ξ is a scalar that indexes the size of shocks. The solution of model (1) is a "policy function" $\Omega_{t+1} = F(\Omega_t, \varepsilon_{t+1}, \xi)$, such that $E_t G(F(\Omega_t, \varepsilon_{t+1}, \xi), \Omega_t, \varepsilon_{t+1}) = 0 \ \forall \Omega_t$. This paper focuses on second-order accurate model solutions, namely on second-order Taylor series expansions of the policy function around a deterministic stead state, i.e. around $\xi = 0$ and a point Ω such that $\Omega = F(\Omega, 0, 0)$. Let $\omega_t \equiv \Omega_t - \Omega$. For a qx1 column vector x whose *i*-th element is denoted x^i, let

$$P(x) \equiv vech(xx') = ((x^1)^2, x^1 x^2, .., x^1 x^q, (x^2)^2, x^2 x^3, .., x^2 x^q, ..., (x^{q-1})^2, x^{q-1} x^{n_x}, (x^q)^2),$$

be a vector consisting of all squares and cross-products of the elements of *x*.[3] The second-order accurate model solution can be written as

$$\omega_{t+1} = F_0 \xi^2 + F_1 \omega_t + F_2 \varepsilon_{t+1} + F_{11} P(\omega_t) + F_{12} \cdot (\omega_t \otimes \varepsilon_{t+1}) + F_{22} P(\varepsilon_{t+1}), \tag{2}$$

where $F_0, F_1, F_2, F_{11}, F_{12}, F_{22}$ are vectors/matrices that are functions of structural model parameters, but that do not depend on ξ (Sims (2000), Schmitt-Grohé and Uribe (2004)). The first-order accurate (linearized) model solution is:

$$\omega_{t+1}^{(1)} = F_1 \omega_t^{(1)} + F_2 \varepsilon_{t+1}. \tag{3}$$

The superscript $^{(1)}$ denotes a variable solved to first-order accuracy. It is assumed that all eigenvalues of F_1 are strictly inside the unit circle, i.e. that the linearized model is stationary.

[2] The literature has discussed 'Extended Kalman filters', i.e. Kalman filters applied to linear approximations of non-linear models; e.g., Harvey (1989).

[3] For a square matrix *M*, vech(*M*) is the column vector obtained by vertically stacking the elements of *M* that are on or below the main diagonal.

Pruning

As discussed above, I use the 'pruning' scheme of Kim et al. (2008) under which second-order terms are replaced by products of the linearized solution--i.e. $P(\omega_t)$ and $\omega_t \otimes \varepsilon_{t+1}$ are substituted by $P(\omega_t^{(1)})$ and $\omega_t^{(1)} \otimes \varepsilon_{t+1}$, respectively. With pruning, the solution (2) is thus replaced by:

$$\omega_{t+1} = F_0 \xi^2 + F_1 \omega_t + F_2 \varepsilon_{t+1} + F_{11} P(\omega_t^{(1)}) + F_{12} \omega_t^{(1)} \otimes \varepsilon_{t+1} + F_{22} P(\varepsilon_{t+1}). \quad (4)$$

Note that $P(\omega_t)=P(\omega_t^{(1)})$ and $\omega_t \otimes \varepsilon_{t+1}=\omega_t^{(1)} \otimes \varepsilon_{t+1}$ hold, up to second-order accuracy. Thus, (4) is a valid second-order accurate solution. In repeated applications of (2), third and higher-order terms of state variables appear; e.g., when ω_{t+1} is quadratic in ω_t, then ω_{t+2} is quartic in ω_t; pruning removes these higher-order terms. The motivation for pruning is that (2) has extraneous steady states (not present in the original model)--some of these steady states mark transitions to unstable behavior. Large shocks can thus move the model into an unstable region. Pruning overcomes this problem. If the first-order solution is stable, then the pruned second-order solution (4) too is stable. The subsequent discussion assumes that the **true** DGP is given by the pruned system (4).

Augmented state equation

The law of motion of $P(\omega_t^{(1)})$ can be expressed as $P(\omega_{t+1}^{(1)})=K_{11}P(\omega_t^{(1)})+K_{12}\omega_t^{(1)} \otimes \varepsilon_{t+1}+K_{22}P(\varepsilon_{t+1})$, where K_{11}, K_{12}, K_{22} are matrices that are functions of F_1 and F_2. Stacking this matrix equation, as well as (3) and (4) gives the following state equation:

$$\begin{bmatrix} \omega_{t+1} \\ P(\omega_{t+1}^{(1)}) \\ \omega_{t+1}^{(1)} \end{bmatrix} = \begin{bmatrix} F_0 \xi^2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} F_1 & F_{11} & 0 \\ 0 & K_{11} & 0 \\ 0 & 0 & F_1 \end{bmatrix} \begin{bmatrix} \omega_t \\ P(\omega_t^{(1)}) \\ \omega_t^{(1)} \end{bmatrix} + \begin{bmatrix} F_2 \\ 0 \\ F_2 \end{bmatrix} \varepsilon_{t+1} + \begin{bmatrix} F_{12} \\ K_{12} \\ 0 \end{bmatrix} (\omega_t^{(1)} \otimes \varepsilon_{t+1}) + \begin{bmatrix} F_{22} \\ K_{22} \\ 0 \end{bmatrix} P(\varepsilon_{t+1}). \quad (5)$$

(5) can be written as: $Z_{t+1}=g_0+G_1 Z_t+G_2 \varepsilon_{t+1}+G_{12}(\omega_t^{(1)} \otimes \varepsilon_{t+1})+G_{22}P(\varepsilon_{t+1})$, with $Z_{t+1} \equiv (\omega_{t+1}', P(\omega_{t+1}^{(1)})', \omega_{t+1}^{(1)\prime})'$, while g_0, G_1, G_2, G_{12} and G_{22} are the first to fifth coefficient vectors/matrices on the right-hand side of (5), respectively. Thus,

$$Z_{t+1}= G_0+G_1 Z_t+u_{t+1}, \quad (6)$$

where $G_0 \equiv g_0+G_{22}E(P(e_{t+1}))$, while $u_{t+1} \equiv G_2 \varepsilon_{t+1}+G_{12}(\omega_t^{(1)} \otimes \varepsilon_{t+1})+G_{22}[P(\varepsilon_{t+1})-E(P(\varepsilon_{t+1}))]$ is a serially uncorrelated, mean zero, disturbance. Note that the state equation (6) is *linear* in the augmented

state vector Z_t consisting of the second- and first-order accurate variables, and of the squares and cross-products of first-order accurate variables. [4]

Observation equation

At *t=1,..,T,* the analyst observes n_y variables that are linear functions of the state vector ω_t plus i.i.d. measurement error that is independent of the state vector, at all leads and lags: $y_t=\gamma\omega_t+\psi_t$, where γ is an n_yxn matrix and $\psi_t \sim N(0,\Sigma_\psi)$ is an n_yx1 vector of measurement errors; Σ_Ψ is a diagonal matrix. The observation equation can be written as:

$$y_t=\Gamma Z_t+\psi_t, \text{ with } \Gamma\equiv(\gamma,0). \tag{7}$$

The filter

Let $\Upsilon^t\equiv\{y_\tau\}_{\tau=1}^t$ be the observables known at date *t*; $X_{t,\tau}\equiv \mathrm{E}(X_t|\Upsilon^\tau)$ and $V_{t,\tau}^X\equiv E([X_t-X_{t,\tau}][X_t-X_{t,\tau}]'|\Upsilon^\tau)$ denote the conditional mean and variance of the column vector X_t, given Υ^τ. Unconditional means and variances are denoted by $E(X_t)$ and $V(X_t)$.

