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Duopolistic Competition and Monetary Policy

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Keywords

Duopoly, monetary policy, strategic complementarities, New Keynesian model

JEL Classification

D43, E32, E52

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Duopolistic Competition and Monetary Policy

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Abstract

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1 Introduction

The model of monopolistic competition developed by Dixit and Stiglitz (1977) provides a simple way to analyze firms' price-setting behavior, which in turn enables us to study the effects of monetary policy in tractable macroeconomic models (see also Blanchard and Kiyotaki 1987). However, the model does not incorporate the strategic behaviors of oligopolistic firms. Firms in the Dixit–Stiglitz model do not need to consider how their price will influence their rival firm's price and then how a rival firm's price influences their optimal price. This setup has a ground if goods produced by firms are perfectly differentiated, consumers have strong preferences for diversity, and the number of rival firms is sufficiently large. However, in reality, the degree of differentiation of goods is not necessarily high. An identical product is sold at different retailers and almost identical products are produced by different manufacturers. Although preferences for diversity make demand for products always positive in the Dixit–Stiglitz model—firms can sell their goods even if the price is extremely high—, consumers in reality never purchase all the products that exist in the market (e.g., they never purchase all the automobiles even if their characteristics are different). Firms compete with a finite number of rival firms, monitoring a limited number of rival firms' (particularly, a market leader's) pricing closely. These observations motivate us to consider the role of strategic pricing of oligopolistic firms in macroeconomic models.

In this study, we aim to investigate how implications for monetary policy change when we incorporate strategic pricing behaviors of oligopolistic firms into the model. Specifically, we construct a macroeconomic model with price stickiness by incorporating duopolistic competition as described by Hotelling (1929) while maintaining model simplicity. In each product line, two firms located in geographically separated places exist and consumers are distributed evenly between the two firms.[1] The firms optimally determine their prices to maximize their present-valued profits under Calvo-type price stickiness. Importantly, this duopoly model entails the feature of Bertrand competition, and thus, firms are price setters. Unlike a Cournot model, in which firms choose a quantity, this feature allows us to incorporate sticky prices as in a standard New Keynesian model (e.g., Woodford 2003, and Gali 2015) while still taking account of strategic pricing behaviors. In this setting, pricing is a strategic complementarity. An increase (a decrease) in a rival firm's price induces a firm to raise (lower) its price. To the best of our knowledge, this is the first study to consider a macroeconomic model characterized by Hotelling's (1929) duopolistic competition and price stickiness.

The model we propose is simple. Although it may ignore many important features,

[1]The Hotelling (1929) part of the model in this paper is based on that presented in Armstrong (2006). Accordingly, we replicate Armstrong's (2006) results in the model without sticky prices. It should be noted that we do not intend to claim that our duopolistic competition model fits the actual economy better than models based on monopolistic competition. Which approximates the actual market structure better depends on industries/products/regions.

model tractability enables us to extend the model in several directions. As an illustration, we extend the model by allowing heterogeneous transport costs. When some consumers have greater transport costs for shopping than others and firms cannot observe consumers' type, a mixed strategy equilibrium arises. This implies that the higher and lower prices constitute the regular and sale prices, respectively.

Our main findings are as follows. First, the steady-state price level (i.e., price markup) under sticky prices is different from that under flexible prices. In the model, the optimal reset price positively depends on the price set by the rival firm in the previous period. In addition, one's own price influences both the price set by the rival firm in the following periods as well as the firm's own future profits. This, in turn, influences the optimal reset price in the current period. Because of this dynamic (intertemporal) strategic complementarity effect, the steady-state price level increases as price stickiness increases, unless price is extremely sticky.[2] By contrast, the steady-state price level is independent of price stickiness in a standard New Keynesian model. Furthermore, it is shown that the difference of the steady-state price level under sticky prices and flexible prices depends on consumers' transport costs in a non-monotonic manner. There is a finite level of transport costs that maximize the ratio of the steady-state price level under sticky prices to that under flexible prices.

Second, consumers' transport costs influence not only demand elasticity but also inflation dynamics. This finding differs from that based on the standard New Keynesian model with monopolistic competition, whereby the elasticity of substitution does not influence inflation dynamics.

Third, the real effect of monetary policy is more pronounced in our model, compared with a model based on monopolistic competition. Because of the strategic complementarity, the expectation that the rival firm may not revise its price today under price stickiness discourages a firm from revising its price aggressively today, which decreases the nominal effect of monetary policy (i.e., price response) and increases the real effect of monetary policy (i.e., output response). However, the difference of the impulse responses between duopolistic and monopolistic competition models is not large. Namely, the output response implied by the former model is approximately one third larger than that implied by the latter model.

Finally, we find that the real effect of monetary policy decreases considerably under duopoly if there is a mixed strategy equilibrium (i.e., if sales exist). In the model, we assume that setting the lower price (i.e., the sale price) is flexible while setting the higher price (i.e., the regular price) is sticky. We find that sales exemplify the strategic complementarity rather than the strategic substitutability. In other words, an increase in the frequency of sales leads to a higher profit from choosing to hold a sale, instead of selling at a regular price. Because of the strategic complementarity, the fact that the sale price is flexible induces firms to change the regular price aggressively when there is a money supply shock. This increases the nominal

[2] See Jun and Vives (2004) and Mongey (2017) for the terminology of *static* and *dynamic* (intertemporal) strategic effect.

effect of monetary policy and decreases the real effect of monetary policy. Quantitatively, the existence of sales is important. The real effect of monetary policy diminishes to almost one tenth, compared with the scenario where transport costs are homogeneous and no sales exist in equilibrium.

The three studies most similar to this one are Faia (2012), Mongey (2017), and Wang and Werning (2020), who investigate monetary policy under oligopolistic competition. Especially, Mongey (2017) constructs a model of duopolistic competition. Namely, two firms exist in each sector and they compete strategically by optimally changing prices in the context of a menu cost model of price adjustment. Mongey (2017) shows that a dynamic strategic complementarity decreases responses of price adjustment and increases the real effect of monetary policy, which is consistent with the findings of our study. A difference between these three studies and the present study is that the oligopolistic competition models applied in the previous studies maintain the key framework of monopolistic competition. Specifically, Faia (2012) and Wang and Werning (2020) modify the Dixit–Stiglitz model so that the number of firms is finite, whereas Mongey (2017) introduces not only cross- but also within-sector elasticity of substitution to incorporate duopoly. By contrast, our model is based on Hotelling's (1929) location model. This leads to different implications for monetary policy and competition policy.[3]

This work is also related to the one that investigates the role of temporary sales in considering the effects of monetary policy. It is well known that temporary sales are held frequently and that price stickiness decreases considerably once we include them in the measurement of price stickiness (see, e.g., Bils and Klenow 2004 for the United States and Sudo, Ueda, and Watanabe 2014 for Japan). However, the effect of temporary sales is often ignored in macroeconomic studies, with a small number of exceptions such as Guimaraes and Sheedy (2011), Sudo et al. (2018), and Kryvtsov and Vincent (forthcoming). The most relevant study in this strand of work is Guimaraes and Sheedy (2011). These authors construct a New Keynesian model with heterogeneous elasticity of substitution and show that sales exist in equilibrium but that the real effect of monetary policy is more or less unchanged. Our result is markedly different, showing that the real effect of monetary policy decreases considerably. This difference arises because sales are a source of strategic complementarity, whereas they are a strategic substitute in the Guimaraes and Sheedy (2011) model. More fundamentally, the difference stems from a difference in a competition environment between Guimaraes and Sheedy (2011) and our study. Similar to Faia (2012) and Mongey (2017), Guimaraes and Sheedy (2011) maintain the key framework of monopolistic competition, whereas our study is based on Hotelling (1929).

Among a number of contributions to the development of New Keynesian models, our study is related to the work that stresses the importance of strategic complementarities and

[3] Nirei and Scheinkman (2020) construct a comparable model, in which competition among a finite number of firms generates inflation volatility.

real rigidity (see, e.g., Ball and Romer 1990, Kimball 1995, Woodford 2003, Christiano, Eichenbaum, and Evans 2005, Levin, Lopez-Salido, and Yun 2007, Angeletos and La'o 2009, Aoki, Ichiue, and Okuda 2019, and L'Huillier 2020). For example, Angeletos and La'o (2009) show that the combination of noisy information and strategic complementarities yields lasting effects of nominal shocks on inflation and real output. L'Huillier (2020) considers a strategic interaction between a monopolistic firm and uninformed consumers. Our study provides a new insight on the source of strategic complementarities, from the perspective of an oligopoly. Furthermore, model implications are different. These studies show that strategic complementarities increase the real effect of monetary policy, which is consistent with our finding based on the simple model. However, when the model is extended to incorporate a large heterogeneity in consumers' transport costs, the fact that sales are a strategic complementarity decreases the real effect of monetary policy rather than increases it.

This contribution builds on the extant literature studying relations between strategic pricing and sticky prices. Fershtman and Kamien (1987) consider a duopolistic competition model, as we do, under the assumption that prices do not adjust instantaneously to the price indicated by a demand function. They study how the sticky price assumption changes Nash equilibrium strategies. Their work is different from this study in that their model is based on Cournot competition, and thus, firms are not price setters and instead optimize their output. Price changes are governed by an exogenous reduced-form equation. Accordingly, the author's research contribution does not directly identify implications for the macroeconomy. For further research on this topic, see also Maskin and Tirole (1988), Slade (1999), Bhaskar (2002), Fehr and Tyran (2008), and Chen, Korpeoglu, and Spear (2017). These studies investigate whether price endogenously becomes sticky in a market where firms compete strategically.

The remainder of this paper is structured as follows. Section 2 presents a basic model and discusses its implications. Section 3 extends the basic model to incorporate unobserved heterogeneity in transport costs. Section 4 concludes.

2 Model

2.1 Setup

Firms In each product line $j \in [0, 1]$, there exist two firms A and B. They are situated at each end of the unit interval [0, 1]. To produce one unit of product, firms require one unit of labor, which costs nominal wage W_t. Firm A and B set output prices p^A and p^B, respectively. No entry or exit is considered.

Household A head of household maximizes the following preference:[4]

$$U = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\log C_t - (L_t + \tau D_t)\right],$$

where L_t represents labor supply, and aggregate consumption C_t and shopping distance D_t are given by

$$\log C = \int_0^1 \log c^j dj \tag{1}$$

$$D = \int_0^1 d^j dj. \tag{2}$$

Here, c^j and d^j represent consumption and shopping distance, respectively, for a product line $j \in [0,1]$. Parameter $\beta \in (0,1)$ is the subjective discount factor and parameter τ is the transport cost incurred per unit of distance.

The budget constraint is given by

$$M_t + B_t + P_t C_t \leq M_{t-1} + R_{t-1} B_{t-1} + W_t L_t + \Pi_t + T_t, \tag{3}$$

where M_t, B_t, P_t, R_t, Π_t, and T_t represent money supply, nominal bonds, aggregate price, nominal interest rate, dividends from firms, and lump-sum transfer, respectively. Furthermore, we assume that nominal spending must be equal to the money supply:[5]

$$P_t C_t = M_t. \tag{4}$$

The first-order conditions lead to

$$\frac{1}{C_t} = \mathbb{E}_t \left[\beta \frac{P_t}{P_{t+1}} R_t \frac{1}{C_{t+1}}\right] \tag{5}$$

$$W_t = M_t = P_t C_t. \tag{6}$$

A household is comprised of an infinite number of consumers who are uniformly located along the interval [0, 1]. A consumer located at $x \in [0,1]$ is a distance x from firm A and $1-x$ from firm B. Because of the unit elasticity in equation (1), a consumer spends M in a nominal term to purchase from either firm A or firm B. Thus, $c = M/p^i$ if the consumer buys from firm $i = A, B$, where p^i represents firm i's price. The consumer's net surplus is written as

$$u^i = \log c^i - \tau d^i = \log M - \log p^i - \tau d^i, \tag{7}$$

[4]We assume a log utility in consumption and linear disutility of labor supply as in Nakamura and Steinsson (2010) and Midrigan (2011). See also Hansen (1985) and Rogerson (1988).

[5]One may assume that the monetary authority targets a path for nominal output to ensure $P_t C_t = M_t$. As we show below, we have $M_t = W_t$, which makes money supply shocks equivalent to changes in nominal marginal costs. See Nakamura and Steinsson (2010) and Midrigan (2011).

where d^i represents the distance the consumer travels to firm i. Although we call τ the transport cost throughout the paper, this parameter also represents a consumer's choosiness, that is, how much he/she dislikes buying from his/her less preferred firm. When τ is high, the consumer is loyal to his/her preferred firm. When τ is low, the consumer cares about the prices sold in the two firms, acting as a bargain hunter.

Goods Market Clearing The goods market is cleared as $Y_t(= L_t) = C_t$.

Money Supply Money supply is exogenous and given by

$$\begin{aligned} \log(M_t/M_{t-1}) &= \varepsilon_t \\ &= \rho\varepsilon_{t-1} + \mu_t, \end{aligned} \tag{8}$$

where μ_t is an i.i.d. shock to money supply. Money supply has zero trend growth.

2.2 Steady State without Price Stickiness

Before we introduce price stickiness, we consider the equilibrium in steady state.

Firms' Pricing Denote the prices of firm A and firm B by p^A and p^B, respectively. A consumer will buy from firm A if

$$\log p^A + \tau x \leq \log p^B + \tau(1-x). \tag{9}$$

Each consumer purchases the amount of M/p^A if they purchase from firm A. Since the total number of consumers who purchase from firm A cannot exceed one or be negative, firm A's profit is written as

$$\Pi(p^A, p^B) = \begin{cases} 0 & \text{if } \frac{\log p^A - \log p^B}{\tau} \geq 1/2 \\ (p^A - W)\left(\frac{1}{2} - \frac{\log p^A - \log p^B}{\tau}\right)\frac{M}{p^A} & \text{if } -1/2 < \frac{\log p^A - \log p^B}{\tau} < 1/2 \\ (p^A - W)\frac{M}{p^A} & \text{if } \frac{\log p^A - \log p^B}{\tau} \leq -1/2. \end{cases} \tag{10}$$

Appendix A.1 shows that the best response of p^A given p^B is expressed as follows:

$$p^A(p^B) = \begin{cases} p^{A*}(p^B) & \text{if } p^B < \exp\left[\frac{1}{2}\tau + \log\{W(\tau+1)\}\right] \\ \exp(\log p^B - 1/2 \cdot \tau) & \text{if } p^B \geq \exp\left[\frac{1}{2}\tau + \log\{W(\tau+1)\}\right], \end{cases} \tag{11}$$

where $p^{A*}(p^B)$ satisfies $p^A + W\log p^A = W\left(\frac{1}{2}\tau + 1 + \log p^B\right)$. It can be proved that $p^A(p^B)$ increases as p^B increases, showing the static strategic complementarity. According to equation (10), a firm's profit is non-decreasing as the rival firm sets a higher price. Thus, a higher price set by the rival firm allows a firm to raise its price and thereby, increase its profit. Figure 1 shows the best response, where we assume $\tau = 0.25$ (whereby the reason for this assumption is explained below) and $W = 1$. This shows that the best response function has an upward slope.

The best response intersects with the 45° line once. The profit maximization with symmetric $p = p^A = p^B$ leads to the following steady-state price:

$$p = (1 + \tau/2)\,W. \tag{12}$$

Around the steady state, the slope of the best response price or the degree of static strategic complementarities equals $1/(2 + \tau/2)$. Thus, it is positive and increases as τ decreases.[6]

Welfare In equilibrium, each consumer spends M/p for consumption C and supplies labor L for the same amount. Shopping distance D equals $2\int_0^{1/2} x dx = 1/4$. Thus, household utility U can be expressed as

$$\begin{aligned} U &= \{\log(M/p) - (M/p + \tau/4)\}/(1-\beta) \\ &= -\left\{\log(1+\tau/2) + (1+\tau/2)^{-1} + \tau/4\right\}/(1-\beta). \end{aligned} \tag{13}$$

This suggests that utility decreases when τ increases. An increase in τ decreases consumption and increases disutility from going shopping. At the same time, it increases utility because the labor supply decreases. Overall, the former effect dominates the latter.

Demand Elasticity The demand elasticity near the steady state is shown to be

$$1 + 2/\tau. \tag{14}$$

When τ goes to infinity, the demand elasticity converges to one.

2.3 Pricing under Price Stickiness

We assume Calvo-type price stickiness. Both firms A and B can reset their prices with a probability of $1 - \theta \in (0, 1)$. Specifically, we introduce the following assumption. With the probability of θ, firms survive in the next period but the price is kept fixed. With the probability of $1 - \theta$, an old firm exits the market and a new firm enters in its place and sets its price. Thus, only a new firm can optimize price.[7]We limit our analysis by assuming

[6]If we assume a non-unit elasticity between different product lines (given by σ as in the Dixit–Stiglitz model), the optimal price is expressed as the lower of $(1 + \tau/2)\,W$ and $\sigma/(\sigma - 1)W$. Thus, the optimal price is unchanged as long as $\sigma/(\sigma - 1) \geq (1 + \tau/2)$.

[7]We assume a finite life span of firms because, in this type of model, firms have to optimize their reset price by taking account of the fact that their reset price today influences the optimal reset price in the future. Below, we consider the effects of this only to the extent that a firm's reset price today influences the rival firm's reset price in the following periods. In the following simulation, we find that a firm responds to its own past price to a much smaller extent than it responds to its rival firm's past price, and thus, the effect of this assumption appears to be quantitatively small.

that the Markov perfect equilibrium concept applies. Each firm's action (i.e., price setting decision) depends on a state consisting only of the following three variables: its price in the previous period, the rival firm's price in the previous period, and a shock to money supply. We exclude collusive pricing, although the folk theorem suggests that dynamic setting can generate multiple collusive equilibria.

When firm A has a chance to set its price at t, it sets $\bar{p}_t^A$ to maximize

$$\max \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \beta^k \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \left(1 - \frac{W_{t+k}}{\bar{p}_t^A}\right) \left(\frac{1}{2} - \frac{\log\bar{p}_t^A - \log p_{t+k}^B}{\tau}\right) \frac{M_{t+k}}{M_t}, \tag{15}$$

where Λ_t represents the stochastic discount factor given by C_t^{-1}. Solving this optimization problem is more complex than solving a similar problem in a standard New Keynesian model, because we have to explicitly consider the path of the prices set by the rival firm. Noting that p_{t+k}^B equals $\bar{p}_{t+k}^B$ with the probability of $1-\theta$, $\bar{p}_{t+k-1}^B$ with the probability of $\theta(1-\theta),\cdots$, $\bar{p}_t^B$ with the probability of $\theta^k(1-\theta)$, and p_{t-1}^B with the probability of θ^{k+1} for $k \geq 0$, we have

$$\begin{aligned}
&\max \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\left(1 - \frac{M_{t+k}}{\bar{p}_t^A}\right) \theta^{k+1} \left(\frac{1}{2} - \frac{\log\bar{p}_t^A - \log p_{t-1}^B}{\tau}\right)\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\left(1 - \frac{M_{t+k}}{\bar{p}_t^A}\right) \sum_{k'=0}^{k} (1-\theta)\theta^{k-k'} \left(\frac{1}{2} - \frac{\log\bar{p}_t^A - \log\bar{p}_{t+k'}^B}{\tau}\right)\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t}.
\end{aligned} \tag{16}$$

Hereafter, we assume $W_0 = M_0 = 1$ in the initial period. The first-order condition for the optimal $\bar{p}_t^A$ is given by

$$\begin{aligned}
0 =& \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(\frac{1}{\bar{p}_t^A}\right)^2 M_{t+k} \left[\theta^{k+1} \left(\frac{1}{2} - \frac{\log\bar{p}_t^A - \log p_{t-1}^B}{\tau}\right)\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(\frac{1}{\bar{p}_t^A}\right)^2 M_{t+k} \left[\sum_{k'=0}^{k} (1-\theta)\theta^{k-k'} \left(\frac{1}{2} - \frac{\log\bar{p}_t^A - \log\bar{p}_{t+k'}^B}{\tau}\right)\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(1 - \frac{M_{t+k}}{\bar{p}_t^A}\right) \left(-\frac{1}{\tau\bar{p}_t^A}\right) \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(1 - \frac{M_{t+k}}{\bar{p}_t^A}\right) \left[\sum_{k'=0}^{k} (1-\theta)\theta^{k-k'} \frac{\partial\log\bar{p}_{t+k'}^B / \partial\log\bar{p}_t^A}{\tau\bar{p}_t^A}\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t}.
\end{aligned}$$

Firm A has to take account of how its reset price at t influences the rival firm B's reset price at $t+k'$, which is given by $\partial\log\bar{p}_{t+k'}^B / \partial\log\bar{p}_t^A$. In log-linearization, let us denote $\bar{p}_t^A \equiv pM_t e^{\hat{p}_t^{A*}}$, $p_t^B \equiv pM_t e^{\hat{p}_t^B}$ as well as $\partial\log\bar{p}_{t+k}^B / \partial\log\bar{p}_t^A \equiv \Gamma^*$ for any $k \geq 1$. The coefficient Γ^* will be defined in detail later. It is independent of k because we consider a case in which the price of firm A is unchanged at $\bar{p}_t^A$). In what follows, we show the main results while relegating detailed derivations to Appendix A.2.

Steady State In the steady state, price equals

$$p = 1 + \frac{1}{2}\tau \left(1 - \frac{(1-\theta)(1+\theta-\theta^2\beta)}{1-\theta^2\beta} \theta\beta\Gamma^*\right)^{-1}. \tag{17}$$

Unless Γ^* is zero, the steady state under nominal rigidity is different from that without nominal rigidity. Firms take account of the effect of their price on the rival firm's price in the following periods. Specifically, if Γ^* is positive, there is a dynamic strategic complementarity. An increase in firm A's price increases firm B's price in the following periods. This effect increases the steady-state price level. The above equation also shows that the steady-state price level becomes identical with that in the scenario without nominal rigidity in the limit of $\theta \to 0$.