Given $Z_{t,t},V_{t,t}^Z$, the 1st and 2nd conditional moments of the augmented state vector Z_t conditional on Υ^t, we can compute moments of Z_{t+1} conditional on Υ^t using (6):

$$Z_{t+1,t} = G_0 + G_1 Z_{t,t}, \tag{8}$$

$$V_{t+1,t}^Z = G_1 V_{t,t}^Z G_1' + V_{t+1,t}^u, \text{ with} \tag{9}$$

$$V_{t+1,t}^u\equiv G_2 V_\varepsilon G_2' + G_{12}(\omega_{t,t}^{(1)}\otimes\Sigma_\varepsilon)G_2' + G_2(\omega_{t,t}^{(1)}\otimes\Sigma_\varepsilon)'G_{12}' + G_{12}\{(V_{t,t}^{\omega^{(1)}}+\omega_{t,t}^{(1)}\omega_{t,t}^{(1)\prime})\otimes\Sigma_\varepsilon\}G_{12}' + G_{22}V(P(\varepsilon_{t+1}))G_{22}' \tag{10}$$

(see Appendix).

To generate $Z_{t+1,t+1},V_{t+1,t+1}^Z$, I apply the *linear* updating equation of the standard Kalman filter (e.g., Hamilton (1994, ch.13)) to the state-space representation (6),(7):

$$Z_{t+1,t+1}=Z_{t+1,t}+\phi_t\cdot(y_{t+1}-y_{t+1,t}), \text{ with } y_{t+1,t}=\Gamma Z_{t+1,t}, \tag{11}$$

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[4] Aruoba, Bocola and Schorfheide (2012) estimate a pruned *univariate* quadratic time series model, using particle filter methods. These authors discard the term that is quadratic in ε_{t+1} on the right-hand side of (4). By contrast, the paper here allows for non-zero coefficients on second-order terms in ε_{t+1}, and it develops a deterministic filter that can be applied to *multivariate* models. Andreasen et al. (2013) also derive a multivariate pruned state equation for second-order accurate models (I learnt about that paper after the present research was completed); these authors use that equation to derive a simulated methods of moments estimator for DSGE models; by contrast, the paper here focuses on filtering latent state variables.

and $\phi_t = V^Z_{t+1,t}\Gamma'\{\Gamma V^Z_{t+1,t}\Gamma + \Sigma_\psi\}^{-1}$, $\quad V^Z_{t+1,t+1} = V^Z_{t+1,t} - V^Z_{t+1,t}\Gamma'\{\Gamma V^Z_{t+1,t}\Gamma' + \Sigma_\psi\}^{-1}\Gamma V^Z_{t+1,t}$. (12)

The filter is started with the unconditional mean and variance of Z_0: $Z_{0,0}=E(Z_0)$, $V^Z_{0,0}\equiv V(Z_0)$; $Z_{t+1,t+1}$ and $V^Z_{t+1,t+1}$ for $t\geq 0$ are computed by iterating on (8)-(12). Henceforth, I refer to this filter as the '**KalmanQ**' filter. Computer code that implements KalmanQ is available from the author.

$E(Z_0), V(Z_0)$ and $V^u_{t+1,t}$ can be computed exactly; see the Appendix. The linear updating formula (11) would be an *exact* algorithm for computing the conditional expectation $Z_{t+1,t+1}$, if Z_{t+1} and the observables were (jointly) Gaussian (then $Z_{t+1,t+1}$ would be a *linear* function of the data). This condition is not met in the second-order model, as the disturbance u_{t+1} of the state equation (6) is non-Gaussian. However, as shown below, the KalmanQ filter closely tracks the true latent variables.[5]

When the *linearized* model is the true DGP (i.e. when $F_0=0, F_{11}=0, F_{12}=0, F_{22}=0$), then the filter here is identical to the conventional linear Kalman filter, and the updating formula (11) holds *exactly*. In the presence of second-order model terms, KalmanQ is more accurate than a conventional Kalman filter that assumes that the linearized model (3) is the true DGP; see below.

Quasi-maximum likelihood estimation of model parameters

If model *parameters* are unknown, then a quasi-maximum likelihood (QML) estimate of those parameters can be obtained by maximizing the function $L(\Upsilon^T|\theta)\equiv\sum_{t=1}^{T}\ln h(y_t|y_{t,t-1}(\theta);V^y_{t,t-1}(\theta))$, with respect to the vector of unknown parameters, θ. Here $h(y|\mu;V)$ is the multivariate normal density with mean μ and variance V. For a given θ, $y_{t,t-1}(\theta)\equiv\Gamma Z_{t,t-1}(\theta)$ is the prediction of y_t generated by KalmanQ, based on date t-1 information, Υ^{t-1}; $V^y_{t,t-1}(\theta)\equiv\Gamma V^Z_{t,t-1}(\theta)\Gamma'+\Sigma_\psi$ is the conditional variance of y_t, given Υ^{t-1}. Under conditions discussed in Hamilton (1994, ch.13), the QML estimator θ^{QML}_T is asymptotically normal: $\sqrt{T}(\theta^{QML}_T-\theta_0)\rightarrow N(0,(J_2(J_1)^{-1}J_2)^{-1})$, where θ_0 is the true

[5] Without Gaussianity, $Z_{t+1,t+1}$ is a non-linear function of data Υ^{t+1}: $Z_{t+1,t+1}=\phi(y_{t+1},\Upsilon^t)$. (11) can be viewed as a linear approximation of this function: $Z_{t+1,t+1}\cong\phi_t\cdot(y_{t+1}-y)$, with $\phi_t\equiv\partial\phi(y_{t+1},\Upsilon^t)/\partial y_{t+1}|_{y_{t+1}=y}$. By the Law of Iterated Expectations, $Z_{t+1,t}=E(Z_{t+1,t+1}|\Upsilon^t)$, and thus: $Z_{t+1,t+1}-Z_{t+1,t}\cong\phi_t\cdot(y_{t+1}-y_{t+1,t})$. Recall that the observable y_{t+1} is a *linear* function of Z_{t+1} (see (7)); this may help to explain the good performance of the linear updating rule.

parameter vector and $J_1 \equiv \text{plim}\, T^{-1}\sum_{t=1}^{T}\eta_t(\theta_0)\eta_t(\theta_0)'$, with $\eta_t(\theta_0) \equiv \partial \log(h_t(\theta_0))/\partial\theta$, $h_t(\theta) \equiv h(y_t|y_{t,t-1}(\theta_0);V^z_{t,t-1}(\theta_0))$ and $J_2 \equiv \text{plim}\, T^{-1}\sum_{t=1}^{T}\partial^2 \log(h_t(\theta_0))/\partial\theta\partial\theta'$.