Log-linearization around the Steady State The optimal reset prices are expressed in the following forms:

$$p_t^{A*} = \Gamma \hat{p}_{t-1}^A + \Gamma^* \hat{p}_{t-1}^B + \Gamma^\varepsilon \varepsilon_t \tag{18}$$

$$p_t^{B*} = \Gamma \hat{p}_{t-1}^B + \Gamma^* \hat{p}_{t-1}^A + \Gamma^\varepsilon \varepsilon_t, \tag{19}$$

$$\partial \log \bar{p}_{t+k}^B / \partial \log \bar{p}_t^A = \partial p_{t+k}^{B*} / \partial p_t^{A*} = \Gamma^*, \tag{20}$$

where Γ, Γ^*, and Γ^ε represent coefficients to be determined to satisfy

$$\begin{aligned}
0 =& \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \log(M_{t+k}/M_t) \left(\frac{1}{2}\right) \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \, (-1) \log(M_{t+k}/M_t) \left(-\frac{1}{\tau}\right) \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \, (-1) \log(M_{t+k}/M_t)(1-\theta^{k+1}) \left(\frac{\Gamma^*}{\tau}\right) \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\theta^{k+1} \left(-\frac{p_t^{A*} - \hat{p}_{t-1}^B + \log(M_t/M_{t-1})}{\tau}\right)\right] \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\sum_{k'=0}^{k} (1-\theta)\theta^{k-k'} \left(-\frac{p_t^{A*} - p_{t+k'}^{B*} - \log(M_{t+k'}/M_t)}{\tau}\right)\right] \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \left(pp_t^{A*}\right) \left(-\frac{1}{\tau}\right) \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \left(pp_t^{A*}\right) (1-\theta^{k+1}) \left(\frac{\Gamma^*}{\tau}\right).
\end{aligned} \tag{21}$$

Note that, if nominal price is unchanged, the log-linearized price denoted by $\hat{p}$ changes by the amount of the change in the aggregate money supply times -1: $\hat{p}_t = \hat{p}_{t-1} - \varepsilon_t$.

Inflation Dynamics We define $\kappa \equiv \theta + (1-\theta)(\Gamma + \Gamma^*)$ and the inflation rate $\pi_t \equiv \log(P_t/P_{t-1}) \simeq \varepsilon_t + \hat{P}_t - \hat{P}_{t-1}$. Here, $\hat{P}_t$ represents the log-linearized aggregate price whereas $\hat{p}_t$ represents the log-linearized individual price. We can obtain the following AR(∞) form for inflation dynamics:

$$\pi_t = (1-\theta)(1+\Gamma^\varepsilon)\mu_t + \left(\kappa + \rho - \frac{\kappa + (1-\theta)\Gamma^\varepsilon - \theta}{(1-\theta)(1+\Gamma^\varepsilon)}\right) \pi_{t-1} + O(\pi_{t-2}), \tag{22}$$

where $O(\pi_{t-2})$ represents the term consisting of π_{t-2-j} for $j = 0, 1, \cdots$. This suggests that inflation dynamics are influenced by Γ, Γ^*, and Γ^ε, which are, in turn, influenced by the transport cost, τ.

Aggregate Output Aggregate output is given by $Y_t = M_t/P_t$. The log-linearization yields

$$\hat{Y}_t = -\hat{P}_t. \tag{23}$$

Welfare Household intertemporal utility is expressed as

$$\begin{aligned} U &= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\log C_t - (L_t + \tau D_t)\right] \\ &= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\log(M_t/P_t) - M_t/P_t - \tau D_t\right]. \end{aligned}$$

The first and second terms in welfare are approximated up to the second order as

$$\log(M_t/P_t) - M_t/P_t = -\log(1+\tau/2) - \frac{1}{1+\tau/2} - \frac{\tau/2}{1+\tau/2}\hat{P}_t - \frac{1/2}{1+\tau/2}\hat{P}_t^2. \tag{24}$$

This suggests that an increase in the aggregate price decreases utility both in the first and second orders, while a decrease in the aggregate price increases utility in the first order but decreases it in the second order.

The third term in utility, shopping distance D_t, is approximated up to the second order as

$$D_t = \frac{1}{4} + \left(\frac{\hat{p}_t^A - \hat{p}_t^B}{\tau}\right)^2. \tag{25}$$

This suggests that price stickiness decreases household intertemporal utility. Some consumers have to walk a longer distance (an increase in d) when there is a price difference between firms A and B. This is a new effect to consider, compared to standard monopolistic competition models.

2.4 Comparison with a Dixit–Stiglitz Monopolistic Competition Model

It is valuable to compare the implications of our model with those derived in a standard New Keynesian model based on Dixit–Stiglitz monopolistic competition. Here, we assume that there is one monopolistic firm, instead of two, in each product line j. Across product lines, the elasticity of substitution is not one but $\sigma > 1$. Consumption is aggregated following the Dixit-Stiglitz form of aggregation:

$$C_t = \left\{ \int_0^1 C_t(j)^{\frac{\sigma-1}{\sigma}} dj \right\}^{\frac{\sigma}{\sigma-1}}, \tag{26}$$

instead of equation (1). This yields the demand and price index given by $Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\sigma} Y_t$ and $P_t = \left\{ \int_0^1 P_t(j)^{1-\sigma} dj \right\}^{\frac{1}{1-\sigma}}$, respectively, where $C_t(j) = Y_t(j)$.

Table 1 summarizes the comparison between the duopolistic model and the Dixit–Stiglitz monopolistic competition model.

Steady State A firm maximizes its profit given by $(P_t(j)Y_t(j) - W_t Y_t(j))/P_t$ by optimally choosing price $P_t(j)$. The optimal price is given by

$$P_t(j) = \sigma/(\sigma-1) \cdot W_t. \tag{27}$$

The demand elasticity is σ and price markup (the ratio of output price to the marginal cost) is $\sigma/(\sigma-1)$. The degree of strategic complementarity is zero because the optimal price is independent of other firms' price P_t.

By contrast, in the duopolistic competition model, the demand elasticity is $1 + 2/\tau$ and price markup is $1 + \tau/2$ in the absence of price stickiness. This suggests that the two models have the same values for both the demand elasticity and price markup when τ equals $2/(\sigma-1)$. For example, $\sigma = 9$ is assumed in Gali (2015), which implies $\tau = 0.25$.

Inflation Dynamics Appendix A.3 shows that the inflation dynamics are represented by

$$\pi_t = \frac{1-\theta}{1-\rho\theta\beta}\mu_t + (\theta + \rho - \rho\theta\beta)\,\pi_{t-1} + O(\pi_{t-2}). \tag{28}$$

This suggests that the elasticity of substitution σ does not affect the inflation dynamics. By contrast, in the duopolistic competition model, transport cost τ, which influences the demand elasticity, also influences inflation dynamics.

2.5 Simulation

A time unit is a quarter. In the benchmark, we normalize $W = 1$. We set transport cost $\tau = 0.25$, consistent with $\sigma = 9$, as assumed in Gali (2015). Price stickiness is set at $\theta = 0.75$, which implies price revisions occur once per year. We also use $\rho = 0.85$ and $\beta = 0.99$.

Policy Function Figure 2 shows policy functions for the optimal reset price, represented by the coefficients Γ, Γ^*, and Γ^ε in equation (18). The horizontal axis represents transport cost τ in a log scale. For comparison, we plot policy functions in the Dixit–Stiglitz model, where $\Gamma = \Gamma^* = 0$. The figure shows that both Γ and Γ^* are positive. This suggests that a firm revises its price upward when its previous price was high (i.e., $\Gamma > 0$) or its rival's previous price was high (i.e., $\Gamma^* > 0$). This strategic complementarity, particularly owing to positive Γ^*, leads to a higher markup in the steady state under sticky prices, as illustrated in equation (17). Quantitatively, the size of Γ^* is around ten times larger than the size of Γ, showing that a firm's own past price is much less important than its rival firm's price. While Γ and Γ^* increase, Γ^ε decreases compared with that in the Dixit–Stiglitz model. That is, the price responds to an aggregate shock to a smaller extent. This strategic complementarity of price setting increases the real effect of monetary policy.

The lower right-hand panel of Figure 2 shows the coefficient on inflation in the previous period (π_{t-1}) for the AR(∞) form of inflation dynamics (π_t), given by equation (22). The coefficient is lower than that based on the Dixit–Stiglitz model, implying a lower persistence, although we need to check impulse responses because π_t depends on the shock in the current period (μ_t) and on inflation in the previous periods (π_{t-2},π_{t-3}, $\cdots$). Moreover, we find that the coefficient on the current shock (μ_t) for the AR(∞) form is also lower in our model than that based on the Dixit–Stiglitz model, implying that the slope of the Phillips curve is lower.

This figure suggests that this duopolistic competition model nests the Dixit–Stiglitz model in terms of dynamics. As τ increases to infinity, the policy functions converge to those in the Dixit–Stiglitz model. When τ is high, consumers purchase goods from the firm that is located closer to them, irrespective of prices. Thus, the situation resembles that described by the Dixit–Stiglitz model of monopolistic competition. However, this duopolistic competition model does not cover all features depicted by the Dixit–Stiglitz model because the steady-state markup based on the former model is infinite when τ is infinite. In order for the duopolistic competition model to fully cover the features captured by the Dixit–Stiglitz model in terms of both dynamics and the steady state, the non-unit elasticity of substitution between product lines must be introduced.

Figure 3 shows policy functions for the optimal reset price, where the horizontal axis represents price stickiness θ. The figure shows that both Γ and Γ^* are positive unless $\theta = 0$ and depend on θ. Especially, Γ^* increases monotonically as θ increases, showing that an increase in price stickiness increases the degree of dynamic strategic complementarity. By contrast, the response of Γ to θ is not monotonic: Γ increases with θ when θ is small; however, when θ is around 0.75, Γ decreases with θ. The size of Γ^ε is increasing with θ, suggesting that price stickiness increases the price response to an aggregate shock.

Steady-State Price Figure 4 shows the steady-state price under sticky prices given by equation (17). The steady-state price is plotted as a ratio to the price under flexible prices given by equation (12). The upper panel shows that transport cost τ influences the ratio of the steady-state price under sticky prices to the price under flexible prices in a non-monotonic manner. There are two forces. On the one hand, high transport cost τ increases steady-state price under flexible prices. Specifically, when τ is zero, firms earn no markup, and thus, price stickiness does not matter for the ratio of the steady-state price under sticky prices to that under flexible prices. On the other hand, high transport cost τ weakens the dynamic strategic complementarity (i.e., Γ^* decreases as τ increases). As a result of equation (17), this decreases the steady-state price under sticky prices. Consequently, there is a certain τ that maximizes the ratio of the steady-state price under sticky prices, relative to that under flexible prices.

The lower panel of Figure 4 shows how the degree of price stickiness (θ) influences the ratio of the steady-state price under sticky prices to that under flexible prices. Similar to the upper panel, there is a certain θ that maximizes it. When θ is not too high, an increase in θ magnifies the importance of reacting to $\hat{p}^B_{t-1}$ for firm A because firm B is more likely to keep its price unchanged. This strengthens the dynamic strategic complementarity and, in turn, increases the steady-state price under sticky prices. However, when θ is very large, firm A's price today is less likely to influence firm B's price tomorrow because firm B is less likely to reset its price. This weakens the effect of dynamic strategic complementarity and, in turn, decreases the steady-state price under sticky prices.

Impulse Responses Figure 5 shows the impulse response functions to a positive money supply shock ($\mu_t = 1$ at $t = 1$) for aggregate inflation rate π_t and output $\hat{Y}_t$. As expected, the strategic complementarity of price setting increases the real effect of monetary policy, while it decreases the nominal effect on inflation. The real effect of monetary policy in this model is larger by approximately one third than that in the Dixit–Stiglitz model.

3 Pricing under Consumers' Unobservable Heterogeneity

Some consumers may have access to a car and be more mobile, whereas others may not be, for example, because they are aged, unhealthy, or busy working. In this section, we extend the previous model to incorporate consumers' heterogeneity in terms of not only their location (x) but also their transport cost (τ) (see Armstrong 2006). As we explained in 2.1, transport cost τ also represents a consumer's choosiness. Thus, the heterogeneity of τ also represents how some consumers are loyal to a particular firm (brand) (i.e., price-insensitive) whereas others are bargain hunters (i.e., price-sensitive). Then, the former customers have a higher τ than the latter. This setup enriches the model implications, especially in terms of accounting for the existence of sales (temporary price reductions).

3.1 Setup

Consumers are heterogeneous in terms of not only their locations but also their transport cost τ. Specifically, τ takes τ_L with the probability of α or τ_H otherwise $(0 < \tau_L < \tau_H)$. This probability is independent of consumer locations. We begin by neglecting price stickiness. If firms A and B can observe consumers' transport cost one by one, they set their price differently: $p = (1 + \tau_L/2)\, W$ for consumers with τ_L and $p = (1 + \tau_H/2)\, W$ for consumers with τ_H. In other words, firms charge a higher price for price-insensitive loyal consumers and a lower price for price-sensitive bargain hunters. From now on, we suppose that firms A and B cannot observe consumers' transport cost, but know the distribution characterized by τ_H,τ_L, and α correctly.

3.2 Steady State without Price Stickiness

Recall that firm A's profit is proportional to

$$\Pi^A(p^A, p^B) = \mathbb{E}\left[\left(1 - \frac{W}{p^A}\right)\left(\frac{1}{2} - \frac{\log p^A - \log p^B}{\tau}\right)\right],$$

if $-1/2 < \frac{\log p^A - \log p^B}{\tau} < 1/2$, and the first-order condition is

$$0 = W\left(\frac{1}{2} - \mathbb{E}\left[\frac{\log p^A - \log p^B}{\tau}\right]\right) - \left(p^A - W\right)\mathbb{E}\left[1/\tau\right].$$

Pure Strategy One possible option for a pricing strategy is a pure strategy. Then, the prices of firms A and B are symmetric, $p^* = p^A = p^B$, which satisfies

$$p^* = \left\{1 + 1/2 \cdot \left(\mathbb{E}\left[1/\tau\right]\right)^{-1}\right\} W, \tag{29}$$

where the harmonic mean of τ is given by

$$\left(\mathbb{E}\left[1/\tau\right]\right)^{-1} = \left\{\alpha(1/\tau_L) + (1-\alpha)(1/\tau_H)\right\}^{-1}. \tag{30}$$

Suppose that firm A deviates to choose p^d. The pure strategy equilibrium holds if $\Pi^A(p^*, p^*) > \Pi^A(p^d, p^*)$ for any p^d. Given firm B's price p^*, firm A may be able to increase its profit by giving up revenues from price-sensitive bargain hunters and charging a higher price. In this case, the profit becomes

$$\Pi^A(p^d, p^*) = (1-\alpha)\left(1 - \frac{W}{p^d}\right)\left(\frac{1}{2} - \frac{\log p^d - \log p^*}{\tau_H}\right),$$

if $\frac{\log p^d - \log p^*}{\tau_L} > 1/2$. The deviating price p^d should satisfy

$$0 = W\left(\frac{1}{2} - \frac{\log p^d - \log p^*}{\tau_H}\right) - \left(p^d - W\right) \cdot 1/\tau_H. \tag{31}$$

The condition is rewritten as

$$\frac{1}{2}\left(1-\frac{W}{p^*}\right) > (1-\alpha)\left(1-\frac{W}{p^d}\right)\left(\frac{1}{2}-\frac{\log p^d - \log p^*}{\tau_H}\right). \tag{32}$$

We calculate the condition of τ_H,τ_L, and α , which must be met for the pure strategy equilibrium to exist numerically.

Mixed Strategy (Regular and Sales) Suppose that firm B chooses a mixed strategy, in which price is p_H^B with the probability of $1-s^B$ and p_L^B with the probability of s^B ($p_H^B > p_L^B$). Hence, s^B represents the frequency of holding sales. Firm A also chooses the mixed strategy characterized by p_H^A, p_L^A, and s^A. Suppose that the price difference between p_H and p_L is large enough for price-sensitive bargain hunters to travel to the more distant firm ($\frac{\log p_H - \log p_L}{\tau_L} > 1/2$) but not for price-insensitive loyal customers ($\frac{\log p_H - \log p_L}{\tau_H} < 1/2$). In other words, price-sensitive bargain hunters purchase at the firm selling at the lower price p_L if the competitor firm sets the higher price, irrespective of their locations.

When $p^A = p_H^A$, firm A's expected profit is written as

$$\Pi^A(p_H^A) = (1-s^B)\mathbb{E}\left[\left(1-\frac{W}{p_H^A}\right)\left(\frac{1}{2}-\frac{\log p_H^A - \log p_H^B}{\tau}\right)\right]$$
$$+ s^B(1-\alpha)\left(1-\frac{W}{p_H^A}\right)\left(\frac{1}{2}-\frac{\log p_H^A - \log p_L^B}{\tau_H}\right).$$

If firm B sets p_L^B, firm A earns zero sales from τ_L consumers. The first-order condition with respect to p_H^A yields

$$0 = \frac{1-s}{2}W - (1-s)\left(p_H - W\right)\mathbb{E}\left[1/\tau\right]$$
$$+ s(1-\alpha)W\left(\frac{1}{2}-\frac{\log p_H - \log p_L}{\tau_H}\right) - s(1-\alpha)\left(p_H - W\right)\frac{1}{\tau_H} \tag{33}$$

given symmetry.

When $p^A = p_L^A$, firm A's expected profit is

$$\Pi^A(p_L^A) = (1-s^B)\left\{\alpha\left(1-\frac{W}{p_L^A}\right) + (1-\alpha)\left(1-\frac{W}{p_L^A}\right)\left(\frac{1}{2}-\frac{\log p_L^A - \log p_H^B}{\tau_H}\right)\right\}$$
$$+ s^B\mathbb{E}\left[\left(1-\frac{W}{p_L^A}\right)\left(\frac{1}{2}-\frac{\log p_L^A - \log p_L^B}{\tau}\right)\right].$$

If firm B sets p_H^B, firm A earns unit sales from τ_L consumers. The first-order condition with respect to p_L^A yields

$$\begin{aligned} 0 =& (1-s)\alpha W \\ &+ (1-s)(1-\alpha)W\left(\frac{1}{2} - \frac{\log p_L - \log p_H}{\tau_H}\right) \\ &- (1-s)(1-\alpha)\left(p_L - W\right)\frac{1}{\tau_H} \\ &+ sW\frac{1}{2} - s\left(p_L - W\right)\mathbb{E}\left[1/\tau\right]. \end{aligned} \tag{34}$$

Furthermore, we should have indifference of $\Pi^A(p_H^A) = \Pi^A(p_L^A)$, which yields

$$\begin{aligned} &(1-s)\left(1-\frac{W}{p_H}\right)\frac{1}{2} + s(1-\alpha)\left(1-\frac{W}{p_H}\right)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{\tau_H}\right) \\ =&(1-s)\left\{\alpha\left(1-\frac{W}{p_L}\right) + (1-\alpha)\left(1-\frac{W}{p_L}\right)\left(\frac{1}{2} - \frac{\log p_L - \log p_H}{\tau_H}\right)\right\} \\ &+ s\left(1-\frac{W}{p_L}\right)\frac{1}{2}. \end{aligned} \tag{35}$$

Both the left- and right-hand sides of the equation are decreasing with s, suggesting that an increase in the frequency of sales decreases the firm profit, irrespective of whether the firms choose the higher price. Furthermore, it can be shown that the difference in profit, $\Pi^A(p_H^A) - \Pi^A(p_L^A)$, is decreasing with s. This suggests that, as the frequency of sales increases, the profit from choosing the lower price increases more than that from choosing the higher price. Thus, there is a strategic complementarity.

Equations (33) to (35) give the solutions for p_H, p_L, and s.

Welfare First, we consider the case of a pure strategy. Firms A and B set the price to p^*. Each consumer spends M/p^* for consumption C, while also supplying labor L for the same amount. Shopping distance D equals $2\int_0^{1/2} x dx = 1/4$, but disutility from shopping differs between consumers with τ_H and τ_L. Household utility U^{pure} becomes

$$U^{pure} = \left\{\log(M/p^*) - (M/p^* + \mathbb{E}\left[\tau\right]/4)\right\}/(1-\beta). \tag{36}$$

The heterogeneity of transport costs influence the equilibrium price level. To see this, we increase deviations between τ_L and τ_H while keeping the harmonic mean of τ fixed. Specifically, we define Δ_τ so that $\tau_L = \tau \cdot (1 + \Delta_\tau/\alpha)^{-1}$ and $\tau_H = \tau \cdot (1 - \Delta_\tau/(1-\alpha))^{-1}$, which maintains $(\mathbb{E}\left[1/\tau\right])^{-1} = \tau$. Then, U^{pure} in the above equation depends on Δ_τ only through the term of $\mathbb{E}\left[\tau\right]$ because p^* is independent of Δ_τ. Moreover, we find

$$\begin{aligned} \mathbb{E}\left[\tau\right] &= (\alpha\tau_L + (1-\alpha)\tau_H) \\ &= \tau\left(\frac{\alpha}{1+\Delta_\tau/\alpha} + \frac{1-\alpha}{1-\Delta_\tau/(1-\alpha)}\right). \end{aligned}$$

The derivative of $\mathbb{E}[\tau]$ with respect to Δ_τ is

$$\tau \frac{(1-\Delta_\tau/(1-\alpha)+1+\Delta_\tau/\alpha)\,(1/(1-\alpha)+1/\alpha)}{(1+\Delta_\tau/\alpha)^2(1-\Delta_\tau/(1-\alpha))^2}\Delta_\tau > 0$$

if $\Delta_\tau \ll 1$. Thus, larger deviations in transport costs increase the mean of τ, which decreases utility.

Second, we consider the case of a mixed strategy. When one firm sets p_L and the other sets p_H, all consumers with τ_L purchase from the former firm, so $D = \int_0^1 x dx = 1/2$. As for consumers with τ_H, the fraction of $x^{HL} \equiv \frac{1}{2} - \frac{\log p_L - \log p_H}{\tau_H}$ purchases from p_L firm and the $1 - x^{HL}$ fraction purchases from p_H. Thus, $D = \int_0^{x^{HL}} x dx + \int_{x^{HL}}^1 (1-x)dx = ((x^{HL})^2 + (1-x^{HL})^2)/2$. Household utility is given by

$$\begin{aligned}
U^{mixed} &= s\left\{\log(M/p^L) - M/p^L\right\}/(1-\beta)\\
&+ (1-s)\left\{\log(M/p^H) - M/p^H\right\}/(1-\beta)\\
&- \left(s^2 + (1-s)^2\right)\mathbb{E}[\tau]/4/(1-\beta)\\
&- 2s(1-s)\left\{\alpha\tau_L/2 + (1-\alpha)\tau_H((x^{HL})^2 + (1-x^{HL})^2)/2\right\}/(1-\beta). \qquad (37)
\end{aligned}$$

The mixed strategy has three effects on utility. First, setting the higher price p_H decreases utility by decreasing consumption. Second, setting the lower price p_L increases utility by increasing consumption. Third, the price dispersion decreases utility by increasing shopping distance.

3.3 Pricing under Consumers' Unobservable Heterogeneity and Price Stickiness

In the next step, we add Calvo-type price stickiness. Both firms A and B compete in multiple periods and can reset their prices at the probability of $1-\theta \in (0,1)$.

Pure Strategy First, we consider a case in which the pure strategy equilibrium holds. Importantly, the condition for the pure strategy equilibrium to hold, which was shown in the previous subsection, is relaxed. In the dynamic setup where two firms compete repeatedly, the incentive to deviate from the pure strategy decreases. Nevertheless, we can obtain implications of price stickiness similar to those in Section 2.3.

Under price stickiness, the pure strategy is expressed by

$$p^* = 1 + \frac{1}{2}\left(\mathbb{E}[1/\tau]\right)^{-1}\left(1 - \frac{(1-\theta)(1+\theta-\theta^2\beta)}{1-\theta^2\beta}\theta\beta\Gamma^*\right)^{-1}. \qquad (38)$$

Importantly, p^* increases by the term of Γ^* under price stickiness, which increases firm profit and decreases the incentive to deviate from this strategy.[8]

[8] See Appendix B.2 for details.

Mixed Strategy Next, we consider the case of a mixed strategy. It is often stated in the literature that regular prices are sticky, while sale prices are highly flexible. Specifically, as in Guimaraes and Sheedy (2011), we assume that the higher price is subject to Calvo-type price stickiness, while the lower price is perfectly flexible. Specifically, we assume that the lower price $p_{L,t}$ is set at $p_L W_t$, where p_L stands for the steady-state lower price when $W_0 = 1$. In other words, the lower price is indexed to the aggregate wage level (money supply) fully. It is of note, however, that this price is not necessarily optimal since it does not consider price history. With the probability of $1 - \theta$, firms can revise the higher price. Firms may also choose to set the lower price, and even in this case, we assume that the rival firm can observe the higher price.

In order for the mixed strategy equilibrium to hold, the two choices (higher price and lower price) must yield the same payoff. However, this is very restrictive, particularly when we impose the equality of the payoff both when firms can revise their higher price and when they cannot. If the payoff from not revising the higher price is the same as the payoff from choosing the lower price, the payoff from revising the higher price is likely to exceed that from choosing the lower price, because not revising the higher price is suboptimal.

In this study, we assume that a certain constraint (such as limited information-processing capacity) prevents firms from optimizing both the higher price ($p_{H,t}$) and the frequency of sales (s_t) simultaneously. When firms can revise their higher price, the frequency of sales is kept constant at a steady-state level (s) and the equality of payoffs does not hold unless the economy is in steady state. When firms cannot revise their higher price, the equality of payoffs holds; firms can flexibly optimize the frequency of sales to achieve this.

We define the optimal higher price set by firm A in period t as $\bar{p}^A_{H,t}$. Further, we define the frequency of sales when firm B revises the higher price as $s^r_t = s$ and the frequency of sales when firm B does not revise the higher price as s^n_t. Then, firm A optimizes $\bar{p}^A_{H,t}$ as

$$\begin{aligned}
\max \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t & \left[\left(1 - \frac{M_{t+k}}{\bar{p}^A_{H,t}}\right) (\theta s^n_{t+k} + (1-\theta)s) \left(\frac{1}{2} - \frac{\log \bar{p}^A_{H,t} - \log p^B_{L,t+k}}{\tau}\right) \right. \\
&+ \left(1 - \frac{M_{t+k}}{\bar{p}^A_{H,t}}\right) \theta (1 - s^n_{t+k}) \left(\frac{1}{2} - \frac{\log \bar{p}^A_{H,t} - \log p^B_{H,t+k-1}}{\tau}\right) \\
&+ \left. \left(1 - \frac{M_{t+k}}{\bar{p}^A_{H,t}}\right) (1-\theta)(1-s) \left(\frac{1}{2} - \frac{\log \bar{p}^A_{H,t} - \log \bar{p}^B_{H,t+k}}{\tau}\right) \right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t}.
\end{aligned}$$

Further, in log-linearization, we define $p^A_{H,t} \equiv p_H M_t e^{\hat{p}^A_t}$,$\bar{p}^A_{H,t} \equiv p_H M_t e^{p^{A*}_t}$, and $s^n_t \equiv s e^{\hat{s}^n_t}$ as well as $\partial \log \bar{p}^B_{H,t+k} / \partial \log \bar{p}^A_{H,t} \equiv \Gamma^*$ and $\partial \log s^n_{t+k} / \partial \log \bar{p}^A_{H,t} \equiv \Lambda^{n*}$.

In Appendix B.2, we explain the condition for the mixed strategy equilibrium to hold, which comes from the first-order condition for the optimal setting $\bar{p}^A_{H,t}$ and the aforementioned condition of equality of profits from choosing the higher price and the lower price when the higher price is not revised. Furthermore, we derive aggregate inflation and output for this scenario in Appendix B.2.

3.4 Simulation

For the simulation, we use the parameter values explained in Section 2.5. The only exception is the transport cost. Instead of the homogeneous value of $\tau = 0.25$, we use various combinations for the values of τ_H, τ_L, and α.

Pure and Mixed Strategy Region under Flexible Prices We show numerical results under flexible prices (i.e., $\theta \to 0$). First, Figure 6 illustrates the parameter region in which the equilibrium is characterized by either the pure or mixed strategy. We fix τ_L either at 0.01 or 0.1. The figure shows that the pure strategy equilibrium is more likely to arise as the difference between τ_L and τ_H becomes smaller (i.e., τ_L is larger and τ_H is smaller) or the probability of τ_L becomes higher. The mixed strategy equilibrium is more likely to arise in the opposite case. The line with circles shows the combination of τ_H and α which is required to keep the harmonic mean of τ at 0.25. This line intersects with the boundary at the point dividing the pure and mixed strategy equilibrium. This suggests that if a harmonic-mean-preserving difference for τ_H and τ_L exceeds a certain level, the equilibrium changes from the pure strategy equilibrium to the mixed strategy equilibrium.

Second, Figure 7 shows how the equilibrium price changes when τ_H changes. Here, we fix τ_L at 0.1 and adjust α so that the harmonic mean of τ is unchanged at 0.25. When τ_H is low (close to τ_L), the pure strategy constitutes the equilibrium, and the equilibrium price is independent of τ_H because the harmonic mean of τ is unchanged. When τ_H is high, the mixed strategy constitutes the equilibrium, generating two possible equilibrium prices. The figure shows that, compared with the pure strategy, the mixed strategy leads to higher equilibrium prices. Interestingly, the equilibrium price is higher even for the lower price of the two. As τ_H increases (i.e., the harmonic-mean-preserving difference increases), the two equilibrium prices increase. Meanwhile, the probability of sales (i.e., the probability of choosing the lower price rather than the higher price) decreases. It should also be noted that in this model, the size of the sale discount, $(p_H - p_L)/p_H$, is approximately 15%, whereas the frequency of sales is around 80%. Compared with the data (e.g., see Sudo et al. 2018), the former is almost the same, but the latter is considerably higher (the actual frequency of sales is around 25%).

Third, Figure 8 shows how utility changes when τ_H changes. Again, we fix τ_L at 0.1 and adjust α so that the harmonic mean of τ is unchanged at 0.25. The figure demonstrates that utility decreases monotonically as the harmonic-mean-preserving difference increases. This result is consistent with what we discussed in Section 3.2.

Impulse Responses under Sticky Prices Next, we simulate the model under price stickiness ($\theta = 0.75$). We set τ_L and τ_H at 0.1 and 10, respectively, whereas α is chosen to make the harmonic mean of τ equal 0.25 (i.e., $\alpha = 0.39$). In this case, the mixed strategy serves as the equilibrium. We numerically calculate policy functions with respect to pricing and then calculate the impulse responses to a positive money supply shock ($\mu_t = 1$ at $t = 1$).

The solid line with dots in Figure 9 shows the simulation results. All the variables are log-linearized from their steady state values.[9] The figure shows that a positive money supply shock increases the inflation rate and output. The higher price is negative, suggesting that price stickiness prevents some firms from adjusting their price upward. Meanwhile, these firms optimally adjust the frequency of sales. Specifically, they decrease the frequency of sales.

For comparison, we plot the impulse responses when τ is homogeneous at 0.25. In this case, equilibrium is characterized by the pure strategy. Although the inflation rate and output exhibit the same qualitative pattern, the magnitude of the change in output is considerably different. Namely, the output under the mixed strategy increases by less than one tenth.

The reason for this is the presence of strategic complementarities. The lower left-hand panel shows that, although the aggregate higher price under the mixed strategy is negative, the extent to which it deviates from the steady state is smaller than the extent to which the aggregate price deviates under the pure strategy. Under the mixed strategy, the lower (i.e., sale) price is revised upward fully in response to the positive money supply shock. Combined with the strategic complementarity effect, this induces firms to increase their higher price more when they can reset it. Furthermore, the decrease in the frequency of sales increases the aggregate price. Therefore, nominal prices are adjusted upward more strongly, which weakens the real effect of monetary policy.

Recall that we assume a certain constraint, which prevents firms from optimizing both the higher price and the frequency of sales simultaneously. Firms optimize the frequency of sales only when they cannot revise their higher price. To investigate the quantitative importance of this assumption, in Figure 9, we also demonstrate simulation results when the frequency of sales is kept constant. Technically, we assume that the equality of profits from choosing the higher price and lower price does not necessarily hold except for in the steady state. The figure shows that, although the frequency of sales does not decrease in response to the shock, the other three variables (i.e., the inflation rate, output, and the higher price) hardly change. Thus, although there could be many different models regarding the mixed strategy under price stickiness, this result suggests that quantitative implications may be more or less the same.

Comparison with Guimaraes and Sheedy (2011) This result is markedly different from that reported in Guimaraes and Sheedy (2011). In their model, the existence of tem-

[9] The lower left-hand panel shows the log-linearized aggregate higher price. Specifically, by denoting the log-linearized optimal reset higher price by p_t^*, we can express the aggregate higher price in period t as $\hat{p}_t = \theta(\hat{p}_{t-1} - \varepsilon_t) + (1 - \theta)p_t^*$.

porary sales hardly changes the real effect of monetary policy. Their model setup is different from ours in that the former is based on the Dixit–Stiglitz model and thus, firms are monopolistic. There exist two types of consumers whose elasticity of substitution is different. Model outcomes are also different. In their model, two price equilibria arise, but strictly speaking, they are not a mixed-strategy equilibrium as in our model. In their model, each firm can set multiple prices in each period, choosing a higher price for a certain fraction of products and a lower price for the other. In our model, each firm sets one price in each period, consequently choosing a higher price with a certain probability and a lower price otherwise. In the Guimaraes and Sheedy (2011) model, sales are strategic substitutes; that is, if other firms choose a higher price rather than a lower price more frequently, a firm will be better off if it chooses a lower price rather than a higher price more frequently. This exemplifies the strategic substitutes scenario and leads to the result that the real effect of monetary policy hardly changes despite sales.

4 Concluding Remarks

In this study, we provided a tractable macroeconomic model incorporating duopolistic competition and price stickiness. We found that implications for monetary policy change when we incorporate strategic pricing behaviors of oligopolistic firms. Firms' pricing entails a dynamic (intertemporal) strategic complementarity. The optimal reset price positively depends on the price set by a rival firm in the previous period. This property increases the real effect of monetary policy slightly, compared with a standard monopolistic competition model in which a strategic complementarity is absent. However, when transport costs are heterogeneous, a mixed strategy equilibrium may arise, under which the real effect of monetary policy is weakened considerably.

Although our model is simple and ignores many important features, model tractability may enable us to incorporate them in the future. The first feature is competition across different product lines. Whereas our model assumed a unit elasticity of substitution for different products, it would be possible to use a generalized value for the elasticity of substitution as in the Dixit–Stiglitz model, which generates competition across different product lines. This feature is also relevant to product bundling. In reality, shoppers purchase more than one variety of product per visit to a retailer. They decide whether to purchase from firm A or B based on a multitude of other products, not just based on the single product offered by A and B. Thus, firms A and B must optimize the prices of multiple products simultaneously. The second feature of the model is competition between more than two firms. In this regard, Salop's circular location model is promising. This feature also allows us to incorporate entry/exit and the optimal location choice, which this study ignores. The third feature is asymmetry between two firms. Whereas two firms were assumed to have the same technology, actual firms are diverse in technology. Some are more competitive than others.

The fourth feature is nonlinearity. Although log-linearization of the model simplified our analysis, the degree of price increase in response to a positive money supply shock may be greatly different from the degree of price decrease in response to a negative money supply shock of equal size.

Investigating these extensions is likely to improve the fit of the model, particularly with respect to the frequency of sales in the case when a mixed strategy equilibrium holds. Moreover, since the first three features concern competition, it is interesting to see how such a model extension can help connect recent developments in firm dynamics (e.g., an increase in markup, a decrease in business dynamism) with those in inflation dynamics (e.g., a decrease in the inflation rate in developed countries, the disappearing Phillips curve). Specifically, the fourth feature may enable the following questions to be addressed. Which is more likely to occur, hyperinflation or deflation, when strategic pricing matters? When an inflation rate deviates from a target upward or downward, which is more difficult to reanchor the inflation rate?

Finally, recent information-technology developments have enabled firms to collect consumers' preferences at an individual level at a low cost. In our model, this development corresponds to a situation in which firms observe heterogeneous τ's and/or x's. This allows firms to set customer-dependent prices, also known as third-degree price discrimination. In future, it would be interesting to study whether this occurs in reality and what the implications for monetary and competition policy are (see, e.g., Armstrong 2006).

References

[1] Angeletos, George-Marios, and Jennifer La'O (2009) "Incomplete Information, Higher-order Beliefs and Price inertia," *Journal of Monetary Economics*, 56S, 19–37.

[2] Aoki, Kosuke, Hibiki Ichiue, and Tatsushi Okuda (2019) "Consumers' Price Beliefs, Central Bank Communication, and Inflation Dynamics." Bank of Japan Working Paper Series 19-E-14.

[3] Armstrong, Mark (2006) "Recent Developments in the Economics of Price Discrimination." In R. Blundell, W. Newey, and T. Persson (Eds.), *Advances in Economics and Econometrics: Theory and Applications*, Ninth World Congress (Econometric Society Monographs, pp. 97-141). Cambridge: Cambridge University Press.

[4] Ball, Laurence, and David Romer (1990) "Real Rigidities and the Non-Neutrality of Money." *Review of Economic Studies*, 57(2), 183–203.

[5] Bhaskar, Venkataraman (2002) "On Endogenously Staggered Prices." *Review of Economic Studies*, 69, 97–116.

[6] Bils, Mark, and Peter J. Klenow (2004) "Some Evidence on the Importance of Sticky Prices." *Journal of Political Economy*, 112, 947–985.

[7] Blanchard, Olivier Jean, and Nobuhiro Kiyotaki (1987) "Monopolistic Competition and the Effects of Aggregate Demand." *American Economic Review*, 77(4), 647–666.

[8] Chen, Guo, C. Gizem Korpeoglu, and Stephen E. Spear (2017) "Price Stickiness and Markup Variations in Market Games." *Journal of Mathematical Economics*, 72, 95–103.

[9] Christiano, Lawrence, Martin Eichenbaum, and Charles Evans (2005) "Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy." *Journal of Political Economy*, 113(1), 1–45.

[10] Dixit, Avinash, and Joseph Stiglitz. (1977) "Monopolistic Competition and Optimum Product Diversity." *American Economic Review*, 67(3), 297–308.

[11] Faia, Ester (2012) "Oligopolistic Competition and Optimal Monetary Policy." *Journal of Economic Dynamics and Control*, 36(11), 1760–1774.

[12] Fehr, Ernst, and Jean-Robert Tyran. (2008) "Limited Rationality and Strategic Interaction: The Impact of the Strategic Environment on Nominal Inertia." *Econometrica*, 76(2), 353–394.

[13] Fershtman, Chaim, and Morton I. Kamien (1987) "Dynamic Duopolistic Competition with Sticky Prices." *Econometrica*, 55(5), 1151–1164.

[14] Gali, Jordi (2015) *Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework and Its Applications.* Princeton: Princeton University Press, edition 2.

[15] Guimaraes, Bernardo, and Kevin Sheedy (2011) "Sales and Monetary Policy." *American Economic Review*, 101, 844–876.

[16] Hansen, Gary D. (1985) "Indivisible Labor and the Business Cycle." *Journal of Monetary Economics*, 16, 309–327.

[17] Hotelling, Harold (1929) "Stability in Competition." *Economic Journal*, 39(153), 41–57.

[18] Jun, Byoung, and Xavier Vives (2004) "Strategic Incentives in Dynamic Duopoly," *Journal of Economic Theory*, 116(2), 249–281.

[19] Kryvtsov, Oleksiy, and Nicolas Vincent (forthcoming) "The Cyclicality of Sales and Aggregate Price Flexibility." *Review of Economic Studies*.

[20] Levin, Andrew T., Jose David Lopez-Salido, and Tack Yun (2007) "Strategic Complementarities and Optimal Monetary Policy." Kiel Working Papers 1355.

[21] L'Huillier, Jean-Paul (2020) "Consumers Imperfect Information and Price Rigidities," *American Economic Journal: Macroeconomics*, 12(2), 94–123.

[22] Maskin, Erin, and Jean Tirole (1988) "A Theory of Dynamic Oligopoly, I: Overview and Quantity Competition with Large Fixed Costs," *Econometrica*, 56(3), 549–569.

[23] Midrigan, Virgiliu (2011) "Menu Costs, Multi-product Firms, and Aggregate Fluctuations." *Econometrica*, 79(4), 1139–1180.

[24] Mongey, Simon (2017) "Market Structure and Monetary Non-neutrality." Federal Reserve Bank of Minneapolis Staff Report 558.

[25] Nakamura, Emi, and Jon Steinsson (2010) "Monetary Non-neutrality in a Multisector Menu Cost Model." *Quarterly Journal of Economics*, 125(3), 961–1013.

[26] Nirei, Makoto, and Jose A. Scheinkman (2020) "Menu Costs and the Volatility of Inflation." Presented at Econometric Society World Congress 2020.

[27] Rogerson, Richard (1988) "Indivisble Labor, Lotteries and Equilibrium." *Journal of Monetary Economics*, 21, 3–16.

[28] Slade, Margaret E. (1999) "Sticky Prices in a Dynamic Oligopoly: An Investigation of (s,S) Thresholds." *International Journal of Industrial Organization*. 17(4), 477–511.

[29] Sudo, Nao, Kozo Ueda, and Kota Watanabe (2014) "Micro Price Dynamics during Japan's Lost Decades." *Asian Economic Policy Review*, 9, 46–64.

[30] Sudo, Nao, Kozo Ueda, Kota Watanabe, and Tsutomu Watanabe (2018) "Working Less and Bargain Hunting More: Macroimplications of Sales during Japan's Lost Decades," *Journal of Money, Credit and Banking*, 50(2–3), 449–478.

[31] Wang, Olivier, and Ivan Werning (2020) "Dynamic Oligopoly and Price Stickiness," NBER Working Paper 27536.

[32] Woodford, Michael (2003) *Interest and Prices: Foundations of a Theory of Monetary Policy*, Princeton: Princeton University Press.

Table 1: Comparison with a Dixit–Stiglitz Monopolistic Competition Model

	Duopoly	Monopoly
Steady-state price markup (p/W)	$1+\frac{1}{2}\tau\left(1-\frac{(1-\theta)(1+\theta-\theta^2\beta)}{1-\theta^2\beta}\theta\beta\Gamma^*\right)^{-1}$	$\sigma/(\sigma-1)$
Demand elasticity	$1+2/\tau$	σ
Degree of strategic complementarity	$1/(2+\tau/2)$	0
Dependence on demand elasticity for inflation dynamics	Yes	No

Note: The demand elasticity and degree of strategic complementarity under duopoly are those in the absence of price stickiness.

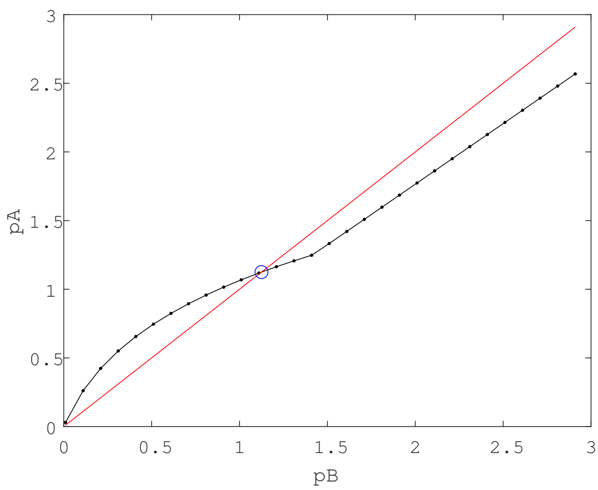

Figure 1: Best Response

Note: The solid line with dots represent firm A's best response of price (p^A) given firm B's price (p^B), where W and τ equal 1 and 0.25, respectively. The solid line shows the 45° line.