3. Monte Carlo evidence

3.1. A textbook RBC model

The method is tested for a basic RBC model. Assume a representative infinitely-lived household whose date *t* expected lifetime utility V_t is given by $V_t=\{\frac{1}{1-\sigma}C_t^{1-\sigma}-\frac{1}{1+1/\eta}N_t^{1+1/\eta}\}+\lambda_t\beta E_t V_{t+1}$, where C_t and N_t are consumption and hours worked, at *t*, respectively. $\sigma>0$ and $\eta>0$ are the risk aversion coefficient and the (Frisch) labor supply elasticity. λ_t is an exogenous taste (discount factor) shock of unit unconditional mean. $0<\beta<1$ is the steady state subjective discount factor. The household maximizes expected life-time utility subject to the resource constraint $C_t+K_{t+1}=\theta_t K_t^{\alpha}N_t^{1-\alpha}+(1-\delta)K_t$, where K_{t+1} is the end-of-period *t* capital stock; $Y_t=\theta_t K_t^{\alpha}N_t^{1-\alpha}$ is output. $0<\alpha,\delta<1$ are the capital share and the capital depreciation rate, respectively. $\theta_t>0$ is exogenous total factor productivity (TFP). The household's first-order conditions are:

$$\lambda_t E_t \beta (C_{t+1}/C_t)^{-\sigma}(\theta_{t+1}\alpha K_{t+1}^{\alpha-1}N_{t+1}^{1-\alpha}+1-\delta)=1, \quad C_t^{-\sigma}(1-\alpha)\theta_t K_t^{\alpha}N_t^{-\alpha}=N_t^{1/\eta}.$$

The forcing variables follow independent autoregressive processes: $\ln(\theta_t)=\rho_\theta \ln(\theta_{t-1})+\varepsilon_{\theta,t}$ $\ln(\lambda_t)=\rho_\lambda \ln(\lambda_{t-1})+\varepsilon_{\lambda,t}$, $0<\rho_\lambda,\rho_\theta<1$, where $\varepsilon_{\theta,t}$ and $\varepsilon_{\lambda,t}$ are normal i.i.d. white noises with standard deviations σ_θ and σ_λ, respectively.

The numerical simulations discussed below assume $\beta=0.99, \eta=4, \alpha=0.3, \delta=0.025$, $\rho_\theta=\rho_\psi=0.99$; parameter values in that range are standard in quarterly macro models. The parameter ξ that scales the size of the shocks is normalized as $\xi=1$. The risk aversion coefficient is set at a high value, $\sigma=10$, so that the model has enough curvature to allow for non-negligible differences between the second-order accurate and linear model approximations. One model variant assumes shocks that are much larger than the shocks in standard macro models, in order to generate big differences between the two approximations: $\sigma_\theta=0.20$, $\sigma_\psi=0.01$. I refer to this variant as the 'big shocks' variant. I also consider a second 'small shocks' variant, in which the standard deviations of shocks are twenty time smaller: $\sigma_\theta=0.01$, $\sigma_\psi=0.0005$ (conventional RBC

models assume that the standard deviation of TFP innovations is about 1%; e.g., Kollmann (1996)). [6]

The observables are assumed to be GDP, consumption, investment and hours worked; measurement error is added to the logs of these variables. Measurement error has a standard deviation of 0.04 (0.002) in the model variant with big (small) shocks.

Chris Sims' MATLAB program gensys2 is used to compute first- and second-order accurate model solutions. The model is approximated in terms of logged variables (the state- and observation equations are expressed in terms of logged variables).

Predicted standard deviations

Table 1 reports unconditional standard deviations of 7 logged variables (GDP, consumption, investment, capital, hours, TFP and the taste shock λ) generated by the first- and second-order approximations.[7] Model variants with both shocks, and variants with just one type of shock, are considered; moments for non-HP filtered variables are shown, as well as moments of HP filtered variables (smoothing parameter: 1600).

In the **'big shocks'** model variant, the standard deviations of endogenous variables are huge; e.g., with both shocks, the standard deviation of (non-HP filtered) GDP is 176% (82%) under the second-order (first-order) approximation; GDP is thus about twice as volatile under the second-order approximation (than in the linearized model).[8] The capital stock, investment and hours worked (non-HP filtered) are about one-half more volatile under the second-order approximation than under the linear approximation. By contrast, consumption volatility is similar across the two approximations. Consumption is much less volatile than GDP, due to the assumed high risk aversion of the household. The preference shock (λ) is the main source of fluctuations in the capital stock, GDP and investment; TFP shocks are the main drivers of consumption. The correlation between the second- and first-order approximations of a given variable is noticeably below unity, in the model variant with big shocks: e.g., about 0.7 for capital and investment, and 0.5 for GDP.

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[6] The relative size of the TFP and taste shocks assumed here (i.e. σ_θ 20-times larger than σ_ψ) ensures that each shock accounts for a non-negligible share of the variance of the endogenous variables; see below.

[7] The statistics are shown for variables without measurement error. The ranking of volatilities generated by the two approximations and shocks is not affected by the presence of measurement error.

[8] HP filtered variables are markedly less volatile than non-HP filtered variables; however, volatility remains much higher under the second-order approximation than under the linear approximation, in the 'big shocks' variant. E.g. the standard dev. of HP filtered GDP is 47% (23%) under the second- (first-) order approximation.

The **'small shocks'** model variant generates much smaller standard deviations of endogenous variables that are roughly in line with predicted moments reported in the RBC literature (e.g., Kollmann (1996)); e.g., the predicted standard deviation of HP-filtered GDP and investment are about 1% and 5%, respectively (with both shocks). With small shocks, it remains true that variables are more volatile in the second-order model than in the linearized model, however, the difference is barely noticeable. E.g., the ratio of the GDP [investment] standard dev. across the $2^{nd}/1^{st}$ order approximations is merely 1.005 [1.002].

Filter accuracy

I generate 50 simulation runs of T=500 and of T=100 periods, using the second-order (pruned) state equation of the RBC model. Each run is initialized at the unconditional mean of the state vector. I apply the KalmanQ filter to the simulated series (with measurement error). I also use a conventional Kalman filter, referred to as '**KalmanL**', that treats the linearized model (3) as the true DGP. In addition, the standard particle filter--referred to a '**PF(*p*)**', where p is the number of particles--as described in An and Schorfheide (2007) is applied to the pruned state equation (4); for the simulation runs with T=500 periods, 100,000 particles are employed; for runs with T=100 periods, versions of the PF with 100,000 and with 500,000 particles are used.[9] Accuracy is evaluated for the 7 logged variables considered in Table 1.