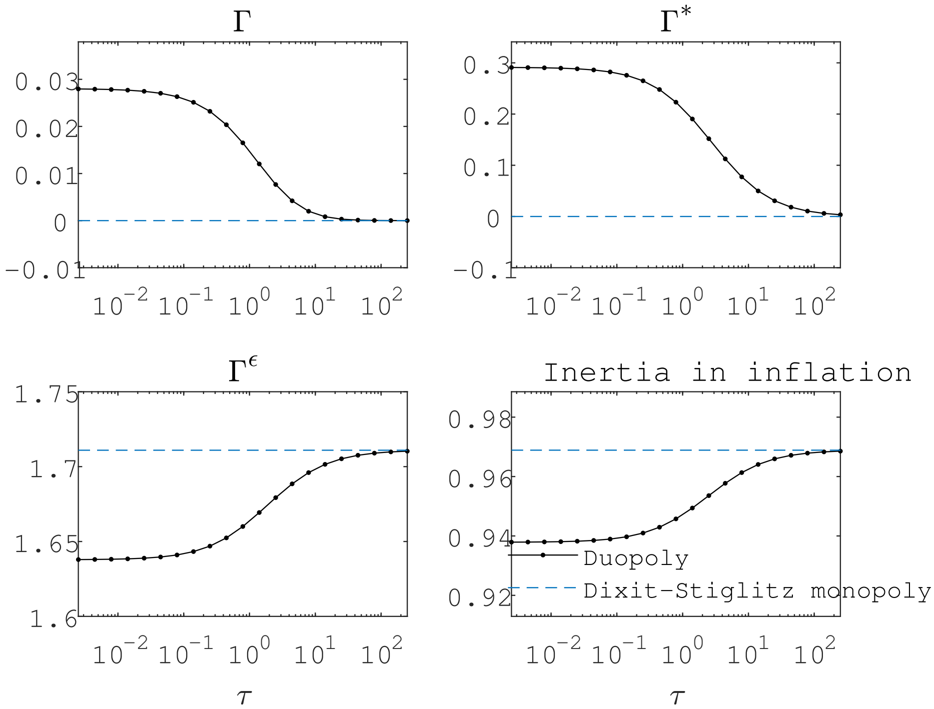

Figure 2: Policy Functions under Price Stickiness: Dependence on Transport Costs

Note: The figure shows the coefficients of policy functions for the optimal reset price by firm A given by $p_t^{A*} = \Gamma \hat{p}_{t-1}^A + \Gamma^* \hat{p}_{t-1}^B + \Gamma^\varepsilon \varepsilon_t$. The lower right-hand panel shows the coefficient on past inflation (π_{t-1}) for the equation of inflation (π_t). The horizontal axis represents transport cost (τ, log scale).

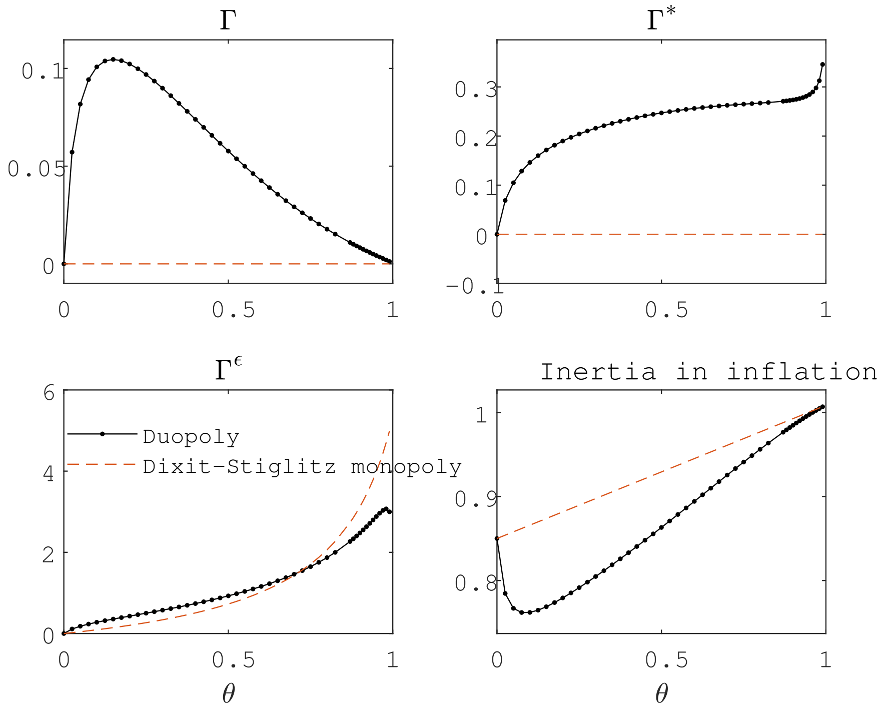

Figure 3: Policy Functions under Price Stickiness: Dependence on Price Stickiness

Note: The figure shows the coefficients of policy functions for the optimal reset price by firm A given by $p_t^{A*} = \Gamma \hat{p}_{t-1}^A + \Gamma^* \hat{p}_{t-1}^B + \Gamma^\varepsilon \varepsilon_t$. The lower right-hand panel shows the coefficient on past inflation (π_{t-1}) for the equation of inflation (π_t). The horizontal axis represents price stickiness (θ).

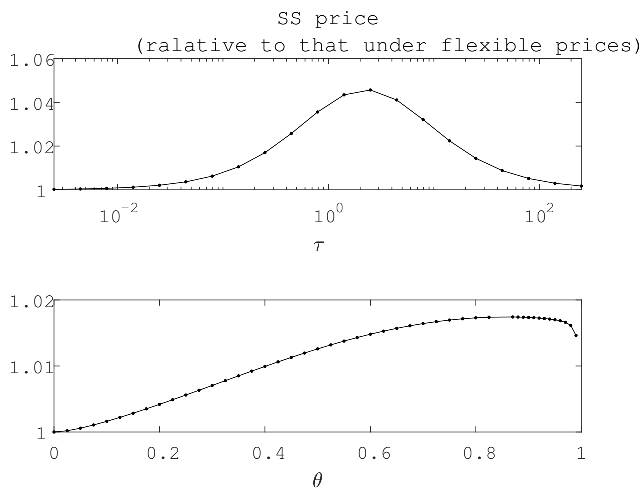

Figure 4: Steady-State Price under Price Stickiness

Note: The vertical axis represents the ratio of the steady-state price under sticky prices to that under flexible prices. The horizontal axis represents transport cost (τ, log scale) and price stickiness (θ) in the upper and lower panels, respectively.

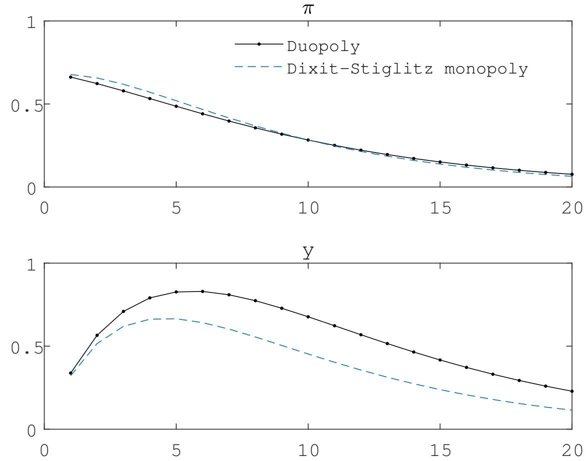

Figure 5: Impulse Responses to Money Supply Shock

Note: The horizontal axis represents quarters after a positive money supply shock occurs at $t = 1$.

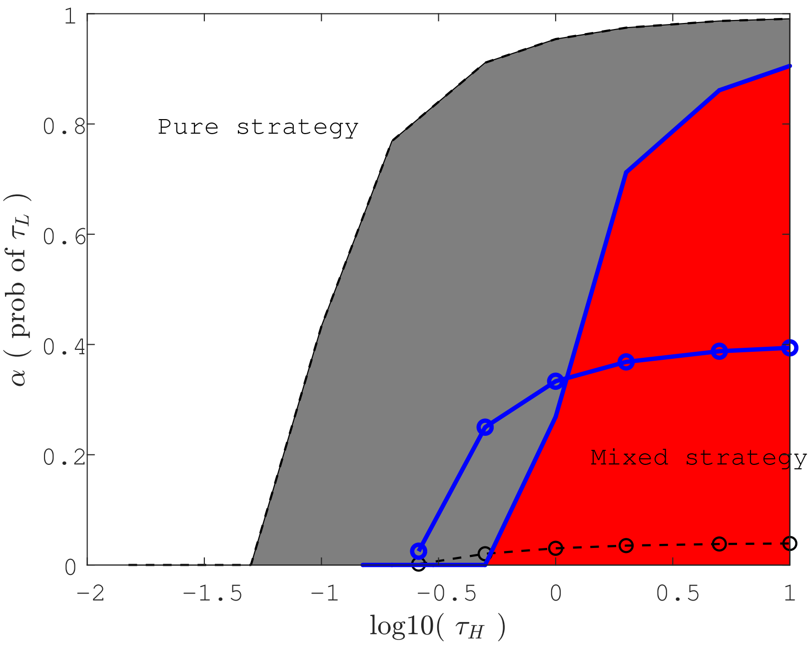

Figure 6: Equilibrium Region under Consumers' Unobservable Heterogeneity and Flexible Prices

Note: The thick solid line (in blue) and the thin dashed line (in black) represent the boundary between pure and mixed strategy when the parameter τ_L is set at 0.01 and 0.1, respectively. The thick solid line with circles (in blue) and the thin dashed line with circles (in black) indicate the combination of τ_H and α that keeps the harmonic mean of τ at 0.25, respectively.

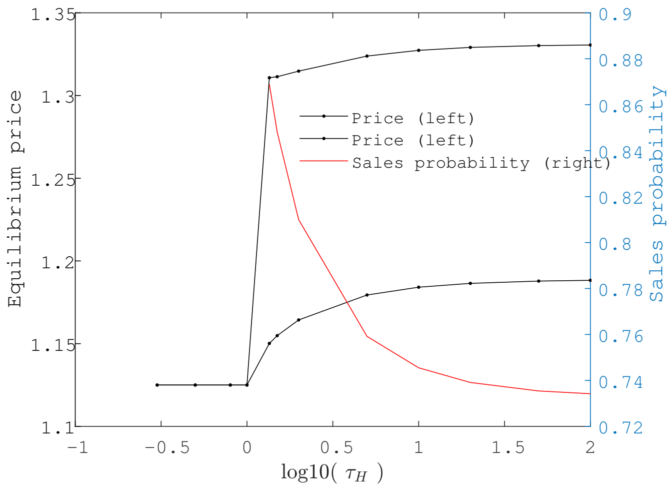

Figure 7: Equilibrium Price under Consumers' Unobservable Heterogeneity and Flexible Prices
Note: The parameter τ_L is set at 0.1, and α is chosen to keep the harmonic mean of τ at 0.25.

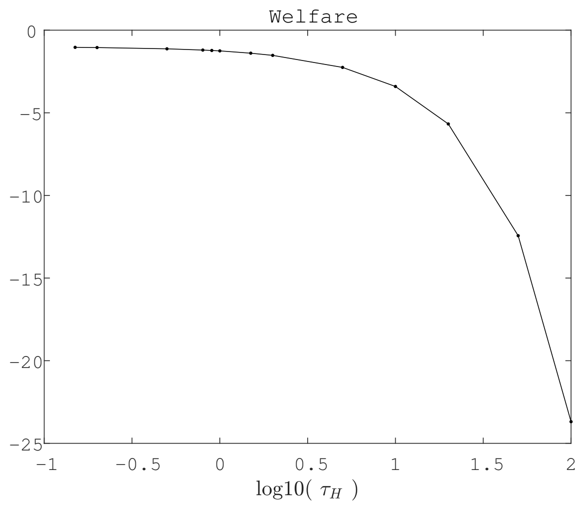

Figure 8: Household Utility under Consumers' Unobservable Heterogeneity and Flexible Prices
Note: The parameter τ_L is set at 0.1, and α is chosen to keep the harmonic mean of τ at 0.25.

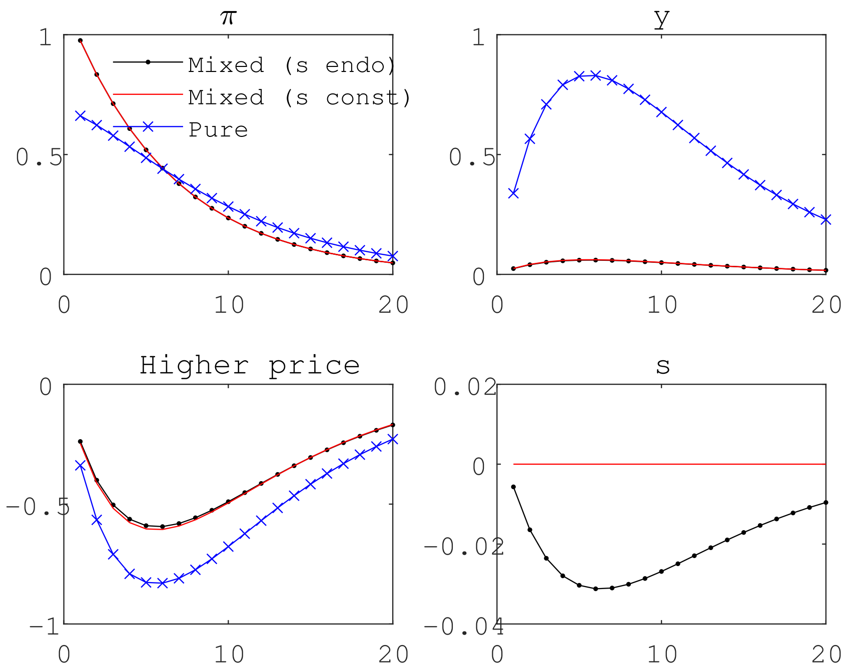

Figure 9: Impulse Responses to Money Supply Shock

Note: For the pure strategy, τ is homogeneous and equals 0.25. For the mixed strategy, τ_L and τ_H equal 0.1 and 10, respectively, whereas the parameter α is chosen to make the harmonic mean of τ equal 0.25 (i.e., $\alpha = 0.39$).

Appendix for "Duopolistic Competition and Monetary Policy"

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January 6, 2021

A Model Details

A.1 Best Response

Firm A's profit is given by

$$\Pi(p^A, p^B) = \begin{cases} 0 & \text{if } \frac{\log p^A - \log p^B}{\tau} \geq 1/2 \\ (p^A - W)\left(\frac{1}{2} - \frac{\log p^A - \log p^B}{\tau}\right)\frac{M}{p^A} & \text{if } -1/2 < \frac{\log p^A - \log p^B}{\tau} < 1/2 \\ (p^A - W)\frac{M}{p^A} & \text{if } \frac{\log p^A - \log p^B}{\tau} \leq -1/2. \end{cases} \tag{1}$$

Thus, the derivative of firm A's profit with respect to p^A given firm B's price p^B is given by

$$\Pi(p^A, p^B) = \begin{cases} 0 & \text{if } \frac{\log p^A - \log p^B}{\tau} \geq 1/2 \\ W\frac{M}{(p^A)^2}\left(\frac{1}{2} - \frac{\log p^A - \log p^B}{\tau}\right) + (p^A - W)\left(-\frac{1}{\tau p^A}\right)\frac{M}{p^A} & \text{if } -1/2 < \frac{\log p^A - \log p^B}{\tau} < 1/2 \\ W\frac{M}{(p^A)^2} & \text{if } \frac{\log p^A - \log p^B}{\tau} \leq -1/2. \end{cases}$$

If $-1/2 < \frac{\log p^A - \log p^B}{\tau} < 1/2$ (the second line in equation (1)), the derivative is zero when p^A satisfies

$$W\left(\frac{1}{2} - \frac{\log p^A - \log p^B}{\tau}\right) = (p^A - W)\left(\frac{1}{\tau}\right)$$

$$p^A + W\log p^A = W\left(\frac{1}{2}\tau + 1 + \log p^B\right). \tag{2}$$

We define such p^A by $p^{A*}(p^B)$. Since the left-hand side of the equation increases with p^A monotonically from $-\infty$ to ∞, such $p^{A*}(p^B)$ is uniquely determined. Moreover it is clear that $p^{A*}(p^B)$ is increasing with p^B. When p^B is low, this $p^{A*}(p^B)$ falls in the range of $\frac{\log p^A - \log p^B}{\tau} \geq 1/2$ (the first line in equation (1)), which causes firm A to earn zero revenue and profit. In this case, the best response is arbitrary (i.e. not limited to $p^{A*}(p^B)$; $p^A = W$ is a best response as well), but we simply assume that $p^{A*}(p^B)$ is a best response. If $\frac{\log p^A - \log p^B}{\tau} \leq -1/2$ (the third line in equation (1)), firm A should choose as high price as possible, which equals $\exp(\log p^B - 1/2 \cdot \tau)$. Note that the equality of $-1/2 = \frac{\log p^{A*}(p^B) - \log p^B}{\tau}$ holds when $p^A = W(\tau + 1)$ and $p^B = \exp\left[\frac{1}{2}\tau + \log\{W(\tau+1)\}\right]$. Further, strict inequaltiy of $-1/2 < \frac{\log p^{A*}(p^B) - \log p^B}{\tau}$ holds when $p^B < \exp\left[\frac{1}{2}\tau + \log\{W(\tau+1)\}\right]$.

In sum, the best response of p^A given p^B is expressed as follows:

$$p^A(p^B) = \begin{cases} p^{A*}(p^B) & \text{if } p^B < \exp\left[\frac{1}{2}\tau + \log\{W(\tau+1)\}\right] \\ \exp(\log p^B - 1/2 \cdot \tau) & \text{if } p^B \geq \exp\left[\frac{1}{2}\tau + \log\{W(\tau+1)\}\right] \end{cases} \tag{3}$$

Differentiating $p^{A*}(p^B)$ with respect to p^B around the steady-state value of $p = p^A = p^B$ yields $p^{*'} + W/p^* \cdot p^{*'} = W/p^B$, which, in turn, becomes

$$p^{*'} = 1/(2 + \tau/2). \tag{4}$$

Thus, the degree of strategic complementarity is positive.

Demand Elasticity We calculate the demand elasticity near the steady state. When firm B's price is p, demand for firm A's output is

$$\left(\frac{1}{2} - \frac{\log p^A - \log p}{\tau}\right)\frac{M}{p^A},$$

and thus, the derivative of demand with respect to a change in p^A is

$$\left(-\frac{1}{\tau p}\right)\frac{M}{p}-\left(\frac{1}{2}\right)\frac{M}{p^2}=-\left(\frac{1}{\tau}+\frac{1}{2}\right)\frac{W}{p^2}.$$

Multiplying p divided by demand at the steady state (i.e., $1/2\cdot W/p$), we obtain the demand elasticity as

$$1+2/\tau. \tag{5}$$

A.2 Pricing under Price Stickiness

When firm A has a chance to set its price at t, it sets $\bar{p}_t^A$ to maximize

$$\begin{aligned}\max\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t&\left[\left(1-\frac{M_{t+k}}{\bar{p}_t^A}\right)\theta^{k+1}\left(\frac{1}{2}-\frac{\log\bar{p}_t^A-\log p_{t-1}^B}{\tau}\right)\right]\cdot\frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}}\frac{M_{t+k}}{M_t}\\+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t&\left[\left(1-\frac{M_{t+k}}{\bar{p}_t^A}\right)\sum_{k'=0}^{k}(1-\theta)\theta^{k-k'}\left(\frac{1}{2}-\frac{\log\bar{p}_t^A-\log\bar{p}_{t+k'}^B}{\tau}\right)\right]\cdot\frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}}\frac{M_{t+k}}{M_t}.\end{aligned}\tag{6}$$

Hereafter, we assume $W_0=M_0=1$ in the initial period. The first-order condition for the optimal $\bar{p}_t^A$ is given by

$$\begin{aligned}0=&\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(\frac{1}{\bar{p}_t^A}\right)^2M_{t+k}\left[\theta^{k+1}\left(\frac{1}{2}-\frac{\log\bar{p}_t^A-\log p_{t-1}^B}{\tau}\right)\right]\cdot\frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}}\frac{M_{t+k}}{M_t}\\&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(\frac{1}{\bar{p}_t^A}\right)^2M_{t+k}\left[\sum_{k'=0}^{k}(1-\theta)\theta^{k-k'}\left(\frac{1}{2}-\frac{\log\bar{p}_t^A-\log\bar{p}_{t+k'}^B}{\tau}\right)\right]\cdot\frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}}\frac{M_{t+k}}{M_t}\\&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(1-\frac{M_{t+k}}{\bar{p}_t^A}\right)\left(-\frac{1}{\tau\bar{p}_t^A}\right)\cdot\frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}}\frac{M_{t+k}}{M_t}\\&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(1-\frac{M_{t+k}}{\bar{p}_t^A}\right)\left[\sum_{k'=0}^{k}(1-\theta)\theta^{k-k'}\frac{\partial\log\bar{p}_{t+k'}^B/\partial\log\bar{p}_t^A}{\tau\bar{p}_t^A}\right]\cdot\frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}}\frac{M_{t+k}}{M_t}.\end{aligned}$$

In log-linearization, let us denote $\bar{p}_t^A\equiv pM_te^{p_t^{A*}}$, $p_t^B\equiv pM_te^{\hat{p}_t^B}$ as well as $\partial\log\bar{p}_{t+k}^B/\partial\log\bar{p}_t^A\equiv\Gamma^*$ for any $k\geq 1$ (which we will define in detail later; it is independent of k because we consider a case in which the price of firm A is unchanged at $\bar{p}_t^A$). Note that $\partial\log\bar{p}_t^B/\partial\log\bar{p}_t^A=0$ because firm B does not know at t that firm A revises its price at t. Then, the log-linearization leads to

$$\begin{aligned}0=&\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(\frac{1}{pe^{p_t^{A*}}}\right)\frac{M_{t+k}}{M_t}\left[\theta^{k+1}\left(\frac{1}{2}-\frac{p_t^{A*}-\hat{p}_{t-1}^B+\log(M_t/M_{t-1})}{\tau}\right)\right]\cdot\frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}}\frac{M_{t+k}}{M_t}\\&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(\frac{1}{pe^{p_t^{A*}}}\right)\frac{M_{t+k}}{M_t}\left[\sum_{k'=0}^{k}(1-\theta)\theta^{k-k'}\left(\frac{1}{2}-\frac{p_t^{A*}-p_{t+k'}^{B*}-\log(M_{t+k'}/M_t)}{\tau}\right)\right]\cdot\frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}}\frac{M_{t+k}}{M_t}\\&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(1-\frac{M_{t+k}/M_t}{pe^{p_t^{A*}}}\right)\left(-\frac{1}{\tau}\right)\cdot\frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}}\frac{M_{t+k}}{M_t}\\&+\sum_{k=1}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(1-\frac{M_{t+k}/M_t}{pe^{p_t^{A*}}}\right)(1-\theta^{k+1})\frac{\Gamma^*}{\tau}\cdot\frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}}\frac{M_{t+k}}{M_t}.\end{aligned}$$

Steady State In the steady state, it becomes

$$\begin{aligned}0=&\sum_{k=0}^{\infty}\theta^k\beta^k\left(\frac{1}{p}\right)\cdot\frac{1}{2}+\sum_{k=0}^{\infty}\theta^k\beta^k\left(1-\frac{1}{p}\right)\cdot\left(-\frac{1}{\tau}\right)\\&+\sum_{k=1}^{\infty}\theta^k\beta^k\left(1-\frac{1}{p}\right)(1-\theta^{k+1})\left(\frac{\Gamma^*}{\tau}\right),\end{aligned}$$

which yields

$$p = 1+\frac{1}{2}\tau\left(1-\frac{(1-\theta)(1+\theta-\theta^2\beta)}{1-\theta^2\beta}\theta\beta\Gamma^*\right)^{-1}. \tag{7}$$

Log-linearization The log-linearization proceeds as

$$\begin{aligned}
0 =& \sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\frac{M_{t+k}}{M_t}\left[\theta^{k+1}\left(\frac{1}{2}-\frac{p_t^{A*}-\hat{p}_{t-1}^B+\log(M_t/M_{t-1})}{\tau}\right)\right]\cdot\frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}}\frac{M_{t+k}}{M_t}\\
&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\frac{M_{t+k}}{M_t}\left[\sum_{k'=0}^{k}(1-\theta)\theta^{k-k'}\left(\frac{1}{2}-\frac{p_t^{A*}-p_{t+k'}^{B*}-\log(M_{t+k'}/M_t)}{\tau}\right)\right]\cdot\frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}}\frac{M_{t+k}}{M_t}\\
&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(p+pp_t^{A*}-M_{t+k}/M_t\right)\left(-\frac{1}{\tau}\right)\cdot\frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}}\frac{M_{t+k}}{M_t}\\
&+\sum_{k=1}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(p_H+p_Hp_t^{A*}-M_{t+k}/M_t\right)(1-\theta^{k+1})\frac{\Gamma^*}{\tau}\cdot\frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}}\frac{M_{t+k}}{M_t}.
\end{aligned}$$

The term $\frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}}\frac{M_{t+k}}{M_t}$ equals one because $P_tC_t = M_t$. Thus, we have

$$\begin{aligned}
0 =& \sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\log(M_{t+k}/M_t)\left(\frac{1}{2}\right)\\
&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\,(-1)\log(M_{t+k}/M_t)\left(-\frac{1}{\tau}\right)\\
&+\sum_{k=1}^{\infty}\theta^k\beta^k\mathbb{E}_t\,(-1)\log(M_{t+k}/M_t)(1-\theta^{k+1})\left(\frac{\Gamma^*}{\tau}\right)\\
&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left[\theta^{k+1}\left(-\frac{p_t^{A*}-\hat{p}_{t-1}^B+\log(M_t/M_{t-1})}{\tau}\right)\right]\\
&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left[\sum_{k'=0}^{k}(1-\theta)\theta^{k-k'}\left(-\frac{p_t^{A*}-p_{t+k'}^{B*}-\log(M_{t+k'}/M_t)}{\tau}\right)\right]\\
&+\sum_{k=0}^{\infty}\theta^k\beta^k\left(pp_t^{A*}\right)\left(-\frac{1}{\tau}\right)\\
&+\sum_{k=1}^{\infty}\theta^k\beta^k\left(pp_t^{A*}\right)(1-\theta^{k+1})\left(\frac{\Gamma^*}{\tau}\right) && (8)\\
=&A_t+B_t, && (9)
\end{aligned}$$

where

$$
\begin{aligned}
A_t \equiv & \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \log(M_{t+k}/M_t) \left(\frac{1}{2}\right) \\
& + \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \,(-1) \log(M_{t+k}/M_t) \left(-\frac{1}{\tau}\right) \\
& + \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \,(-1) \log(M_{t+k}/M_t)(1-\theta^{k+1}) \left(\frac{\Gamma^*}{\tau}\right) \\
& + \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\theta^{k+1} \left(-\frac{p_t^{A*} - \hat{p}_{t-1}^B + \log(M_t/M_{t-1})}{\tau}\right)\right] \\
& + \sum_{k=0}^{\infty} \theta^k \beta^k \left(pp_t^{A*}\right) \left(-\frac{1}{\tau}\right) \\
& + \sum_{k=1}^{\infty} \theta^k \beta^k \left(pp_t^{A*}\right) (1-\theta^{k+1}) \left(\frac{\Gamma^*}{\tau}\right),
\end{aligned}
$$

$$
B_t \equiv \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\sum_{k'=0}^{k} (1-\theta)\theta^{k-k'} \left(-\frac{p_t^{A*} - p_{t+k'}^{B*} - \log(M_{t+k'}/M_t)}{\tau}\right)\right].
$$

Note that for $k \geq 1$, we have

$$
\mathbb{E}_t[\log(M_{t+k}/M_t)] = \sum_{k'=1}^{k} \mathbb{E}_t \varepsilon_{t+k'} = \rho(1-\rho^k)/(1-\rho) \cdot \varepsilon_t,
$$

$$
\sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \log(M_{t+k}/M_t) = \frac{\rho}{1-\rho} \left(\frac{\theta\beta}{1-\theta\beta} - \frac{\theta\beta\rho}{1-\theta\beta\rho}\right) \varepsilon_t.
$$

As for A_t, we have

$$
\begin{aligned}
A_t = & \frac{\rho}{1-\rho} \left(\frac{\theta\beta}{1-\theta\beta} - \frac{\theta\beta\rho}{1-\theta\beta\rho}\right) \varepsilon_t \left(\frac{1}{2}\right) \\
& + (-1) \frac{\rho}{1-\rho} \left(\frac{\theta\beta}{1-\theta\beta} - \frac{\theta\beta\rho}{1-\theta\beta\rho}\right) \varepsilon_t \left(-\frac{1}{\tau}\right) \\
& + (-1) \frac{\rho}{1-\rho} \left(\frac{\theta\beta}{1-\theta\beta} - \frac{\theta\beta\rho}{1-\theta\beta\rho}\right) \varepsilon_t \left(\frac{\Gamma^*}{\tau}\right) \\
& - (-1) \frac{\rho}{1-\rho} \theta \left(\frac{\theta^2\beta}{1-\theta^2\beta} - \frac{\theta^2\beta\rho}{1-\theta^2\beta\rho}\right) \varepsilon_t \left(\frac{\Gamma^*}{\tau}\right) \\
& + \frac{\theta}{1-\theta^2\beta} (p_t^{A*} - \hat{p}_{t-1}^B + \varepsilon_t) \left(-\frac{1}{\tau}\right) \\
& + \frac{1}{1-\theta\beta} pp_t^{A*} \left(-\frac{1}{\tau}\right) \\
& + \frac{\theta\beta}{1-\theta\beta} pp_t^{A*} \left(\frac{\Gamma^*}{\tau}\right) - \frac{\theta^3\beta}{1-\theta^2\beta} pp_t^{A*} \left(\frac{\Gamma^*}{\tau}\right)
\end{aligned}
\tag{10}
$$

As for B_t, we have

$$
\begin{aligned}
B_t &= \sum_{k=0}^{\infty} \theta^k \beta^k \left[(1-\theta^{k+1}) \left(-p_t^{A*}\right) \left(\frac{1}{\tau}\right) \right] \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\sum_{k'=0}^{k} (1-\theta)\theta^{k-k'} p_{t+k'}^{B*} \left(\frac{1}{\tau}\right) \right] \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \left[\sum_{k'=0}^{k} (1-\theta)\theta^{k-k'} \left(\frac{\rho(1-\rho^{k'}) \cdot \varepsilon_t}{1-\rho} \right) \left(\frac{1}{\tau}\right) \right] \\
&= \left(\frac{1}{1-\theta\beta} - \frac{\theta}{1-\theta^2\beta} \right) \left(-p_t^{A*}\right) \left(\frac{1}{\tau}\right) \\
&+ B_t' \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \left[1 - \theta^{k+1} - \frac{\rho^{k+1} - \theta^{k+1}}{\rho - \theta} (1-\theta) \right] \frac{\rho \varepsilon_t}{1-\rho} \left(\frac{1}{\tau}\right) \\
&= B_t' \\
&+ \theta \left(-p_t^{A*}\right) \left(\frac{1}{1-\theta\beta} - \frac{1}{1-\theta^2\beta} \right) \left(\frac{1}{\tau}\right) \\
&+ \left[\frac{1}{1-\theta\beta} - \frac{1-\theta}{\rho-\theta} \frac{\rho}{1-\theta\beta\rho} + \frac{1-\rho}{\rho-\theta} \frac{\theta}{1-\theta^2\beta} \right] \frac{\rho\varepsilon_t}{1-\rho} \left(\frac{1}{\tau}\right) \qquad (11)
\end{aligned}
$$

if $\rho \neq \theta$, where

$$
\begin{aligned}
B_t' &= \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\sum_{k'=0}^{k} (1-\theta)\theta^{k-k'} p_{t+k'}^{B*} \right] \left(\frac{1}{\tau}\right) \\
\mathbb{E}_t \left[B_{t+1}' \right] &= \sum_{k=1}^{\infty} \theta^{k-1} \beta^{k-1} \mathbb{E}_t \left[\sum_{k'=1}^{k} (1-\theta)\theta^{k-k'} p_{t+k'}^{B*} \right] \left(\frac{1}{\tau}\right) \\
B_t' &= \theta\beta \mathbb{E}_t \left[B_{t+1}' \right] + \sum_{k=0}^{\infty} \theta^k \beta^k (1-\theta) \theta^k \mathbb{E}_t \left[p_t^{B*} \right] \left(\frac{1}{\tau}\right) \\
&= \theta\beta \mathbb{E}_t \left[B_{t+1}' \right] + \frac{1-\theta}{1-\theta^2\beta} \mathbb{E}_t \left[p_t^{B*} \right] \left(\frac{1}{\tau}\right). \qquad (12)
\end{aligned}
$$

We futher have

$$
\begin{aligned}
B_t' &= \Lambda^B \hat{p}_{t-1}^A + \Lambda^{B*} \hat{p}_{t-1}^B + \Lambda^{B\varepsilon} \varepsilon_t \qquad (13) \\
\mathbb{E}_t \left[B_{t+1}' \right] &= \mathbb{E}_t \left[\Lambda^B \hat{p}_t^A + \Lambda^{B*} \hat{p}_t^B + \Lambda^{B\varepsilon} \varepsilon_{t+1} \right] \\
&= \Lambda^B p_t^{A*} + \Lambda^{B*} \left\{ \theta \left(\hat{p}_{t-1}^B - \varepsilon_t \right) + (1-\theta) p_t^{B*} \right\} + \Lambda^{B\varepsilon} \rho \varepsilon_t \\
&= \Lambda^B \left(\Gamma \hat{p}_{t-1}^A + \Gamma^* \hat{p}_{t-1}^B + \Gamma^\varepsilon \varepsilon_t \right) \\
&+ \Lambda^{B*} \left\{ \theta \left(\hat{p}_{t-1}^B - \varepsilon_t \right) + (1-\theta) \left(\Gamma \hat{p}_{t-1}^B + \Gamma^* \hat{p}_{t-1}^A + \Gamma^\varepsilon \varepsilon_t \right) \right\} \\
&+ \Lambda^{B\varepsilon} \rho \varepsilon_t \\
&= \left(\Lambda^B \Gamma + (1-\theta) \Lambda^{B*} \Gamma^* \right) \hat{p}_{t-1}^A \\
&+ \left(\Lambda^B \Gamma^* + \theta \Lambda^{B*} + (1-\theta) \Lambda^{B*} \Gamma \right) \hat{p}_{t-1}^B \\
&+ \left(\Lambda^B \Gamma^\varepsilon - \theta \Lambda^{B*} + (1-\theta) \Lambda^{B*} \Gamma^\varepsilon + \Lambda^{B\varepsilon} \rho \right) \varepsilon_t.
\end{aligned}
$$

It should be noted that the optimal prices are expressed in the following forms:

$$p_t^{A*} = \Gamma \hat{p}_{t-1}^A + \Gamma^* \hat{p}_{t-1}^B + \Gamma^\varepsilon \varepsilon_t \tag{14}$$

$$p_t^{B*} = \Gamma \hat{p}_{t-1}^B + \Gamma^* \hat{p}_{t-1}^A + \Gamma^\varepsilon \varepsilon_t, \tag{15}$$

$$\partial \log \bar{p}_{t+k}^B / \partial \log \bar{p}_t^A = \partial p_{t+k}^{B*} / \partial p_t^{A*} = \Gamma^*, \tag{16}$$

where Γ, Γ^*, and Γ^ε represent coefficients to be determined. They are determined by equation (8) or (9) by comparing coefficients on $\hat{p}_{t-1}^A$, $\hat{p}_{t-1}^B$, and ε_t, each. More precisely, we use equation (7) for p and equations (10) and (11) for A_t and B_t.

Inflation Dynamics Aggregate price index is given by

$$\log P_t = \int_0^1 \log p_t^j dj \tag{17}$$

for product line j. For each j, suppose the log-linearized prices set by firm A and B are given by $\hat{p}_t^A$ and $\hat{p}_t^B$, respectively. Then, consumers $x = \frac{1}{2} - \frac{\hat{p}_t^A - \hat{p}_t^B}{\tau}$ buy from firm A at $\hat{p}_t^A$ and $1 - x$ consumers buy from firm B at $\hat{p}_t^B$. Thus, the log-linearized price aggregated at the level of product line j is given by

$$\begin{aligned} & x\hat{p}_t^A + (1-x)\hat{p}_t^B \\ =& \left(\frac{1}{2} - \frac{\hat{p}_t^A - \hat{p}_t^B}{\tau} \right) \hat{p}_t^A + \left(\frac{1}{2} - \frac{\hat{p}_t^B - \hat{p}_t^A}{\tau} \right) \hat{p}_t^B \\ =& \frac{\hat{p}_t^A + \hat{p}_t^B}{2} - \frac{\left(\hat{p}_t^A - \hat{p}_t^B\right)^2}{\tau} \simeq \frac{\hat{p}_t^A + \hat{p}_t^B}{2}. \end{aligned}$$

Under symmetry, the log-linearized aggregate price becomes

$$\begin{aligned} \hat{P}_t &= \int_0^1 \left(\frac{\hat{p}_t^A + \hat{p}_t^B}{2} \right) dj \\ &= \theta \int_0^1 \left(\hat{p}_{t-1} - \varepsilon_t\right) dj + (1-\theta) \int_0^1 p_t^* dj \\ &= \theta \hat{P}_{t-1} - \theta \varepsilon_t + (1-\theta)\left(\Gamma \hat{p}_{t-1} + \Gamma^* \hat{p}_{t-1} + \Gamma^\varepsilon \varepsilon_t\right) \\ &= \kappa \hat{P}_{t-1} + \left\{(1-\theta)\Gamma^\varepsilon - \theta\right\} \varepsilon_t, \end{aligned} \tag{18}$$

where $\kappa \equiv \theta + (1-\theta)(\Gamma + \Gamma^*)$. For the inflation rate $\pi_t \equiv \log(P_t/P_{t-1}) \simeq \varepsilon_t + \hat{P}_t - \hat{P}_{t-1}$, we obtain

$$\begin{aligned} \pi_t - \varepsilon_t &= \kappa \left(\pi_{t-1} - \varepsilon_{t-1}\right) + \left\{(1-\theta)\Gamma^\varepsilon - \theta\right\} \left(\varepsilon_t - \varepsilon_{t-1}\right) \\ \pi_t &= \kappa \pi_{t-1} + (1-\theta)(1+\Gamma^\varepsilon)\varepsilon_t - \left\{\kappa + (1-\theta)\Gamma^\varepsilon - \theta\right\} \varepsilon_{t-1}. \end{aligned} \tag{19}$$

This suggests that inflation dynamics is influenced by Γ, Γ^*, and Γ^ε, which are, in turn, influenced by τ.

The above equation can be further transformed into

$$\pi_t = (\kappa + \rho)\pi_{t-1} - \kappa \rho \pi_{t-2} + (1-\theta)(1+\Gamma^\varepsilon)\mu_t - \left\{\kappa + (1-\theta)\Gamma^\varepsilon - \theta\right\} \mu_{t-1}. \tag{20}$$

Then, we obtain

$$\mu_t = \left(\pi_t - (\kappa+\rho)\pi_{t-1} + \kappa\rho\pi_{t-2}\right)/(1-\theta)/(1+\Gamma^\varepsilon) + \left\{\kappa + (1-\theta)\Gamma^\varepsilon - \theta\right\}/(1-\theta)/(1+\Gamma^\varepsilon) \cdot \mu_{t-1}$$

$$
\begin{aligned}
\pi_t &= (\kappa+\rho)\pi_{t-1} - \kappa\rho\pi_{t-2} + (1-\theta)(1+\Gamma^\varepsilon)\mu_t \\
&\quad - \{\kappa + (1-\theta)\Gamma^\varepsilon - \theta\}\{(\pi_{t-1} - (\kappa+\rho)\pi_{t-2} + \kappa\rho\pi_{t-3})/(1-\theta)/(1+\Gamma^\varepsilon) + \{\kappa + (1-\theta)\Gamma^\varepsilon - \theta\}/(1-\theta)/(1+\Gamma^\varepsilon)\cdot\mu_{t-2}\} \\
&= (1-\theta)(1+\Gamma^\varepsilon)\mu_t + \left(\kappa + \rho - \frac{\kappa + (1-\theta)\Gamma^\varepsilon - \theta}{(1-\theta)(1+\Gamma^\varepsilon)}\right)\pi_{t-1} + O(\pi_{t-2}),
\end{aligned} \tag{21}
$$

where $O(\pi_{t-2})$ represents the term consisting of π_{t-2-j} for $j = 0, 1, \cdots$.

Aggregate Output Aggregate output is given by $Y_t = M_t/P_t$. The log-linearization yields

$$
\hat{Y}_t = -\hat{P}_t. \tag{22}
$$

Welfare Welfare is expressed as

$$
\begin{aligned}
U &= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\log C_t - (L_t + \tau D_t)\right] \\
&= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\log(M_t/P_t) - M_t/P_t - \tau D_t\right].
\end{aligned}
$$

The first and second terms in welfare are approximated up to the second order as

$$
\begin{aligned}
\log(M_t/P_t) - M_t/P_t &= \log[(1+\tau/2)^{-1}e^{-\hat{P}_t}] - (1+\tau/2)^{-1}e^{-\hat{P}_t} \\
&= -\log(1+\tau/2) - (1+\tau/2)^{-1} - \hat{P}_t + (1+\tau/2)^{-1}(\hat{P}_t - \hat{P}_t^2/2) \\
&= -\log(1+\tau/2) - \frac{1}{1+\tau/2} - \frac{\tau/2}{1+\tau/2}\hat{P}_t - \frac{1/2}{1+\tau/2}\hat{P}_t^2.
\end{aligned} \tag{23}
$$

The third term in welfare, shopping distance D_t, is approximated up to the second order as

$$
\begin{aligned}
D_t &= \int_0^{\frac{1}{2} - \frac{\log p_t^A - \log p_t^B}{\tau}} x dx + \int_{\frac{1}{2} - \frac{\log p_t^A - \log p_t^B}{\tau}}^1 (1-x) dx \\
&= \frac{\left(\frac{1}{2} - \frac{\log p_t^A - \log p_t^B}{\tau}\right)^2}{2} + \frac{\left(\frac{1}{2} - \frac{\log p_t^B - \log p_t^A}{\tau}\right)^2}{2} \\
&= \frac{\left(\frac{1}{2} - \frac{\hat{p}_t^A - \hat{p}_t^B}{\tau}\right)^2}{2} + \frac{\left(\frac{1}{2} - \frac{\hat{p}_t^B - \hat{p}_t^A}{\tau}\right)^2}{2} \\
&= \frac{1}{4} + \left(\frac{\hat{p}_t^A - \hat{p}_t^B}{\tau}\right)^2.
\end{aligned} \tag{24}
$$

A.3 Comparison with a Dixit–Stiglitz Model

Consumption is aggregated following the Dixit-Stiglitz form of aggregation:

$$
C_t = \left\{\int_0^1 C_t(j)^{\frac{\sigma-1}{\sigma}} dj\right\}^{\frac{\sigma}{\sigma-1}}. \tag{25}
$$

This yields demand and price index given by $Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\sigma} Y_t$ and $P_t = \left\{\int_0^1 P_t(j)^{1-\sigma} dj\right\}^{\frac{1}{1-\sigma}}$, respectively, where $C_t(j) = Y_t(j)$.