In each simulation run s=1,..,50, the root mean square error (RMSE) is computed, across all (logged) 7 variables, $RMSE_{s,All} \equiv (\frac{1}{7T}\sum_{i=1}^{7}\sum_{t=1}^{T}(\omega_{s,t}^{i}-\omega_{s,t,t}^{i})^2)^{1/2}$, and separately for each individual variable i=1,..7, $RMSE_{s,i} \equiv (\frac{1}{T}\sum_{t=1}^{T}(\omega_{s,t}^{i}-\omega_{s,t,t}^{i})^2)^{1/2}$, where $\omega_{s,t}^{i}$ is the true date t value of variable i in run s, while $\omega_{s,t,t}^{i}$ is the filtered estimate (conditional expectation) of that variable, given the date t information set. Table 2 reports RMSEs that are averaged across simulation runs. In the Panels labeled 'Average RMSEs', Column (1) shows average RMSE, across all 7 variables, $\frac{1}{50}\sum_{s=1}^{50} RMSE_{s,ALL}$, while Cols. (2)-(8) separately show average RMSEs for each individual variable i, $\frac{1}{50}\sum_{s=1}^{50} RMSE_{s,i}$. Also reported are maximum estimation errors across all variables, periods and runs, as well as maximum estimation errors for each variable i (across all periods and all simulation runs); see Panels labeled 'Maximum Errors'. These accuracy measures are

[9] I apply KalmanL to de-meaned series, as the linearized model implies that the unconditional mean of state variables, expressed as differences from steady state, is zero, while variables generated from the second-order model have a non-zero mean. The initial particles used for the particle filter are drawn from a multi-variate normal distribution whose mean and variance are set to unconditional moments of the state vector.

reported for each of the filters (see rows labeled 'KalmanQ', 'PF(100,000)', 'PF(500,000)', and 'KalmanL'). In addition, I report the fraction of simulation runs in which the KalmanQ filter generates lower RMSEs and lower maximum estimation error than the other filters.

Table 2 shows that the KalmanQ filter is more accurate than the other filters, in all (or almost all) simulation runs—this holds for both the 'big shocks' and 'small shocks' model variants. The conventional KalmanL filter, is least accurate.

Average RMSEs generated by KalmanQ are often orders of magnitudes smaller than the RMSE's generated by the particle filter, and that even when 500,000 particles are used. E.g., for the simulation runs of the 'big shocks' model variant with T=100 periods, the average RMSEs for GDP are 0.039, 0.755, 0.527 and 1.488, respectively, for KalmanQ, PF(100,000), PF(500,000), and for KalmanL; the corresponding maximum errors are 0.160, 10.615, 4.548 and 9.826, respectively (see Panel (a.2), Col. (2)).

For the 'small shocks' simulation runs with T=100 periods, the average RMSEs for GDP are 0.0007, 0.0060, 0.0046 and 0.0224, for KalmanQ, PF(100,000), PF(500,000) and KalmanL respectively, while corresponding maximum errors are 0.0033, 0.0319, 0.0224 and 0.0918 (see Panel (b.2)). Not surprisingly, all the filters are more accurate when shocks are small, and thus, the *absolute* accuracy differences between the filters are smaller. However, the *relative* improvement in accuracy from using the KalmanQ filter remains sizable.

Computing time

KalmanQ, the particle filters with 100,000 and 500,000 particles, and KalmanL require 0.03, 14.69, 81.21 and 0.01 seconds, respectively, to filter simulated series of T=100 periods generated by the RBC model, on a desktop computer with a 64-bit operating system and a 3.4 Ghz processor. For a series of T=500 periods, the corresponding computing times are 0.12, 73.72, 401.58 and 0.04 seconds, respectively. Thus, the KalmanQ filter is about 500 (3000) times faster than the particle filter with 100,000 (500,000) particles.

For a sufficiently large number of particles, the particle filter is (asymptotically) an *exact* algorithm for computing the conditional expectation of the state vector. However, the experiments in Table 2 suggest that a very large number of particles (above 500,000) is needed to outperform KalmanQ; the computational cost of using such a large number of particles would be substantial.

Evaluating the QML parameter estimates

For 20 simulations runs of the 'big shocks' and 'small shocks' model variants, I computed quasi-maximum likelihood (QML) estimates of the risk aversion coefficient (σ), the labor supply elasticity (η), the autocorrelations of the forcing variables $(\rho_\theta, \rho_\lambda)$ and the standard deviations of the innovations to the forcing variables $(\sigma_\theta, \sigma_\lambda)$. Table 3 reports the mean and median parameter estimates, and the standard deviation of the estimates, across the sample of 20 estimates per model variant. The parameters are tightly estimated; mean and median parameter estimates are close to the true parameter values.[10]

3.2. State equations with randomly drawn coefficients

Many other Monte Carlo experiments confirmed that the KalmanQ filter is competitive with the particle filter, in terms of accuracy of the estimated state variables. Tables 4 and 5 consider model variants in which the coefficients of the state equation (4) are drawn randomly from normal distributions. In both Tables, Panel (a) pertains to models with n=20 variables, while Panel (b) assumes n=7 variables; I refer to the models in Panels (a) and (b) as 'large models', and as 'small models', respectively. In both set-ups, m=7 independent exogenous innovations, and n_Y=4 observables are assumed. In all variants, the standard deviations of the (independent) exogenous innovations and of measurement errors are set at 1%.[11] The elements of F_0 are independent draws from $N(0,1)$ that are scaled by a common factor so that the largest element of F_0 is $(0.01)^2$ in absolute value. The elements of F_1 are independent draws from $N(0,1)$ that are scaled by a common factor so that the largest eigenvalue of F_1 has an absolute value of 0.99. The elements of F_2 are independent draws from $N(0,1)$. In one set of simulations, referred to as 'strong curvature' simulations, all elements of F_{11}, F_{12}, F_{22} are independent draws from $N(0,1)$; in another set of simulations with 'weak curvature', the elements of F_{11}, F_{12}, F_{22} are independent draws from $N(0,(0.01)^2)$, so that curvature is much smaller, on average. For both the 'large' and 'small' model variants, 50 random 'strong curvature' coefficient sets, and 50 random 'weak curvature' coefficient sets were drawn. For each of the 200 random sets of coefficient, the model was simulated over T=100 periods.

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[10] A more detailed evaluation of the small sample properties of the QML estimator is left for future research.

[11] $\Sigma_\varepsilon=(.01)^2 I_m; \Sigma_\psi=(.01)^2 I_{n_y}$. As before, the parameter ξ that scales the size of the shocks is normalized as ξ=1.

Table 4 reports averaged standard deviations of model variables (elements of ω_t) across the draws for the 'large' and 'small' model variants (see Panels (a) and (b), respectively) with 'strong curvature' (Col. (1)) and with 'weak curvature' (Col. (2)).[12] The 'strong curvature' model variants generate much greater average standard deviations (above 240%) than the 'weak curvature' variants (about 10%). In 'strong curvature' model variants, average predicted volatility is several time larger than in corresponding linear structures (in which $F_0, F_2, F_{11}, F_{12}, F_{22}$ are set to zero). By contrast, in the 'weak curvature' variants, volatility is only slightly higher than in corresponding linear structures.