Pricing under Price Stickiness Under Calvo-type price stickiness, firm j sets $\bar{p}_t$ to maximize

$$\max \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \beta^k \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \left(\bar{p}_t Y_{t+k}(j) - W_{t+k} Y_{t+k}(j)\right).$$

The first-order condition leads to

$$0 = \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} Y_{t+k} \bar{p}_t^{-\sigma-1} P_{t+k}^{\sigma} \left[(1-\sigma)\bar{p}_t + \sigma M_{t+k}\right].$$

In log-linearization, denoting $\bar{p}_t \equiv \frac{\sigma}{\sigma-1} M_t e^{p_t^*}$, we have

$$0 = \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[M_{t+k}/M_t - 1 - p_t^*\right].$$

$$\begin{aligned}
p_t^* &= (1-\theta\beta) \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[M_{t+k}/M_t - 1\right] \\
&= (1-\theta\beta) \sum_{k=0}^{\infty} \theta^k \beta^k \sum_{k'=1}^{k} \mathbb{E}_t \varepsilon_{t+k'} \\
&= (1-\theta\beta) \sum_{k=0}^{\infty} \theta^k \beta^k \sum_{k'=1}^{k} \rho^{k'} \varepsilon_t \\
&= (1-\theta\beta) \sum_{k=0}^{\infty} \theta^k \beta^k \frac{\rho(1-\rho^k)}{1-\rho} \varepsilon_t \\
&= \frac{\rho\theta\beta}{1-\rho\theta\beta} \varepsilon_t.
\end{aligned}$$

Aggregate price is

$$P_t^{1-\varepsilon} = (1-\theta)\bar{p}_t^{1-\sigma} + \theta P_{t-1}^{1-\sigma},$$

which becomes

$$\begin{aligned}
\left(M_t e^{\hat{P}_t}\right)^{1-\sigma} &= (1-\theta)\left(M_t e^{p_t^*}\right)^{1-\sigma} + \theta\left(M_{t-1} e^{\hat{P}_{t-1}}\right)^{1-\sigma} \\
\left(e^{\varepsilon_t + \hat{P}_t}\right)^{1-\sigma} &= (1-\theta)\left(e^{\varepsilon_t + p_t^*}\right)^{1-\sigma} + \theta\left(e^{\hat{P}_{t-1}}\right)^{1-\sigma} \\
\varepsilon_t + \hat{P}_t &= (1-\theta)(\varepsilon_t + p_t^*) + \theta \hat{P}_{t-1} \\
\hat{P}_t &= (1-\theta) p_t^* + \theta \hat{P}_{t-1} - \theta\varepsilon_t.
\end{aligned}$$

Thus, the aggregate inflation rate, $\pi_t \equiv \log(P_t/P_{t-1}) \simeq \varepsilon_t + \hat{P}_t - \hat{P}_{t-1}$, is given by

$$\begin{aligned}
\pi_t - \varepsilon_t &= (1-\theta)\left(p_t^* - p_{t-1}^*\right) + \theta(\pi_{t-1} - \varepsilon_{t-1}) - \theta(\varepsilon_t - \varepsilon_{t-1}) \\
\pi_t &= \theta\pi_{t-1} + \left(\frac{(1-\theta)\rho\theta\beta}{1-\rho\theta\beta} - \theta\right)(\varepsilon_t - \varepsilon_{t-1}) + \varepsilon_t - \theta\varepsilon_{t-1} \\
&= \theta\pi_{t-1} + \frac{1-\theta}{1-\rho\theta\beta}\varepsilon_t - \frac{(1-\theta)\rho\theta\beta}{1-\rho\theta\beta}\varepsilon_{t-1}.
\end{aligned} \tag{26}$$

The above equation can be further transformed into

$$\pi_t = (\theta+\rho)\pi_{t-1} - \theta\rho\pi_{t-2} + \frac{1-\theta}{1-\rho\theta\beta}\mu_t - \frac{(1-\theta)\rho\theta\beta}{1-\rho\theta\beta}\mu_{t-1}. \tag{27}$$

Then, we obtain

$$\mu_t = (\pi_t - (\theta+\rho)\pi_{t-1} + \theta\rho\pi_{t-2})\,\frac{1-\rho\theta\beta}{1-\theta} - \frac{1}{\rho\theta\beta}\mu_{t-1}$$

$$\begin{aligned}\pi_t &= (\theta+\rho)\pi_{t-1} - \theta\rho\pi_{t-2} + \frac{1-\theta}{1-\rho\theta\beta}\mu_t \\ &\quad - \frac{(1-\theta)\rho\theta\beta}{1-\rho\theta\beta}\left\{(\pi_{t-1} - (\theta+\rho)\pi_{t-2} + \theta\rho\pi_{t-3})\,\frac{1-\rho\theta\beta}{1-\theta} - \frac{1}{\rho\theta\beta}\mu_{t-2}\right\} \\ &= \frac{1-\theta}{1-\rho\theta\beta}\mu_t + (\theta+\rho-\rho\theta\beta)\,\pi_{t-1} + O(\pi_{t-2}),\end{aligned} \tag{28}$$

where $O(\pi_{t-2})$ represents the term consisting of π_{t-2-j} for $j = 0, 1, \cdots$.

B Pricing under Consumers' Unobservable Heterogeneity

B.1 Steady State without Price Stickiness

Pure Strategy The optimal price chosen by firms A and B is symmetric, $p^* = p^A = p^B$, which satisfies

$$p^* = \left\{1 + 1/2 \cdot (\mathbb{E}\,[1/\tau])^{-1}\right\} W, \tag{29}$$

where the harmonic mean of τ is given by

$$(\mathbb{E}\,[1/\tau])^{-1} = \frac{1}{\alpha(1/\tau_L) + (1-\alpha)(1/\tau_H)}. \tag{30}$$

Mixed Strategy (Regular and Sales) Suppose that firm B chooses mixed strategy, in which price is p_H^B with the probability of $1 - s^B$ and p_L^B otherwise ($p_H^B > p_L^B$). Firm A also chooses mixed strategy characterized by p_H^A, p_L^A, and s^A.

When $p^A = p_H^A$, firm A's expected profit is written as

$$\begin{aligned}\Pi^A(p_H^A) &= (1-s^B)\mathbb{E}\left[\left(1 - \frac{W}{p_H^A}\right)\left(\frac{1}{2} - \frac{\log p_H^A - \log p_H^B}{\tau}\right)\right] \\ &\quad + s^B(1-\alpha)\left(1 - \frac{W}{p_H^A}\right)\left(\frac{1}{2} - \frac{\log p_H^A - \log p_L^B}{\tau_H}\right).\end{aligned}$$

If firm B sets p_L^B, firm A earns zero sales from τ_L consumers. The first-order condition with respect to p_H^A yields

$$\begin{aligned}0 &= \frac{1-s}{2}W - (1-s)\,(p_H - W)\,\mathbb{E}\,[1/\tau] \\ &\quad + s(1-\alpha)W\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{\tau_H}\right) - s(1-\alpha)\,(p_H - W)\,\frac{1}{\tau_H}\end{aligned} \tag{31}$$

given symmetry.

When $p^A = p_L^A$, firm A's expected profit is

$$\begin{aligned}\Pi^A(p_L^A) = &(1-s^B)\left\{\alpha\left(1-\frac{W}{p_L^A}\right)+(1-\alpha)\left(1-\frac{W}{p_L^A}\right)\left(\frac{1}{2}-\frac{\log p_L^A-\log p_H^B}{\tau_H}\right)\right\}\\ &+s^B\mathbb{E}\left[\left(1-\frac{W}{p_L^A}\right)\left(\frac{1}{2}-\frac{\log p_L^A-\log p_L^B}{\tau}\right)\right].\end{aligned}$$

If firm B sets p_H^B, firm A earns unit sales from τ_L consumers. The first-order condition with respect to p_L^A yields

$$\begin{aligned}0 = &(1-s)\alpha W\\ &+(1-s)(1-\alpha)W\left(\frac{1}{2}-\frac{\log p_L-\log p_H}{\tau_H}\right)\\ &-(1-s)(1-\alpha)\left(p_L-W\right)\frac{1}{\tau_H}\\ &+sW\frac{1}{2}-s\left(p_L-W\right)\mathbb{E}\left[1/\tau\right].\end{aligned} \tag{32}$$

Furthermore, we should have indifference of $\Pi^A(p_H^A)=\Pi^A(p_L^A)$, which yields

$$\begin{aligned}&(1-s)\left(1-\frac{W}{p_H}\right)\frac{1}{2}+s(1-\alpha)\left(1-\frac{W}{p_H}\right)\left(\frac{1}{2}-\frac{\log p_H-\log p_L}{\tau_H}\right)\\ =&(1-s)\left\{\alpha\left(1-\frac{W}{p_L}\right)+(1-\alpha)\left(1-\frac{W}{p_L}\right)\left(\frac{1}{2}-\frac{\log p_L-\log p_H}{\tau_H}\right)\right\}\\ &+s\left(1-\frac{W}{p_L}\right)\frac{1}{2}.\end{aligned} \tag{33}$$

Equations (31) to (33) give the solutions for p_H, p_L, and s.

B.2 Pricing under Consumers' Unobservable Heterogeneity and Price Stickiness

Pure Strategy Under price stickiness, pure strategy is expressed by

$$p^* = 1+\frac{1}{2}\left(\mathbb{E}\left[1/\tau\right]\right)^{-1}\left(1-\frac{(1-\theta)(1+\theta-\theta^2\beta)}{1-\theta^2\beta}\theta\beta\Gamma^*\right)^{-1}. \tag{34}$$

Importantly, p^* increases by the term of Γ^* under price stickiness, which increases firm profit and decreases an incentive to deviate from this strategy.

Suppose that firm B follows a pure strategy by choosing p^*: that is, firm B sets p^* as long as firm A sets p^*, where p^* is the same as that obtained in equation (29). Then, if firm A sets p^*, the present value of profits is given by $(1-W/p^*)/2/(1-\theta\beta)$. Moreover, suppose that if firm A deviates from equilibrium pure strategy by choosing a higher price p^d, firm B sets the best response price of $p(p^d)$ as long as firm A survives and continues to set the same pricewhen firm B has a chance to reset its price. Suppose that $p(p^d)$ is close to p^d such that both firms attract both price-sensitive and insensistive consumers. In this case, the present value of profits becomes the max value of

$$\begin{aligned}&(1-\alpha)\left(1-\frac{W}{p^d}\right)\left(\frac{1}{2}-\frac{\log p^d-\log p^*}{\tau_H}\right)\frac{1}{1-\theta^2\beta}\\ &+\left(1-\frac{W}{p^d}\right)\left(\frac{1}{2}-\left(\mathbb{E}\left[1/\tau\right]\right)\left(\log p^d-\log p(p^d)\right)\right)\left(\frac{\theta\beta}{1-\theta\beta}-\frac{\theta^2\beta}{1-\theta^2\beta}\right)\end{aligned} \tag{35}$$

by choosing the optimal p^d. Thus, the condition for the pure strategy to hold is rewritten as

$$\begin{aligned}
&\left(1-\frac{W}{p^*}\right)\frac{1}{2}\frac{1}{1-\theta\beta} \\
\geq&(1-\alpha)\left(1-\frac{W}{p^d}\right)\left(\frac{1}{2}-\frac{\log p^d-\log p^*}{\tau_H}\right)\frac{1}{1-\theta^2\beta} \\
&+\left(1-\frac{W}{p^d}\right)\left(\frac{1}{2}-\left(\mathbb{E}\left[1/\tau\right]\right)\left(\log p^d-\log p(p^d)\right)\right)\left(\frac{\theta\beta}{1-\theta\beta}-\frac{\theta^2\beta}{1-\theta^2\beta}\right).
\end{aligned}$$

If $p(p^d) > p^*$, firm A earns a larger profit after firm B revises its price from p^* to $p(p^d)$ (i.e., the profit in the first term on the right-hand side is smaller than that in the second term on the right-hand side). Thus, the following condition,

$$\left(1-\frac{W}{p^*}\right)\frac{1}{2}>\left(1-\frac{W}{p^d}\right)\left(\frac{1}{2}-\left(\mathbb{E}\left[1/\tau\right]\right)\left(\log p^d-\log p(p^d)\right)\right), \tag{36}$$

serves as a sufficient condition for the pure strategy to hold, given $p(p^d) > p^*$. The right-hand side of the ineqality represents the profit that takes account of the effect that its own price reset (to p^d) affects its rival firm's price in the following periods under price stickiness. Note that p^* given by equation 34 is the optimal price that is determined to maximize its present-value profit by taking this effect into account. This is why p^* embeds the term Γ^* and is higher under sticky prices ($\theta > 0$) under flexible prices ($\theta = 0$). Furthermore, $p(p^d) > p^*$ because of $\Gamma^* > 0$. Thus, any deviation of price from p^* likely decreases firm profit, and thus, the above inequality is likely to hold.

Mixed Strategy Next, we consider the case of mixed strategy. When firm A has a chance to optimize higher price at t, it sets $\bar{p}^A_{H,t}$ to maximize

$$\max\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}}\left(1-\frac{W_{t+k}}{\bar{p}^A_t}\right)\left(\frac{1}{2}-\frac{\log\bar{p}^A_t-\log p^B_{t+k}}{\tau}\right)M_{t+k}, \tag{37}$$

given that firm A chooses the higher price rather than the lower price in the periods following price revision. Regarding the probability that firm B chooses the lower price at t, s_t, we need to differentiate two cases: one is when firm B revises the higher price ($s^r_t = s$) and the other is when firm B does not (s^n_t). Then, the above equation is rewritten as

$$\begin{aligned}
\max\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t&\left[\left(1-\frac{M_{t+k}}{\bar{p}^A_{H,t}}\right)(\theta s^n_{t+k}+(1-\theta)s)\left(\frac{1}{2}-\frac{\log\bar{p}^A_{H,t}-\log p^B_{L,t+k}}{\tau}\right)\right. \\
&+\left(1-\frac{M_{t+k}}{\bar{p}^A_{H,t}}\right)\theta(1-s^n_{t+k})\left(\frac{1}{2}-\frac{\log\bar{p}^A_{H,t}-\log p^B_{H,t+k-1}}{\tau}\right) \\
&\left.+\left(1-\frac{M_{t+k}}{\bar{p}^A_{H,t}}\right)(1-\theta)(1-s)\left(\frac{1}{2}-\frac{\log\bar{p}^A_{H,t}-\log\bar{p}^B_{H,t+k}}{\tau}\right)\right]\cdot\frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}}\frac{M_{t+k}}{M_t}.
\end{aligned}$$

Noting that $p^B_{H,t+k-1}$ equals $\bar{p}^B_{H,t+k-1}$ with the probability of $1-\theta$, $\bar{p}^B_{H,t+k-2}$ with the probability of $\theta(1-\theta)$,$\cdots$, $\bar{p}^B_{H,t}$ with the probability of $\theta^{k-1}(1-\theta)$, and $p^B_{H,t-1}$ with the probability of θ^k when

$k \geq 1$, we have

$$\max \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\left(1 - \frac{M_{t+k}}{\bar{p}^A_{H,t}}\right) (\theta s^n_{t+k} + (1-\theta)s) \left(\frac{1}{2} - \frac{\log \bar{p}^A_{H,t} - \log p^B_{L,t+k}}{\tau}\right)\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t}$$
$$+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\left(1 - \frac{M_{t+k}}{\bar{p}^A_{H,t}}\right) \theta(1 - s^n_{t+k})\theta^k \left(\frac{1}{2} - \frac{\log \bar{p}^A_{H,t} - \log p^B_{H,t-1}}{\tau}\right)\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t}$$
$$+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\left(1 - \frac{M_{t+k}}{\bar{p}^A_{H,t}}\right) \theta(1 - s^n_{t+k}) \sum_{k'=0}^{k-1} (1-\theta)\theta^{k-1-k'} \left(\frac{1}{2} - \frac{\log \bar{p}^A_{H,t} - \log \bar{p}^B_{H,t+k'}}{\tau}\right)\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t}$$
$$+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\left(1 - \frac{M_{t+k}}{\bar{p}^A_{H,t}}\right) (1-\theta)(1-s) \left(\frac{1}{2} - \frac{\log \bar{p}^A_{H,t} - \log \bar{p}^B_{H,t+k}}{\tau}\right)\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t}. \quad (38)$$

Furthermore, we should have indifference of profits from choosing the higher price and choosing the lower price when the higher price is not revised in period t, which is

$$\theta(1 - s^n_t) \left(1 - \frac{M_t}{p^A_{H,t-1}}\right) \mathbb{E} \left(\frac{1}{2} - \frac{\log p^A_{H,t-1} - \log p^B_{H,t-1}}{\tau}\right)$$
$$+ (1-\theta)(1-s) \left(1 - \frac{M_t}{p^A_{H,t-1}}\right) \mathbb{E} \left(\frac{1}{2} - \frac{\log p^A_{H,t-1} - \log \bar{p}^B_{H,t}}{\tau}\right)$$
$$+ (\theta s^n_t + (1-\theta)s)(1-\alpha) \left(1 - \frac{M_t}{p^A_{H,t-1}}\right) \left(\frac{1}{2} - \frac{\log \bar{p}^A_{H,t-1} - \log (p_L M_t)}{\tau_H}\right)$$
$$= \theta(1 - s^n_t) \left\{ \alpha \left(1 - \frac{1}{p_L}\right) + (1-\alpha)\left(1 - \frac{1}{p_L}\right) \left(\frac{1}{2} - \frac{\log (p_L M_t) - \log p^B_{H,t-1}}{\tau_H}\right)\right\}$$
$$+ (1-\theta)(1-s) \left\{ \alpha \left(1 - \frac{1}{p_L}\right) + (1-\alpha)\left(1 - \frac{1}{p_L}\right) \left(\frac{1}{2} - \frac{\log (p_L M_t) - \log \bar{p}^B_{H,t}}{\tau_H}\right)\right\}$$
$$+ (\theta s^n_t + (1-\theta)s) \left(1 - \frac{1}{p_L}\right) \frac{1}{2}. \quad (39)$$

The first-order condition for the optimal $\bar{p}^A_{H,t}$ is given by

$$
\begin{aligned}
0 = &\sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(\frac{1}{\bar{p}^A_{H,t}}\right)^2 M_{t+k} \left[(\theta s^n_{t+k} + (1-\theta)s) \left(\frac{1}{2} - \frac{\log\bar{p}^A_{H,t} - \log p^B_{L,t+k}}{\tau}\right)\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(\frac{1}{\bar{p}^A_{H,t}}\right)^2 M_{t+k} \left[(1-s^n_{t+k})\theta^{k+1} \left(\frac{1}{2} - \frac{\log\bar{p}^A_{H,t} - \log p^B_{H,t-1}}{\tau}\right)\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(\frac{1}{\bar{p}^A_{H,t}}\right)^2 M_{t+k} \left[(1-s^n_{t+k}) \sum_{k'=0}^{k-1} (1-\theta)\theta^{k-k'} \left(\frac{1}{2} - \frac{\log\bar{p}^A_{H,t} - \log\bar{p}^B_{H,t+k'}}{\tau}\right)\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(\frac{1}{\bar{p}^A_{H,t}}\right)^2 M_{t+k} \left[(1-\theta)(1-s) \left(\frac{1}{2} - \frac{\log\bar{p}^A_{H,t} - \log\bar{p}^B_{H,t+k}}{\tau}\right)\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(1 - \frac{M_{t+k}}{\bar{p}^A_{H,t}}\right) \\
&\quad \cdot \left[(\theta s^n_{t+k} + (1-\theta)s)(1-\alpha) \left(-\frac{1}{\tau_H \bar{p}^A_{H,t}}\right) + \{\theta(1-s^n_{t+k}) + (1-\theta)(1-s)\} \left(-\frac{1}{\tau \bar{p}^A_{H,t}}\right)\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(1 - \frac{M_{t+k}}{\bar{p}^A_{H,t}}\right) \left[(1-s^n_{t+k}) \sum_{k'=0}^{k-1} (1-\theta)\theta^{k-k'} \frac{\partial\log\bar{p}^B_{H,t+k'}/\partial\log\bar{p}^A_{H,t}}{\tau\bar{p}^A_{H,t}}\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(1 - \frac{M_{t+k}}{\bar{p}^A_{H,t}}\right) \left[(1-\theta)(1-s) \frac{\partial\log\bar{p}^B_{H,t+k}/\partial\log\bar{p}^A_{H,t}}{\tau\bar{p}^A_{H,t}}\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(1 - \frac{M_{t+k}}{\bar{p}^A_{H,t}}\right) \frac{\partial\log s^n_{t+k}/\partial\log\bar{p}^A_{H,t}}{\bar{p}^A_{H,t}/s^n_{t+k}} \\
&\quad \cdot \left[\theta \left(\frac{1}{2} - \frac{\log\bar{p}^A_{H,t} - \log p^B_{L,t+k}}{\tau}\right) - \theta^{k+1} \left(\frac{1}{2} - \frac{\log\bar{p}^A_{H,t} - \log p^B_{H,t-1}}{\tau}\right)\right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&- \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(1 - \frac{M_{t+k}}{\bar{p}^A_{H,t}}\right) \frac{\partial\log s^n_{t+k}/\partial\log\bar{p}^A_{H,t}}{\bar{p}^A_{H,t}/s^n_{t+k}} \sum_{k'=0}^{k-1} (1-\theta)\theta^{k-k'} \left(\frac{1}{2} - \frac{\log\bar{p}^A_{H,t} - \log\bar{p}^B_{H,t+k'}}{\tau}\right) \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t}.
\end{aligned}
$$

Let us define $p^A_{H,t} \equiv p_H M_t e^{\hat{p}^A_t}$,$\bar{p}^A_{H,t} \equiv p_H M_t e^{p^{A*}_t}$, and $s^n_t \equiv s e^{\hat{s}^n_t}$ as well as $\partial\log\bar{p}^B_{H,t+k}/\partial\log\bar{p}^A_{H,t} \equiv$

Γ^* and $\partial \text{log} s^n_{t+k} / \partial \text{log} \bar{p}^A_{H,t} \equiv \Lambda^{n*}$. Then, the log-linearization leads to

$$
\begin{aligned}
0 =& \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(\frac{1}{p_H e^{p_t^{A*}}} \right) \frac{M_{t+k}}{M_t} \left[(s + s\theta \hat{s}^n_{t+k})(1-\alpha) \left(\frac{1}{2} - \frac{\text{log}p_H - \text{log}p_L}{\tau_H} - \frac{p_t^{A*} - \log(M_{t+k}/M_t)}{\tau_H} \right) \right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(\frac{1}{p_H e^{p_t^{A*}}} \right) \frac{M_{t+k}}{M_t} \left[(1 - s - s\hat{s}^n_{t+k}) \theta^{k+1} \left(\frac{1}{2} - \frac{p_t^{A*} - \hat{p}^B_{t-1} + \log(M_t/M_{t-1})}{\tau} \right) \right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(\frac{1}{p_H e^{p_t^{A*}}} \right) \frac{M_{t+k}}{M_t} \left[(1 - s - s\hat{s}^n_{t+k}) \sum_{k'=0}^{k-1} (1-\theta)\theta^{k-k'} \left(\frac{1}{2} - \frac{p_t^{A*} - p^{B*}_{t+k'} - \log(M_{t+k'}/M_t)}{\tau} \right) \right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(\frac{1}{p_H e^{p_t^{A*}}} \right) \frac{M_{t+k}}{M_t} \left[(1-\theta)(1-s) \left(\frac{1}{2} - \frac{p_t^{A*} - p^{B*}_{t+k} - \log(M_{t+k}/M_t)}{\tau} \right) \right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(1 - \frac{M_{t+k}/M_t}{p_H e^{p_t^{A*}}} \right) \left[(s + s\theta \hat{s}^n_{t+k})(1-\alpha) \left(-\frac{1}{\tau_H} \right) + \{1 - s - s\theta \hat{s}^n_{t+k}\} \left(-\frac{1}{\tau} \right) \right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(1 - \frac{M_{t+k}/M_t}{p_H e^{p_t^{A*}}} \right) \left[(1 - s - s\hat{s}^n_{t+k}) \theta (1-\theta^k) \frac{\Gamma^*}{\tau} \right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(1 - \frac{M_{t+k}/M_t}{p_H e^{p_t^{A*}}} \right) \left[(1-\theta)(1-s) \frac{\Gamma^*}{\tau} \right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(1 - \frac{M_{t+k}/M_t}{p_H e^{p_t^{A*}}} \right) \Lambda^{n*} (s + s\hat{s}^n_{t+k}) \\
&\cdot \left[\theta(1-\alpha) \left(\frac{1}{2} - \frac{\text{log}p_H - \text{log}p_L}{\tau_H} - \frac{p_t^{A*} - \log(M_{t+k}/M_t)}{\tau_H} \right) - \theta^{k+1} \left(\frac{1}{2} - \frac{p_t^{A*} - \hat{p}^B_{t-1} + \log(M_t/M_{t-1})}{\tau} \right) \right] \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t} \\
&- \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(1 - \frac{M_{t+k}/M_t}{p_H e^{p_t^{A*}}} \right) \Lambda^{n*} (s + s\hat{s}^n_{t+k}) \sum_{k'=0}^{k-1} (1-\theta)\theta^{k-k'} \left(\frac{1}{2} - \frac{p_t^{A*} - p^{B*}_{t+k'} - \log(M_{t+k'}/M_t)}{\tau} \right) \cdot \frac{\Lambda_{t+k}}{\Lambda_t} \frac{P_t}{P_{t+k}} \frac{M_{t+k}}{M_t}.
\end{aligned}
$$

In the steady state, it becomes

$$
\begin{aligned}
0 =& \sum_{k=0}^{\infty} \theta^k \beta^k \left(\frac{1}{p_H} \right) \cdot \left\{ s(1-\alpha) \left(\frac{1}{2} - \frac{\text{log}p_H - \text{log}p_L}{\tau_H} \right) + (1-s) \left(\frac{1}{2} \right) \right\} \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \left(1 - \frac{1}{p_H} \right) \cdot \left[s(1-\alpha) \left(-\frac{1}{\tau_H} \right) + (1-s)\mathbb{E} \left(-\frac{1}{\tau} \right) \right] \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \left(1 - \frac{1}{p_H} \right) (1-s)(1-\theta^{k+1}) \mathbb{E} \left(\frac{\Gamma^*}{\tau} \right) \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \left(1 - \frac{1}{p_H} \right) s\Lambda^{n*} \left[\theta(1-\alpha) \left(\frac{1}{2} - \frac{\text{log}p_H - \text{log}p_L}{\tau_H} \right) - \frac{1}{2}\theta^{k+1} \right] \\
&- \sum_{k=1}^{\infty} \theta^k \beta^k \left(1 - \frac{1}{p_H} \right) s\Lambda^{n*} (1-\theta^{k+1})(1-s)\frac{1}{2},
\end{aligned}
$$

or

$$
\begin{aligned}
0 =& \left(\frac{1}{p_H} \right) \cdot \left\{ s(1-\alpha) \left(\frac{1}{2} - \frac{\text{log}p_H - \text{log}p_L}{\tau_H} \right) + (1-s) \left(\frac{1}{2} \right) \right\} \\
&+ \left(1 - \frac{1}{p_H} \right) \cdot \left[s(1-\alpha) \left(-\frac{1}{\tau_H} \right) + (1-s)\mathbb{E} \left(-\frac{1}{\tau} \right) \right] \\
&+ \left(1 - \frac{1}{p_H} \right) (1-s) \frac{1-\theta}{1-\theta^2\beta} \mathbb{E} \left(\frac{\Gamma^*}{\tau} \right) \\
&+ \theta\beta \left(1 - \frac{1}{p_H} \right) s\Lambda^{n*} \left[\theta(1-\alpha) \left(\frac{1}{2} - \frac{\text{log}p_H - \text{log}p_L}{\tau_H} \right) - \frac{1}{2} \frac{(1-\theta\beta)\theta^2}{1-\theta^2\beta} \right] \\
&- \theta\beta \left(1 - \frac{1}{p_H} \right) s\Lambda^{n*} (1-\theta) \frac{1+\theta-\theta^2\beta}{1-\theta^2\beta} (1-s)\frac{1}{2}.
\end{aligned} \tag{40}
$$

Because of the last three terms (if Γ^* or Λ^{n*} is nonzero), the steady state under nominal rigitidy is different from that without nominal rigditiy. Firms take account of the dynamic effect of its price on the rival firm's price in the following periods.

Note that the log-linearized deviation of the term $\frac{\Lambda_{t+k}}{\Lambda_t}\frac{P_t}{P_{t+k}}\frac{M_{t+k}}{M_t}$ disappears because of the steady state condition. The log-linearization proceeds as

$$\begin{aligned}
0 =& \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \frac{M_{t+k}}{M_t} \left[\left(s + s\theta \hat{s}^n_{t+k}\right)(1-\alpha) \left(\frac{1}{2} - \frac{\log p_H - \log p_L}{\tau_H} - \frac{p_t^{A*} - \log(M_{t+k}/M_t)}{\tau_H} \right) \right] \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \frac{M_{t+k}}{M_t} \left[(1 - s - s\hat{s}^n_{t+k})\theta^{k+1} \left(\frac{1}{2} - \frac{p_t^{A*} - \hat{p}^B_{t-1} + \log(M_t/M_{t-1})}{\tau} \right) \right] \\
&+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \frac{M_{t+k}}{M_t} \left[(1 - s - s\hat{s}^n_{t+k}) \sum_{k'=0}^{k-1} (1-\theta)\theta^{k-k'} \left(\frac{1}{2} - \frac{p_t^{A*} - p^{B*}_{t+k'} - \log(M_{t+k'}/M_t)}{\tau} \right) \right] \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \frac{M_{t+k}}{M_t} \left[(1-\theta)(1-s) \left(\frac{1}{2} - \frac{p_t^{A*} - p^{B*}_{t+k} - \log(M_{t+k}/M_t)}{\tau} \right) \right] \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(p_H + p_H p_t^{A*} - M_{t+k}/M_t \right) \left[\left(s + s\theta \hat{s}^n_{t+k}\right)(1-\alpha) \left(-\frac{1}{\tau_H} \right) + \left\{ 1 - s - s\theta \hat{s}^n_{t+k} \right\} \left(-\frac{1}{\tau} \right) \right] \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(p_H + p_H p_t^{A*} - M_{t+k}/M_t \right) \left[(1 - s - s\hat{s}^n_{t+k})\theta(1-\theta^k)\frac{\Gamma^*}{\tau} \right] \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(p_H + p_H p_t^{A*} - M_{t+k}/M_t \right) \left[(1-\theta)(1-s)\frac{\Gamma^*}{\tau} \right] \\
&+ \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(p_H + p_H p_t^{A*} - M_{t+k}/M_t \right) \Lambda^{n*} \left(s + s\hat{s}^n_{t+k}\right) \\
&\cdot \left[\theta(1-\alpha) \left(\frac{1}{2} - \frac{\log p_H - \log p_L}{\tau_H} - \frac{p_t^{A*} - \log(M_{t+k}/M_t)}{\tau_H} \right) - \theta^{k+1} \left(\frac{1}{2} - \frac{p_t^{A*} - \hat{p}^B_{t-1} + \log(M_t/M_{t-1})}{\tau} \right) \right] \\
&- \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(p_H + p_H p_t^{A*} - M_{t+k}/M_t \right) \Lambda^{n*} \left(s + s\hat{s}^n_{t+k}\right) \sum_{k'=0}^{k-1} (1-\theta)\theta^{k-k'} \left(\frac{1}{2} - \frac{p_t^{A*} - p^{B*}_{t+k'} - \log(M_{t+k'}/M_t)}{\tau} \right),
\end{aligned}$$

$$
\begin{aligned}
0 = &\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\log(M_{t+k}/M_t)\left[s(1-\alpha)\left(\frac{1}{2}-\frac{\log p_H-\log p_L}{\tau_H}\right)+(1-s)\left(\frac{1}{2}\right)\right]\\
&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\,(-1)\log(M_{t+k}/M_t)\left[s(1-\alpha)\left(-\frac{1}{\tau_H}\right)+(1-s)\mathbb{E}\left(-\frac{1}{\tau}\right)\right]\\
&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\,(-1)\log(M_{t+k}/M_t)(1-s)(1-\theta^{k+1})\mathbb{E}\left(\frac{\Gamma^*}{\tau}\right)\\
&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\,(-1)\log(M_{t+k}/M_t)s\Lambda^{n*}\left[\theta(1-\alpha)\left(\frac{1}{2}-\frac{\log p_H-\log p_L}{\tau_H}\right)-\frac{1}{2}\theta^{k+1}\right]\\
&+\sum_{k=1}^{\infty}\theta^k\beta^k\mathbb{E}_t\,(-1)\log(M_{t+k}/M_t)s\Lambda^{n*}\left(-\frac{1}{2}\theta(1-\theta^k)\right)\\
&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(s\theta\hat{s}^n_{t+k}\right)(1-\alpha)\left(\frac{1}{2}-\frac{\log p_H-\log p_L}{\tau_H}\right)\\
&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t(-s\hat{s}^n_{t+k})\theta\frac{1}{2}\\
&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\,(p_H-1)\left[\left(s\theta\hat{s}^n_{t+k}\right)(1-\alpha)\left(-\frac{1}{\tau_H}\right)+\{-s\theta\hat{s}^n_{t+k}\}\left(-\frac{1}{\tau}\right)\right]\\
&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\,(p_H-1)\,(-s\hat{s}^n_{t+k})\theta(1-\theta^k)\frac{\Gamma^*}{\tau}\\
&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\,(p_H-1)\,\Lambda^{n*}\left(s\hat{s}^n_{t+k}\right)\left[\theta(1-\alpha)\left(\frac{1}{2}-\frac{\log p_H-\log p_L}{\tau_H}\right)-\frac{1}{2}\theta^{k+1}\right]\\
&+\sum_{k=1}^{\infty}\theta^k\beta^k\mathbb{E}_t\,(p_H-1)\,\Lambda^{n*}\left(s\hat{s}^n_{t+k}\right)\left(-\frac{1}{2}\theta(1-\theta^k)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left[s(1-\alpha)\left(-\frac{p_t^{A*}-\log(M_{t+k}/M_t)}{\tau_H}\right)\right]\\
&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left[(1-s)\theta^{k+1}\left(-\frac{p_t^{A*}-\hat{p}^B_{t-1}+\log(M_t/M_{t-1})}{\tau}\right)\right]\\
&+\sum_{k=1}^{\infty}\theta^k\beta^k\mathbb{E}_t\left[(1-s)\sum_{k'=0}^{k-1}(1-\theta)\theta^{k-k'}\left(-\frac{p_t^{A*}-p^{B*}_{t+k'}-\log(M_{t+k'}/M_t)}{\tau}\right)\right]\\
&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left[(1-\theta)(1-s)\left(-\frac{p_t^{A*}-p^{B*}_{t+k}-\log(M_{t+k}/M_t)}{\tau}\right)\right]\\
&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(p_Hp_t^{A*}\right)\left[s(1-\alpha)\left(-\frac{1}{\tau_H}\right)+(1-s)\left(-\frac{1}{\tau}\right)\right]\\
&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(p_Hp_t^{A*}\right)(1-\theta^{k+1})\mathbb{E}\left(\frac{\Gamma^*}{\tau}\right)\\
&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(p_Hp_t^{A*}\right)s\Lambda^{n*}\left[\theta(1-\alpha)\left(\frac{1}{2}-\frac{\log p_H-\log p_L}{\tau_H}\right)-\frac{1}{2}\theta^{k+1}\right]\\
&+\sum_{k=1}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(p_Hp_t^{A*}\right)s\Lambda^{n*}\left(-\frac{1}{2}\theta(1-\theta^k)\right)\\
&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\,(p_H-1)\,s\Lambda^{n*}\left[\theta(1-\alpha)\left(-\frac{p_t^{A*}-\log(M_{t+k}/M_t)}{\tau_H}\right)-\theta^{k+1}\left(-\frac{p_t^{A*}-\hat{p}^B_{t-1}+\log(M_t/M_{t-1})}{\tau}\right)\right]\\
&+\sum_{k=1}^{\infty}\theta^k\beta^k\mathbb{E}_t\,(p_H-1)\,s\Lambda^{n*}\sum_{k'=0}^{k-1}(1-\theta)\theta^{k-k'}\left(\frac{p_t^{A*}-p^{B*}_{t+k'}-\log(M_{t+k'}/M_t)}{\tau}\right).
\end{aligned}
$$

Thus, we have

$$0 = A_t + B_t + C_t + D_t + E_t, \tag{41}$$

where

$$\begin{aligned}
A_t \equiv & \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \log(M_{t+k}/M_t) \left[s(1-\alpha) \left(\frac{1}{2} - \frac{\log p_H - \log p_L}{\tau_H} \right) + (1-s) \left(\frac{1}{2} \right) \right] \\
& + \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \,(-1) \log(M_{t+k}/M_t) \left[s(1-\alpha) \left(-\frac{1}{\tau_H} \right) + (1-s) \mathbb{E} \left(-\frac{1}{\tau} \right) \right] \\
& + \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \,(-1) \log(M_{t+k}/M_t)(1-s)(1-\theta^{k+1}) \mathbb{E} \left(\frac{\Gamma^*}{\tau} \right) \\
& + \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \,(-1) \log(M_{t+k}/M_t) s \Lambda^{n*} \theta (1-\alpha) \left(\frac{1}{2} - \frac{\log p_H - \log p_L}{\tau_H} \right) \\
& + \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \,(-1) \log(M_{t+k}/M_t) s \Lambda^{n*} \left(-\frac{1}{2} \theta \right) \\
& + \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[s(1-\alpha) \left(-\frac{p_t^{A*} - \log(M_{t+k}/M_t)}{\tau_H} \right) \right] \\
& + \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[(1-s) \theta^{k+1} \left(-\frac{p_t^{A*} - \hat{p}_{t-1}^B + \log(M_t/M_{t-1})}{\tau} \right) \right] \\
& + \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(p_H p_t^{A*} \right) \left[s(1-\alpha) \left(-\frac{1}{\tau_H} \right) + (1-s) \left(-\frac{1}{\tau} \right) \right] \\
& + \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(p_H p_t^{A*} \right) (1-\theta^{k+1}) \mathbb{E} \left(\frac{\Gamma^*}{\tau} \right) \\
& + \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(p_H p_t^{A*} \right) s \Lambda^{n*} \theta (1-\alpha) \left(\frac{1}{2} - \frac{\log p_H - \log p_L}{\tau_H} \right) \\
& + \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(p_H p_t^{A*} \right) s \Lambda^{n*} \left(-\frac{1}{2} \theta \right) \\
& + \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \,(p_H - 1) s \Lambda^{n*} \left[\theta (1-\alpha) \left(-\frac{p_t^{A*} - \log(M_{t+k}/M_t)}{\tau_H} \right) - \theta^{k+1} \left(-\frac{p_t^{A*} - \hat{p}_{t-1}^B + \log(M_t/M_{t-1})}{\tau} \right) \right],
\end{aligned}$$

$$\begin{aligned}
B_t \equiv & \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(s \theta \hat{s}_{t+k}^n \right) (1-\alpha) \left(\frac{1}{2} - \frac{\log p_H - \log p_L}{\tau_H} \right) \\
& + \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t (-s \hat{s}_{t+k}^n) \frac{1}{2} \theta \\
& + \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \,(p_H - 1) \left[(1-\alpha) \left(-\frac{1}{\tau_H} \right) + \frac{1}{\tau} \right] s \theta \hat{s}_{t+k}^n \\
& + \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \,(p_H - 1) (-s \hat{s}_{t+k}^n) \theta \frac{\Gamma^*}{\tau} \\
& + \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \,(p_H - 1) \Lambda^{n*} \left(s \hat{s}_{t+k}^n \right) \theta (1-\alpha) \left(\frac{1}{2} - \frac{\log p_H - \log p_L}{\tau_H} \right) \\
& + \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \,(p_H - 1) \Lambda^{n*} \left(s \hat{s}_{t+k}^n \right) \left(-\frac{1}{2} \theta \right),
\end{aligned}$$

$$C_t \equiv \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \,(p_H - 1) \,(-s \hat{s}_{t+k}^n)(-\theta^{k+1}) \frac{\Gamma^*}{\tau}$$

$$D_t \equiv \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[(1-\theta)(1-s) \left(-\frac{p_t^{A*} - p_{t+k}^{B*} - \log(M_{t+k}/M_t)}{\tau} \right) \right],$$

and

$$E_t \equiv \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[(1-s) \sum_{k'=0}^{k-1} (1-\theta)\theta^{k-k'} \left(-\frac{p_t^{A*} - p_{t+k'}^{B*} - \log(M_{t+k'}/M_t)}{\tau} \right) \right]$$
$$+ \sum_{k=1}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[(p_H - 1)\, s\Lambda^{n*} \sum_{k'=0}^{k-1} (1-\theta)\theta^{k-k'} \left(\frac{p_t^{A*} - p_{t+k'}^{B*} - \log(M_{t+k'}/M_t)}{\tau} \right) \right].$$

Note that for $k \geq 1$, we have

$$\mathbb{E}_t[\log(M_{t+k}/M_t)] = \sum_{k'=1}^{k} \mathbb{E}_t \varepsilon_{t+k'} = \rho(1-\rho^k)/(1-\rho) \cdot \varepsilon_t,$$

$$M(\varepsilon_t) \equiv \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \log(M_{t+k}/M_t) = \frac{\rho}{1-\rho} \left(\frac{\theta\beta}{1-\theta\beta} - \frac{\theta\beta\rho}{1-\theta\beta\rho} \right) \varepsilon_t.$$

As for A_t, we have

$$\begin{aligned}
A_t &= M(\varepsilon_t) \left[s(1-\alpha) \left(\frac{1}{2} - \frac{\log p_H - \log p_L}{\tau_H} \right) + (1-s) \left(\frac{1}{2} \right) \right] \\
&+ (-1)\, M(\varepsilon_t) \left[s(1-\alpha) \left(-\frac{1}{\tau_H} \right) + (1-s)\mathbb{E} \left(-\frac{1}{\tau} \right) \right] \\
&+ (-1)\, M(\varepsilon_t)(1-s)\mathbb{E} \left(\frac{\Gamma^*}{\tau} \right) \\
&- (-1)\, \frac{\rho}{1-\rho}\theta \left(\frac{\theta^2\beta}{1-\theta^2\beta} - \frac{\theta^2\beta\rho}{1-\theta^2\beta\rho} \right) \varepsilon_t (1-s)\mathbb{E} \left(\frac{\Gamma^*}{\tau} \right) \\
&+ (-1)\, M(\varepsilon_t) s\Lambda^{n*}\theta(1-\alpha) \left(\frac{1}{2} - \frac{\log p_H - \log p_L}{\tau_H} \right) \\
&+ (-1)\, M(\varepsilon_t) s\Lambda^{n*} \left(-\frac{1}{2}\theta \right) \\
&+ \frac{1}{1-\theta\beta} s(1-\alpha) \left(-\frac{p_t^{A*}}{\tau_H} \right) + M(\varepsilon_t) s(1-\alpha) \frac{1}{\tau_H} \\
&+ \frac{\theta}{1-\theta^2\beta} (1-s)(p_t^{A*} - \hat{p}_{t-1}^{B} + \varepsilon_t)\mathbb{E} \left(-\frac{1}{\tau} \right) \\
&+ \frac{1}{1-\theta\beta} p_H p_t^{A*} \left[s(1-\alpha) \left(-\frac{1}{\tau_H} \right) + (1-s)\mathbb{E} \left(-\frac{1}{\tau} \right) \right] \\
&+ \frac{1}{1-\theta\beta} p_H p_t^{A*} \mathbb{E} \left(\frac{\Gamma^*}{\tau} \right) - \frac{\theta}{1-\theta^2\beta} p_H p_t^{A*} \mathbb{E} \left(\frac{\Gamma^*}{\tau} \right) \\
&+ \frac{1}{1-\theta\beta} p_H p_t^{A*} s\Lambda^{n*}\theta(1-\alpha) \left(\frac{1}{2} - \frac{\log p_H - \log p_L}{\tau_H} \right) \\
&+ \frac{1}{1-\theta\beta} p_H p_t^{A*} s\Lambda^{n*} \left(-\frac{1}{2}\theta \right) \\
&+ \frac{1}{1-\theta\beta} (p_H - 1)\, s\Lambda^{n*}\theta(1-\alpha) \left(-\frac{p_t^{A*}}{\tau_H} \right) \\
&+ (p_H - 1)\, s\Lambda^{n*}\theta(1-\alpha) M(\varepsilon_t) \left(\frac{1}{\tau_H} \right) \\
&+ \frac{\theta}{1-\theta^2\beta} (p_H - 1)\, s\Lambda^{n*}(p_t^{A*} - \hat{p}_{t-1}^{B} + \varepsilon_t)\mathbb{E} \left(\frac{1}{\tau} \right). \qquad (42)
\end{aligned}$$