Table 5 evaluates the accuracy of the KalmanQ, PF(100,000) and KalmanL filters, for each of the four model classes (large/small models with strong/weak curvature). For each model class, the KalmanQ filter generates the lowest average RMSEs and the lowest maximum errors. For example, for the 'large models', the average RMSEs of KalmanQ, PF(100,000) and KalmanL are 0.155, 31.243 and 2.978, respectively, under 'strong curvature', and 0.031, 0.085 and 0.052, respectively under 'weak curvature' (see Panel (a)).

4. Conclusion

This paper has developed a novel approach for the estimation of latent state variables in DSGE models that are solved using a second-order accurate approximation. By contrast to particle filters, no stochastic simulations are needed for the filter here--the present method is thus much faster than particle filters. In Monte Carlo experiments, the filter here generates more accurate estimates of latent state variables than the standard particle filter, especially when the model has strong curvature or when shocks are large. The present filter is also more accurate than a Kalman filter that treats the linearized model as the true DGP. Due to its high speed, the filter presented here is suited for the estimation of model *parameters*; a quasi-maximum likelihood procedure can be used for that purpose.

[12]For each set of coefficients, the standard deviation of each variable was computed; then, standard deviations were averaged across variables and coefficient draws, for each model class (large/small models with strong/weak curvature).

APPENDIX: Computing moments of the state vector (for filter formula)

The **unconditional mean and variance** of the state vector Z_{t+1} of the augmented state equation (5) are given by: $E(Z_{t+1})=(I-G_1)^{-1}G_0$ and $V(Z_{t+1})=G_1V(Z_{t+1})G_1'+V(u_{t+1})$, respectively. Stationarity of Z_{t+1} (which holds under the assumption that all eigenvalues of F_1 are strictly inside the unit circle) implies $E(Z_{t+1})=E(Z_t)$, $V(Z_{t+1})=V(Z_t)$. Once $V(u_{t+1})$ has been determined, $V(Z_{t+1})$ can efficiently be computed using a doubling algorithm. Note that $\omega_t^{(1)}=\sum_{i=0}^{\infty}(F_1)^iF_2\varepsilon_{t-i}$ and recall that

$$u_{t+1}\equiv G_2\varepsilon_{t+1}+G_{12}\omega_t^{(1)}\otimes\varepsilon_{t+1}+G_{22}[P(\varepsilon_{t+1})-E(P(\varepsilon_{t+1}))]. \qquad (A.1)$$

$E(\omega_t^{(1)})=0$, $E(\omega_t^{(1)}\otimes\varepsilon_{t+1})=0$, $E((\omega_t^{(1)}\otimes\varepsilon_{t+1})\varepsilon_{t+1}')=0$, $E((\omega_t^{(1)}\otimes\varepsilon_{t+1})P(\varepsilon_{t+1})')=0$ hold as ε_{t+1} has mean zero and is serially independent. Hence, the covariances between the first and second right-hand side (rhs) terms in (A.1), and between the second and third rhs terms are zero. Normality of ε_{t+1} implies that the unconditional mean of all third order products of elements of ε_{t+1} is zero (Isserlis' theorem): $E\varepsilon_{t+1}^i\varepsilon_{t+1}^j\varepsilon_t^k=0$ for all *i,j,k*=1,..,m, where ε_{t+1}^h is the *h*-th element of ε_{t+1}. Thus the covariance between the first and third rhs terms in (A.1) too is zero. Note that $V(\omega_t^{(1)}\otimes\varepsilon_{t+1})=V(\omega_t^{(1)})\otimes\Sigma_\varepsilon$, with $V(\omega_t^{(1)})=F_1V(\omega_t^{(1)})F_1'+F_2\Sigma_\varepsilon F_2'$. Thus,

$$V(u_{t+1})=G_2\Sigma_\varepsilon G_2'+G_{12}(V(\omega_t^{(1)})\otimes\Sigma_\varepsilon)G_{12}'+G_{22}V(P(\varepsilon_{t+1}))G_{22}'.$$

$V(P(\varepsilon_{t+1}))$, the variance of

$$P(\varepsilon_{t+1})\equiv(\varepsilon_{t+1}^1\varepsilon_{t+1}^1,\varepsilon_{t+1}^1\varepsilon_{t+1}^2,...,\varepsilon_{t+1}^1\varepsilon_{t+1}^m,\varepsilon_{t+1}^2\varepsilon_{t+1}^2,...,\varepsilon_{t+1}^2\varepsilon_{t+1}^m,...,\varepsilon_{t+1}^{m-1}\varepsilon_{t+1}^{m-1},\varepsilon_{t+1}^{m-1}\varepsilon_{t+1}^m,\varepsilon_{t+1}^m\varepsilon_{t+1}^m),$$

can be computed as follows. For i=1,..,m, let e_i be a column vector whose *i*-th element is 1, while all other elements are zero. Thus $\varepsilon_{t+1}^i=\varepsilon_{t+1}'e_i$ and $\varepsilon_{t+1}^j=\varepsilon_{t+1}'e_j$. Hence, $\varepsilon_{t+1}^i\varepsilon_{t+1}^j=\varepsilon_{t+1}'\Lambda_{ij}\varepsilon_{t+1}$, with $\Lambda_{ij}\equiv\frac{1}{2}(e_ie_j'+e_je_i')$. Note that $E(\varepsilon_{t+1}'\Lambda_{ij}\varepsilon_{t+1})=trace(\Lambda_{ij}\Sigma_\varepsilon)$, $Var(\varepsilon_{t+1}'\Lambda_{ij}\varepsilon_{t+1})=2\cdot trace(\Lambda_{ij}\Sigma_\varepsilon\Lambda_{ij}\Sigma_\varepsilon)$ and $Cov(\varepsilon_{t+1}'\Lambda_{ij}\varepsilon_{t+1},\varepsilon_{t+1}'\Lambda_{kl}\varepsilon_{t+1})=2\cdot trace(\Lambda_{ij}\Sigma_\varepsilon\Lambda_{kl}\Sigma_\varepsilon)$ (normality of ε_{t+1} is needed for the second and third formulae); see Magnus (1978). These formulae allow to compute the variance of elements of the vector $P(\varepsilon_{t+1})$ and the covariance between any two elements of that vector, which pins down all elements of the matrix $V(P(\varepsilon_{t+1}))$.

Conditional variance of state-form disturbance

To derive the formula for the conditional variance of u_{t+1} ((10) in main text) these facts are used:

(i) $E((\omega_t^{(1)}\otimes\varepsilon_{t+1})\varepsilon_{t+1}'|\Upsilon^t)=\omega_{t,t}^{(1)}\otimes\Sigma_\varepsilon$, with $\omega_{t,t}^{(1)}\equiv E(\omega_t^{(1)}|\Upsilon^t)$.

(ii) $E((\omega_t^{(1)}\otimes\varepsilon_{t+1})(\omega_t^{(1)}\otimes\varepsilon_{t+1})'|\Upsilon^t)=E((\omega_t^{(1)}\omega_t^{(1)\prime})\otimes(\varepsilon_{t+1}\varepsilon_{t+1}')|\Upsilon^t)=E((\omega_t^{(1)}\omega_t^{(1)\prime})|\Upsilon^t)\otimes\Sigma_\varepsilon=(V_{t,t}^{\omega^{(1)}}+\omega_{t,t}^{(1)}\omega_{t,t}^{(1)\prime})\otimes\Sigma_\varepsilon$.

(Note that $V_{t,t}^{\omega^{(1)}}=E(\omega_t^{(1)}\omega_t^{(1)\prime}|\Upsilon^t)-E(\omega_t^{(1)}|\Upsilon^t)E(\omega_t^{(1)}|\Upsilon^t)'=E(\omega_t^{(1)}\omega_t^{(1)\prime}|\Upsilon^t)-\omega_{t,t}^{(1)}\omega_{t,t}^{(1)\prime}$.)

(iii) $E(P(\varepsilon_{t+1})\varepsilon_{t+1}'|\Upsilon^t)=0$, $E(P(\varepsilon_{t+1})(\omega_t^{(1)}\otimes\varepsilon_{t+1})'|\Upsilon^t)=0$ (due to Isserlis' theorem). Thus, the conditional covariance between the 1st and 3rd rhs terms in (A.1) and between th 2nd and 3rd rhs terms is zero.

References

An, S. and Schorfheide, 2007. Bayesian Analysis of DSGE models. Econometric Reviews 26, 113–172.

Andreasen, M., Fernández-Villaverde, J. and J. Rubio-Ramírez, 2013. The Pruned State-Space System for Non-Linear DSGE Models: Theory and Empirical Applications. NBER WP 18983.

Aruoba, S., L. Bocola and F. Schorfheide, 2012. A New Class of Nonlinear Time Series Models for the Evaluation of DSGE Models. Working Paper, U MD and U Penn.

Collard, F., Juillard, M., 2001. Perturbation methods for rational expectations models. Working Paper, GREMAQ and CEPREMAP.

Del Negro, M. and F. Schorfheide, 2011. Bayesian Macroeconometrics, in: The Oxford Handbook of Bayesian Econometrics (J. Geweke, G. Koop and H. van Dijk, eds.), Oxford University Press, 293-389.

Fernández-Villaverde, J. and J. Rubio-Ramírez, 2007. Estimating Macroeconomic Models: a Likelihood Approach. Review of Economic Studies 74, 1059–1087.

Hamilton, J., 1994. Time Series Analysis. Princeton University Press.

Harvey, A., 1989. Forecasting—Structural Time Series Models and the Kalman Filter. Cambridge University Press.

Jin, H., Judd, K., 2004. Applying PertSolv to Complete Market RBC models. Working paper, Stanford University.

Kim, J., Kim, S., Schaumburg, E. and C. Sims, 2008. Calculating and Using Second-Order Accurate Solutions of Discrete-Time Dynamic Equilibrium Models, Journal of Economic Dynamics and Control 32, 3397-3414.

Kollmann, R., 1996. Incomplete Asset Markets and the Cross-Country Consumption Correlation Puzzle, Journal of Economic Dynamics and Control 20, 945-962.

Kollmann, R., 2002. Monetary Policy Rules in the Open Economy: Effects of Welfare and Business Cycles, Journal of Monetary Economics 49, 989-1015.

Kollmann, R., 2005. Solving Non-Linear Rational Expectations Models: Approximations based on Taylor Expansions, Working Paper, University of Paris XII.

Kollmann, R., H.S. Kim and J. Kim, 2011. Solving the Multi-Country Real Business Cycle Model Using a Perturbation Method, Journal of Economic Dynamics and Control 35, 203-206.

Lombardo, G. and A. Sutherland, 2007. Computing Second-Order Accurate Solutions for Rational Expectation Models Using Linear Solution Methods. Journal of Economic Dynamics and Control 31, 515-530.

Magnus, J., 1978. The Moments of Products of Quadratic Forms in Normal Variables. Statistica Neerlandica 32, 201-210.

Schmitt-Grohé, S. and M. Uribe, 2004. Solving Dynamic General Equilibrium Models Using a Second-Order Approximation to the Policy Function. Journal of Economic Dynamics and Control 28, 755 – 775.

Sims, C., 2000. Second Order Accurate Solution of Discrete Time Dynamic Equilibrium Models. Working Paper, Economics Department, Princeton University

Smets, F. and R. Wouters, 2007. Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach. American Economic Review 97, 586-660.

Table 1. RBC model: predicted standard deviations

	Y	C	I	K	N	θ	λ
	(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*
(a) Model variant with big shocks (σ_θ=0.20, σ_λ=0.01)							
(a.1) Non-HP filtered variables							
Second-order model approximation							
Both shocks	1.757	0.300	5.366	3.400	2.609	1.418	0.071
Just θ shock	0.492	0.264	1.387	0.959	1.761	1.418	0.000
Just λ shock	1.558	0.133	4.799	3.096	1.762	0.000	0.071
Linearized model							
Both shocks	0.817	0.276	3.269	2.364	1.862	1.418	0.071
Just θ shock	0.469	0.264	1.285	0.929	1.751	1.418	0.000
Just λ shock	0.669	0.083	3.006	2.174	0.634	0.000	0.071
(a.2) HP filtered variables							
Second-order model approximation							
Both shocks	0.469	0.053	1.962	0.115	0.688	0.259	0.013
Just θ shock	0.124	0.037	0.483	0.038	0.212	0.259	0.000
Just λ shock	0.420	0.034	1.706	0.104	0.608	0.000	0.013
Linearized model							
Both shocks	0.229	0.041	1.059	0.095	0.350	0.259	0.013
Just θ shock	0.118	0.037	0.416	0.037	0.205	0.259	0.000
Just λ shock	0.196	0.016	0.974	0.087	0.284	0.000	0.013
(b) Model variant with small shocks (σ_θ=0.01, σ_λ=0.0005)							
(b.1) Non-HP filtered variables							
Second-order model approximation							
Both shocks	0.041	0.014	0.164	0.118	0.093	0.071	0.004
Just θ shock	0.023	0.013	0.064	0.046	0.088	0.071	0.000
Just λ shock	0.034	0.004	0.151	0.109	0.032	0.000	0.004
Linearized model							
Both shocks	0.041	0.014	0.163	0.118	0.093	0.071	0.004
Just θ shock	0.023	0.013	0.064	0.046	0.088	0.071	0.000
Just λ shock	0.033	0.004	0.150	0.109	0.032	0.000	0.004
(b.2) HP filtered variables							
Second-order model approximation							
Both shocks	0.011	0.002	0.053	0.005	0.018	0.013	0.001
Just θ shock	0.006	0.002	0.021	0.002	0.010	0.013	0.000
Just λ shock	0.010	0.001	0.049	0.004	0.014	0.000	0.001
Linearized model							
Both shocks	0.011	0.002	0.053	0.005	0.018	0.013	0.001
Just θ shock	0.006	0.002	0.021	0.002	0.010	0.013	0.000
Just λ shock	0.010	0.001	0.049	0.004	0.014	0.000	0.001

Note: Standard deviations (std.) of logged variables (listed above Cols. (1)-(7)), without measurement error, are shown for the RBC model in Section 3.1. The std. were computed using the formulae in the Appendix. Std. are not reported in %.
Panel (a) ('Big shocks') assumes std. of innovations to TFP (θ) and the taste parameter λ of 20% and 1%, respectively. Panel (b) ('Small shocks') sets these std. at 1% and 0.05%, respectively. Rows labeled 'Both shocks'; 'Just θ shock'; and 'Just λ shock' show moments predicted with simultaneous two shocks; with just the TFP shock; and with just the taste shock, respectively. Panels (a.1) and (b.1) report moments of Non-HP filtered variables; Panels (a.2) and (b.2) pertain to HP filtered variables (smoothing parameter: 1600). *Y*: GDP; *C*: consumption; *I*: gross investment; *K*: capital stock; *N*: hours worked.

Table 2. RBC model: accuracy of filters

	All variables	Y	C	I	K	N	θ	λ
	(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*
(a) Model variant with big shocks (σ_θ=0.20, σ_λ=0.01)								
(a.1) 50 simulation runs with T=500 periods								
Average RMSEs								
KalmanQ	0.157	0.038	0.006	0.040	0.387	0.039	0.127	0.022
PF (100,000)	1.189	1.022	0.188	0.907	2.193	1.447	0.873	0.070
KalmanL	1.939	1.659	0.184	1.605	4.183	0.770	1.143	0.082
Maximum Errors								
KalmanQ	3.448	0.209	0.024	0.164	3.448	0.166	1.198	0.236
PF (100,000)	20.188	15.750	1.927	10.451	20.188	19.842	4.236	0.563
KalmanL	14.824	14.824	1.275	7.849	13.071	3.820	12.039	0.568
Fraction of runs in which RMSE is lower for KalmanQ than for other filters								
PF (100,000)	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.98
KalmanL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.96
Fraction of runs in which maximum error is lower for KalmanQ than for other filters								
PF (100,000)	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.98
KalmanL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.94
. . . . .								
(a.2) 50 simulation runs with T=100 periods								
Average RMSEs								
KalmanQ	0.176	0.039	0.006	0.039	0.435	0.039	0.141	0.023
PF (100,000)	0.828	0.755	0.121	0.744	1.370	0.892	0.679	0.049
PF (500,000)	0.597	0.527	0.108	0.499	0.892	0.695	0.658	0.030
KalmanL	1.917	1.448	0.176	2.063	3.955	0.973	0.975	0.067
Maximum Errors								
KalmanQ	2.855	0.160	0.023	0.134	2.855	0.163	0.901	0.226
PF (100,000)	13.298	10.615	1.364	9.944	13.190	13.298	2.846	0.414
PF (500,000)	8.018	4.548	0.560	8.018	5.680	4.577	3.425	0.235
KalmanL	11.866	9.826	0.655	8.399	11.866	3.484	5.093	0.332
Fraction of runs in which RMSE is lower for KalmanQ than for other filters								
PF (100,000)	1.00	1.00	1.00	1.00	0.84	1.00	1.00	0.64
PF (500,000)	1.00	1.00	1.00	1.00	0.84	1.00	1.00	0.68
KalmanL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.98
Fraction of runs in which maximum error is lower for KalmanQ than for other filters								
PF (100,000)	0.96	1.00	1.00	1.00	0.84	1.00	1.00	0.80
PF (500,000)	1.00	1.00	1.00	1.00	0.80	1.00	1.00	0.76
KalmanL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.96
. . . . .								

Table 2.—ctd.

	All variables	Y	C	I	K	N	θ	λ
	(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*
(b) Model variant with small shocks (σ_θ=0.01, σ_λ=0.0005)								
(b.1) 50 simulation runs with T=500 periods								
Average RMSEs								
KalmanQ	0.0022	0.0007	0.0003	0.0019	0.0044	0.0019	0.0022	0.0001
PF (100,000)	0.0222	0.0068	0.0057	0.0109	0.0180	0.0405	0.0343	0.0007
KalmanL	0.0411	0.0163	0.0059	0.0583	0.0585	0.0397	0.0312	0.0018
Maximum Errors								
KalmanQ	0.0475	0.0041	0.0011	0.0074	0.0475	0.0081	0.0163	0.0012
PF (100,000)	0.2437	0.0434	0.0330	0.0723	0.2438	0.2295	0.1939	0.0054
KalmanL	0.2293	0.0753	0.0248	0.2146	0.2294	0.1837	0.1512	0.0074
Fraction of runs in which RMSE is lower for KalmanQ than for other filters								
PF (100,000)	1.00	1.00	1.00	1.00	0.98	1.00	1.00	1.00
KalmanL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Fraction of runs in which maximum error is lower for KalmanQ than for other filters								
PF (100,000)	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
KalmanL	0.98	1.00	1.00	0.98	0.92	1.00	0.98	0.94
. . . . .								
(b.2) 50 simulation runs with T=100 periods								
Average RMSEs								
KalmanQ	0.0042	0.0007	0.0003	0.0019	0.0099	0.0019	0.0035	0.0002
PF (100,000)	0.0244	0.0060	0.0051	0.0096	0.0368	0.0362	0.0313	0.0009
PF (500,000)	0.0223	0.0046	0.0039	0.0074	0.0398	0.0278	0.0271	0.0010
KalmanL	0.0508	0.0224	0.0078	0.0819	0.0716	0.0466	0.0357	0.0019
Maximum Errors								
KalmanQ	0.0496	0.0033	0.0009	0.0073	0.0496	0.0068	0.0171	0.0012
PF (100,000)	0.1866	0.0319	0.0261	0.0551	0.1866	0.1848	0.1473	0.0046
PF (500,000)	0.2591	0.0224	0.0171	0.0402	0.2591	0.1189	0.1143	0.0065
KalmanL	0.3001	0.0918	0.0263	0.3001	0.2900	0.1846	0.1424	0.0083
Fraction of runs in which RMSE is lower for KalmanQ than for other filters								
PF (100,000)	1.00	1.00	1.00	1.00	0.94	1.00	1.00	0.98
PF (500,000)	1.00	1.00	1.00	1.00	0.90	1.00	1.00	0.92
KalmanL	1.00	1.00	1.00	1.00	0.94	1.00	1.00	0.90
Fraction of runs in which maximum error is lower for KalmanQ than for other filters								
PF (100,000)	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
PF (500,000)	1.00	1.00	1.00	1.00	0.92	1.00	1.00	0.96
KalmanL	0.98	1.00	1.00	1.00	0.94	1.00	1.00	0.90

Note: The Table reports average Root Mean Squared Errors (RMSEs) and maximum estimation errors of estimated latent states, across simulation runs (errors are *not* expressed in %), for different variants of the RBC model; accuracy is reported across all logged variables (Col. (1) 'All variables'), and separately for each logged variable (see Cols. (2)-(8) labeled 'Y',…,'λ'); see Sect. 3.1. for further information. KalmanQ: the filter for pruned second-order models developed in this paper; PF(p): standard particle filter with p particles: KalmanL: standard Kalman filter that assumes that the linearized DSGE model is the true DGP. Panel (a) ('Big shocks') assumes std. of TFP and taste shock innovations of 20% and 1%, respectively; panel (b) ('Small shocks') sets these standard deviations at 1% and 0.05%, respectively. Y: GDP; C: consumption; I: gross investment; K: capital stock; N: hours worked; θ: TFP; λ: taste shock.

Table 3. RBC model: distribution of quasi-maximum likelihood estimates of model parameters based on the KalmanQ filter

	σ	η	ρ_θ	ρ_λ	σ_θ	σ_λ
	(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*
(a) Model variant with big shocks (σ_θ=0.20, σ_λ=0.01)						
True parameter value	10.00	4.00	0.99	0.99	20.00%	1.00%
Mean estimate	10.19	4.30	0.99	0.98	19.41%	0.95%
Median estimate	10.01	4.11	0.99	0.98	19.04%	0.89%
Standard dev. of estimates	0.96	1.51	0.003	0.005	2.27%	0.17%
(b) Model variant with small shocks (σ_θ=0.01, σ_λ=0.0005)						
True parameter value	10.00	4.00	0.99	0.99	1.00%	0.05%
Mean estimate	10.21	5.55	0.99	0.71	0.77%	0.18%
Median estimate	10.24	3.40	0.99	0.71	0.76%	0.14%
Standard dev. of estimates	1.17	4.06	0.004	0.20	0.12%	0.11%

Note: The Table reports true parameter values, as well as the mean, median and standard deviation of quasi-maximum likelihood (QML) estimates of structural model parameters based on the KalmanQ filter (developed in this paper), obtained for 20 different simulation runs with T=100 periods. Results are shown for the 'big shocks' variant of the RBC model (Panel (a)), and for the 'small shocks' variant (Panel (b)). σ: risk aversion coefficient (Col. (1)); η: Frisch labor supply elasticity (Col. (2)); ρ_θ [ρ_λ]: autocorrelation of TFP [taste shock] (Cols. (3)-(4)); σ_θ [σ_λ]: standard deviation of TFP [taste shock] innovation (Cols. (5)-(6)).

Table 4. Models with randomly drawn coefficients: standard deviations of variables

	strong curvature	**weak curvature**
	(1)	*(2)*
(a) Large models (*n*=20)		
(a.1) Non-HP filtered variables		
Second-order model	3.061	0.119
Linearized model	0.106	0.106
(a.2) HP filtered variables		
Second-order model	1.089	0.094
Linearized model	0.092	0.092
(b) Small models (*n*=7)		
(b.1) Non-HP filtered variables		
Second-order model	2.446	0.147
Linearized model	0.144	0.144
(b.2) HP filtered variables		
Second-order model	0.678	0.116
Linearized model	0.115	0.115

Note: Panel (a) considers models with *n*=20 variables ('large models'), while Panel (b) assumes *n*=7 variables ('small models'). The coefficients of the state equation (4) are drawn randomly. In 'strong curvature' ['weak curvature'] experiments—see Col. (1) [Col. (2)]--all elements of the matrices of curvature coefficients F_{11}, F_{12}, F_{22} are independent draws from $N(0,1)$ [$N(0,(0.01)^2$]. For both the 'large' and 'small' model variants, 50 random 'strong curvature' coefficient sets, and 50 random 'strong curvature' coefficient sets were drawn. For each of the resulting 200 sets of coefficients, the model was simulated over T=100 periods. For each set of coefficients, the standard deviation of each variable without measurement error was computed; then, standard deviations were averaged across all variables and coefficient draws, for each of the four model class (large/small models with strong/weak curvature). (The averaged standard deviations are *not* reported in %.)

Table 5. Models with randomly drawn coefficients: accuracy of filters

	strong curvature	weak curvature
	(1)	*(2)*
(a) Large models (*n*=20)		
Average RMSEs		
KalmanQ	0.155	0.0308
PF (100,000)	31.243	0.0851
KalmanL	2.978	0.0523
Maximum Errors		
KalmanQ	4.784	0.2789
PF (100,000)	4440.00	9.3808
KalmanL	149.441	1.2315
Fraction of runs in which RMSE is lower for KalmanQ than for other filters		
PF (100,000)	1.00	0.98
KalmanL	1.00	0.94
Fraction of runs in which maximum error is lower for KalmanQ than for other filters		
PF (100,000)	1.00	0.76
KalmanL	0.98	0.74
(b) Small models (*n*=7)		
Average RMSEs		
KalmanQ	0.035	0.0184
PF (100,000)	38.082	0.0186
KalmanL	1.651	0.0409
Maximum Errors		
KalmanQ	1.12	0.0943
PF (100,000)	3263.28	0.0954
KalmanL	37.38	0.1265
Fraction of runs in which RMSE is lower for KalmanQ than for other filters		
PF (100,000)	0.76	0.76
KalmanL	1.00	0.96
Fraction of runs in which maximum error is lower for KalmanQ than for other filters		
PF (100,000)	0.82	0.500
KalmanL	1.00	0.660

Note: The Table reports average Root Mean Squared Errors (RMSE) and maximum estimation errors of estimated latent variables produced by three filters, across simulation runs, for versions of state equation (4) with randomly drawn coefficients. Panel (a) ('Large models') assumes n=20 variables; Panel (b) ('Small model') assumes n=7 variables. In 'strong curvature' ['weak curvature'] experiments (see Column 1 [2]), all elements of the matrices of curvature coefficients F_{11}, F_{12}, F_{22} are independent draws from $N(0,1)$ [$N(0,(0.01)^2$]. For both the 'large' and 'small' model variants, 50 random 'strong curvature' coefficient sets, and 50 random 'weak curvature' coefficient sets were drawn. For each of the resulting 200 sets of coefficient, the model was simulated over T=100 periods. For each set of coefficients, the RMSE and the maximal error was computed, for each of the '*n*' estimated latent variables; then, RMSEs were averaged across variables and coefficient draws, for each of the four model classes (large/small models with strong/weak curvature); maximum errors were likewise determined across all *n* variables, and across all draws, for each of the four model classes. KalmanQ: the filter for pruned second-order models developed in this paper; PF(*p*): standard particle filter with *p* particles: KalmanL: standard Kalman filter that assumes that the linearized model is the true DGP.