As for B_t, we have

$$\begin{aligned}
\theta\beta\mathbb{E}_t B_{t+1} = &\sum_{k=1}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(s\theta\hat{s}^n_{t+k}\right)(1-\alpha)\left(\frac{1}{2}-\frac{\log p_H-\log p_L}{\tau_H}\right)\\
&+\sum_{k=1}^{\infty}\theta^k\beta^k\mathbb{E}_t(-s\hat{s}^n_{t+k})\frac{1}{2}\theta\\
&+\sum_{k=1}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(p_H-1\right)\left[(1-\alpha)\left(-\frac{1}{\tau_H}\right)+\frac{1}{\tau}\right]s\theta\hat{s}^n_{t+k}\\
&+\sum_{k=1}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(p_H-1\right)(-s\hat{s}^n_{t+k})\theta\frac{\Gamma^*}{\tau}\\
&+\sum_{k=2}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(p_H-1\right)\Lambda^{n*}\left(s\hat{s}^n_{t+k}\right)\theta(1-\alpha)\left(\frac{1}{2}-\frac{\log p_H-\log p_L}{\tau_H}\right)\\
&+\sum_{k=1}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(p_H-1\right)\Lambda^{n*}\left(s\hat{s}^n_{t+k}\right)\left(-\frac{1}{2}\theta\right),
\end{aligned}$$

$$\begin{aligned}
B_t = &\,\theta\beta\mathbb{E}_t B_{t+1}\\
&+\left(s\theta\hat{s}^n_t\right)(1-\alpha)\left(\frac{1}{2}-\frac{\log p_H-\log p_L}{\tau_H}\right)\\
&+(-s\hat{s}^n_t)\frac{1}{2}\theta\\
&+\left(p_H-1\right)\left[(1-\alpha)\left(-\frac{1}{\tau_H}\right)+\mathbb{E}\left(\frac{1}{\tau}\right)\right]s\theta\hat{s}^n_t\\
&+\left(p_H-1\right)(-s\hat{s}^n_{t'})\theta\Gamma^*\mathbb{E}\left(\frac{1}{\tau}\right)\\
&+\theta\beta\left(p_H-1\right)\Lambda^{n*}s\mathbb{E}_t\left(\hat{s}^n_{t+1}\right)\theta(1-\alpha)\left(\frac{1}{2}-\frac{\log p_H-\log p_L}{\tau_H}\right)\\
&+\left(p_H-1\right)\Lambda^{n*}s\hat{s}^n_t\left(-\frac{1}{2}\theta\right).
\end{aligned} \tag{43}$$

Similarly, C_t and D_t are given by

$$C_t = \theta^2\beta\mathbb{E}_t C_{t+1} + \left(p_H-1\right)(s\hat{s}^n_t)\theta\Gamma^*\mathbb{E}\left(\frac{1}{\tau}\right). \tag{44}$$

$$\begin{aligned}
D_t \equiv &\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left[(1-\theta)(1-s)\left(-\frac{p^{A*}_t-p^{B*}_{t+k}-\log(M_{t+k}/M_t)}{\tau}\right)\right]\\
= &\,(1-\theta)(1-s)\left[\frac{1}{1-\theta\beta}\left(-p^{A*}_t\right)+\frac{\rho}{1-\rho}\left(\frac{\theta\beta}{1-\theta\beta}-\frac{\theta\beta\rho}{1-\theta\beta\rho}\right)\varepsilon_t\right]\mathbb{E}\left(\frac{1}{\tau}\right)\\
&+\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left[(1-\theta)(1-s)\left(\frac{p^{B*}_{t+k}}{\tau}\right)\right]\\
= &\,(1-\theta)(1-s)\left[\frac{1}{1-\theta\beta}\left(-p^{A*}_t\right)+\frac{\rho}{1-\rho}\left(\frac{\theta\beta}{1-\theta\beta}-\frac{\theta\beta\rho}{1-\theta\beta\rho}\right)\varepsilon_t\right]\mathbb{E}\left(\frac{1}{\tau}\right)+D'_t,
\end{aligned} \tag{45}$$

where

$$D'_t = \theta\beta\mathbb{E}_t D'_{t+1} + (1-\theta)(1-s)\mathbb{E}_t\left[p^{B*}_t\right]\mathbb{E}\left(\frac{1}{\tau}\right). \tag{46}$$

As for E_t, we have

$$
\begin{aligned}
\frac{E_t}{\{(1-s)+(p_H-1)\,s\Lambda^{n*}\}\,\mathbb{E}\left(\frac{1}{\tau}\right)} &= \sum_{k=1}^{\infty}\theta^k\beta^k\left[\theta(1-\theta^k)\left(-p_t^{A*}\right)\right] \\
&+ \sum_{k=1}^{\infty}\theta^k\beta^k\mathbb{E}_t\left[\sum_{k'=0}^{k-1}(1-\theta)\theta^{k-k'}p_{t+k'}^{B*}\right] \\
&+ \sum_{k=1}^{\infty}\theta^k\beta^k\left[\sum_{k'=0}^{k-1}(1-\theta)\theta^{k-k'}\left(\frac{\rho(1-\rho^{k'})\cdot\varepsilon_t}{1-\rho}\right)\right] \\
&= \theta\left(-p_t^{A*}\right)\left(\frac{\theta\beta}{1-\theta\beta}-\frac{\theta^2\beta}{1-\theta^2\beta}\right) \\
&+ E_t' \\
&+ \theta\sum_{k=1}^{\infty}\theta^k\beta^k\left[1-\theta^k-\frac{\rho^k-\theta^k}{\rho-\theta}(1-\theta)\right]\frac{\rho\varepsilon_t}{1-\rho} \\
&= E_t' \\
&+ \left(-p_t^{A*}\right)\left(\frac{\theta^2\beta}{1-\theta\beta}-\frac{\theta^3\beta}{1-\theta^2\beta}\right) \\
&+ \theta\left[\frac{\theta\beta}{1-\theta\beta}-\frac{1-\theta}{\rho-\theta}\frac{\theta\beta\rho}{1-\theta\beta\rho}+\frac{1-\rho}{\rho-\theta}\frac{\theta^2\beta}{1-\theta^2\beta}\right]\frac{\rho\varepsilon_t}{1-\rho}, \qquad (47)
\end{aligned}
$$

where

$$
\begin{aligned}
E_t' &= \sum_{k=1}^{\infty}\theta^k\beta^k\mathbb{E}_t\left[\sum_{k'=0}^{k-1}(1-\theta)\theta^{k-k'}p_{t+k'}^{B*}\right] \\
\mathbb{E}_t\left[E_{t+1}'\right] &= \sum_{k=2}^{\infty}\theta^{k-1}\beta^{k-1}\mathbb{E}_t\left[\sum_{k'=1}^{k-1}(1-\theta)\theta^{k-k'}p_{t+k'}^{B*}\right] \\
E_t' &= \theta\beta\mathbb{E}_t\left[E_{t+1}'\right]+\sum_{k=1}^{\infty}\theta^k\beta^k(1-\theta)\theta^k\mathbb{E}_t\left[p_t^{B*}\right] \\
&= \theta\beta\mathbb{E}_t\left[E_{t+1}'\right]+\frac{(1-\theta)\theta^2\beta}{1-\theta^2\beta}\mathbb{E}_t\left[p_t^{B*}\right]. \qquad (48)
\end{aligned}
$$

The log-linearization of the profit indifference equation yields

$$
\begin{aligned}
&\theta(1-s-s\hat{s}_t^n)\left(1-\frac{1}{p_H}+\frac{\hat{p}_{t-1}^A-\varepsilon_t}{p_H}\right)\mathbb{E}\left(\frac{1}{2}-\frac{\hat{p}_{t-1}^A-\hat{p}_{t-1}^B}{\tau}\right) \\
&+(1-\theta)(1-s)\left(1-\frac{1}{p_H}+\frac{\hat{p}_{t-1}^A-\varepsilon_t}{p_H}\right)\mathbb{E}\left(\frac{1}{2}-\frac{\hat{p}_{t-1}^A-p_t^{B*}-\varepsilon_t}{\tau}\right) \\
&+s(1+\theta\hat{s}_t^n)(1-\alpha)\left(1-\frac{1}{p_H}+\frac{\hat{p}_{t-1}^A-\varepsilon_t}{p_H}\right)\mathbb{E}_t\left(\frac{1}{2}-\frac{\log p_H-\log p_L}{\tau_H}-\frac{\hat{p}_{t-1}^A-\varepsilon_t}{\tau_H}\right) \\
=&\theta(1-s-s\hat{s}_t^n)\left\{\alpha\left(1-\frac{1}{p_L}\right)+(1-\alpha)\left(1-\frac{1}{p_L}\right)\mathbb{E}_t\left(\frac{1}{2}-\frac{\log p_L-\log p_H}{\tau_H}+\frac{\hat{p}_{t-1}^B-\varepsilon_t}{\tau_H}\right)\right\} \\
&+(1-\theta)(1-s)\left\{\alpha\left(1-\frac{1}{p_L}\right)+(1-\alpha)\left(1-\frac{1}{p_L}\right)\mathbb{E}_t\left(\frac{1}{2}-\frac{\log p_L-\log p_H}{\tau_H}+\frac{p_t^{B*}}{\tau_H}\right)\right\} \\
&+s(1+\theta\hat{s}_t^n)\left(1-\frac{1}{p_L}\right)\frac{1}{2}.
\end{aligned}
$$

$$
\begin{aligned}
&-s\theta\hat{s}_t^n\left(1-\frac{1}{p_H}\right)\frac{1}{2}+(1-s)\left(\frac{\hat{p}_{t-1}^A-\varepsilon_t}{p_H}\right)\frac{1}{2}\\
&+(1-s)\left(1-\frac{1}{p_H}\right)\mathbb{E}\left(\frac{-\hat{p}_{t-1}^A+\theta\hat{p}_{t-1}^B+(1-\theta)p_t^{B*}+(1-\theta)\varepsilon_t}{\tau}\right)\\
&+s\theta\hat{s}_t^n(1-\alpha)\left(1-\frac{1}{p_H}\right)\left(\frac{1}{2}-\frac{\log p_H-\log p_L}{\tau_H}\right)\\
&+s(1-\alpha)\left(\frac{\hat{p}_{t-1}^A-\varepsilon_t}{p_H}\right)\left(\frac{1}{2}-\frac{\log p_H-\log p_L}{\tau_H}\right)\\
&+s(1-\alpha)\left(1-\frac{1}{p_H}\right)\left(-\frac{\hat{p}_{t-1}^A-\varepsilon_t}{\tau_H}\right)\\
&=-s\theta\hat{s}_t^n\left\{\alpha\left(1-\frac{1}{p_L}\right)+(1-\alpha)\left(1-\frac{1}{p_L}\right)\left(\frac{1}{2}-\frac{\log p_L-\log p_H}{\tau_H}\right)\right\}\\
&+(1-s)(1-\alpha)\left(1-\frac{1}{p_L}\right)\frac{\theta\hat{p}_{t-1}^B-\theta\varepsilon_t+(1-\theta)p_t^{B*}}{\tau_H}\\
&+s\theta\hat{s}_t^n\left(1-\frac{1}{p_L}\right)\frac{1}{2},
\end{aligned}
$$

$$
\begin{aligned}
&s\theta\hat{s}_t^n\left(\frac{1}{p_L}-\frac{1}{p_H}\right)\left\{\frac{1}{2}-\left(\frac{1}{2}-\frac{\log p_L-\log p_H}{\tau_H}\right)-\alpha\left(\frac{1}{2}-\frac{\log p_H-\log p_L}{\tau_H}\right)\right\}\\
&+\left(\frac{\hat{p}_{t-1}^A-\varepsilon_t}{p_H}\right)\left(\frac{1}{2}(1-s)+s(1-\alpha)\left(\frac{1}{2}-\frac{\log p_H-\log p_L}{\tau_H}\right)\right)\\
&+(1-s)\left(1-\frac{1}{p_H}\right)\mathbb{E}\left(\frac{-\hat{p}_{t-1}^A+\theta\hat{p}_{t-1}^B+(1-\theta)p_t^{B*}+(1-\theta)\varepsilon_t}{\tau}\right)\\
&+s(1-\alpha)\left(1-\frac{1}{p_H}\right)\left(-\frac{\hat{p}_{t-1}^A-\varepsilon_t}{\tau_H}\right)\\
&=(1-s)(1-\alpha)\left(1-\frac{1}{p_L}\right)\frac{\theta\hat{p}_{t-1}^B-\theta\varepsilon_t+(1-\theta)p_t^{B*}}{\tau_H}.
\end{aligned}
\tag{49}
$$

Equations (32), (33), (40), (41), and (49) give solutions for p_H, p_L, s, p_t^{A*} (p_t^{B*}), and $\hat{s}_t^n$, where A_t to E_t are given by equations (42) to (47) and

$$p_t^{A*}=\Gamma\hat{p}_{t-1}^A+\Gamma^*\hat{p}_{t-1}^B+\Gamma^\varepsilon\varepsilon_t \tag{50}$$

$$p_t^{B*}=\Gamma\hat{p}_{t-1}^B+\Gamma^*\hat{p}_{t-1}^A+\Gamma^\varepsilon\varepsilon_t \tag{51}$$

$$
\begin{aligned}
\mathbb{E}_tp_{t+1}^{B*}&=\mathbb{E}_t\left[\Gamma\hat{p}_t^B+\Gamma^*\hat{p}_t^A+\Gamma^\varepsilon\varepsilon_{t+1}\right]\\
&=\theta\Gamma\left(\hat{p}_{t-1}^B-\varepsilon_t\right)+(1-\theta)\Gamma p_t^{B*}+\Gamma^*p_t^{A*}+\rho\Gamma^\varepsilon\varepsilon_t\\
&=\{\theta\Gamma+(1-\theta)\Gamma^2+\Gamma^{*2}\}\hat{p}_{t-1}^B+(2-\theta)\Gamma\Gamma^*\hat{p}_{t-1}^A+\{-\theta\Gamma+(1-\theta)\Gamma\Gamma^\varepsilon+\Gamma^*\Gamma^\varepsilon+\rho\Gamma^\varepsilon\}\varepsilon_t
\end{aligned}
\tag{52}
$$

$$\partial\log\bar{p}_{H,t+1}^B/\partial\log\bar{p}_{H,t}^A=\partial p_{t+1}^{B*}/\partial p_t^{A*}=\Gamma^*, \tag{53}$$

and

$$\hat{s}_t^n\equiv\mathbb{E}_t\hat{s}_t^{n,B}=\Lambda^n\hat{p}_{t-1}^B+\Lambda^{n*}\hat{p}_{t-1}^A+\Lambda^{n\varepsilon}\varepsilon_t \tag{54}$$

$$
\begin{aligned}
\mathbb{E}_t\hat{s}_{t+1}^n&=\mathbb{E}_t\left[\Lambda^n\hat{p}_t^B+\Lambda^{n*}\hat{p}_t^A+\Lambda^{n\varepsilon}\varepsilon_{t+1}\right]\\
&=\theta\Lambda^n\left(\hat{p}_{t-1}^B-\varepsilon_t\right)+(1-\theta)\Lambda^np_t^{B*}+\Lambda^{n*}p_t^{A*}+\rho\Lambda^{n\varepsilon}\varepsilon_t\\
&=\{\theta\Lambda^n+(1-\theta)\Lambda^n\Gamma+\Lambda^{n*}\Gamma^*\}\hat{p}_{t-1}^B+\{(1-\theta)\Lambda^n\Gamma^*+\Lambda^{n*}\Gamma\}\hat{p}_{t-1}^A+\{-\theta\Lambda^n+(1-\theta)\Lambda^n\Gamma^\varepsilon+\Lambda^{n*}\Gamma^\varepsilon+\rho\Lambda^{n\varepsilon}\}\varepsilon_t,
\end{aligned}
\tag{55}
$$

$$\partial\log s_{t+k}^n/\partial\log\bar{p}_{H,t}^A\equiv\Lambda^{n*}. \tag{56}$$

Aggregate Price and Output Aggregate price index is written as

$$
\begin{aligned}
\log P_t &= \int_0^1 \log p_t^j dj \\
&= (1-\theta)(1-s)\log \bar{p}_t \cdot x_1 \\
&+ ((1-\theta)s + \theta s_t^n)\log p_{L,t} \cdot x_2 \\
&+ \theta(1-s_t^n)\log p_{H,t-1} \cdot x_3,
\end{aligned}
$$

where

$$
x_1 = (1-\theta)(1-s)\frac{1}{2} + ((1-\theta)s + \theta s_t^n)(1-\alpha)\left(\frac{1}{2} - \frac{\log \bar{p}_t - \log p_{L,t}}{\tau_H}\right) + \theta(1-s_t^n)\left(\frac{1}{2} - (\log \bar{p}_t - \log p_{H,t-1})\,\mathbb{E}\left(\frac{1}{\tau}\right)\right)
$$

$$
\begin{aligned}
x_2 = &(1-\theta)(1-s)\left(\alpha + (1-\alpha)\left(\frac{1}{2} - \frac{\log p_{L,t} - \log \bar{p}_t}{\tau_H}\right)\right) \\
&+ ((1-\theta)s + \theta s_t^n)\frac{1}{2} + \theta(1-s_t^n)\left(\alpha + (1-\alpha)\left(\frac{1}{2} - \frac{\log p_{L,t} - \log p_{H,t-1}}{\tau_H}\right)\right)
\end{aligned}
$$

$$
x_3 = (1-\theta)(1-s)\left(\frac{1}{2} - (\log p_{H,t-1} - \log \bar{p}_t)\,\mathbb{E}\left(\frac{1}{\tau}\right)\right) + ((1-\theta)s + \theta s_t^n)(1-\alpha)\left(\frac{1}{2} - \frac{\log p_{H,t-1} - \log p_{L,t}}{\tau_H}\right) + \theta(1-s_t^n)\frac{1}{2}.
$$

Thus, the log-linearized aggregate price divided by M_t, which equals minus log-linearized aggregate output, is given by

$$
\begin{aligned}
-\log Y_t = \log P_t - \log M_t &= (1-\theta)(1-s)(\log p_H + p_t^*) \cdot x_1 \\
&+ s(1+\theta \hat{s}_t^n)(\log p_L) \cdot x_2 \\
&+ \theta(1-s-s\hat{s}_t^n)(\log p_H - \varepsilon_t + \hat{p}_{t-1}) \cdot x_3,
\end{aligned}
$$

where

$$
\begin{aligned}
x_1 &= (1-\theta)(1-s)\frac{1}{2} + s(1+\theta\hat{s}_t^n)(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{\tau_H} - \frac{p_t^*}{\tau_H}\right) \\
&+ \theta(1-s-s\hat{s}_t^n)\left(\frac{1}{2} - (p_t^* - \hat{p}_{t-1} + \varepsilon_t)\,\mathbb{E}\left(\frac{1}{\tau}\right)\right) \\
&= (1-s)\frac{1}{2} + s(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{\tau_H}\right) \\
&+ \theta s\hat{s}_t^n(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{\tau_H}\right) - \theta s\hat{s}_t^n\frac{1}{2} - s(1-\alpha)\frac{p_t^*}{\tau_H} + \theta(1-s)(-p_t^* + \hat{p}_{t-1} - \varepsilon_t)\,\mathbb{E}\left(\frac{1}{\tau}\right)
\end{aligned}
$$

$$
\begin{aligned}
x_2 &= (1-\theta)(1-s)\left(\alpha + (1-\alpha)\left(\frac{1}{2} - \frac{\log p_L - \log p_H}{\tau_H} + \frac{p_t^*}{\tau_H}\right)\right) + s(1+\theta\hat{s}_t^n)\frac{1}{2} \\
&+ \theta(1-s-s\hat{s}_t^n)\left(\alpha + (1-\alpha)\left(\frac{1}{2} - \frac{\log p_L - \log p_H}{\tau_H} + \frac{\hat{p}_{t-1} - \varepsilon_t}{\tau_H}\right)\right) \\
&= (1-s)\left(\alpha + (1-\alpha)\left(\frac{1}{2} - \frac{\log p_L - \log p_H}{\tau_H}\right)\right) + s\frac{1}{2} \\
&+ (1-\theta)(1-s)(1-\alpha)\frac{p_t^*}{\tau_H} + \theta s\hat{s}_t^n\frac{1}{2} + \theta(-s\hat{s}_t^n)\left(\alpha + (1-\alpha)\left(\frac{1}{2} - \frac{\log p_L - \log p_H}{\tau_H}\right)\right) + \theta(1-s)(1-\alpha)\left(\frac{\hat{p}_{t-1} - \varepsilon_t}{\tau_H}\right)
\end{aligned}
$$

$$
\begin{aligned}
x_3 &= (1-\theta)(1-s)\left(\frac{1}{2} - (\hat{p}_{t-1} - p_t^* - \varepsilon_t)\,\mathbb{E}\left(\frac{1}{\tau}\right)\right) + s(1+\theta\hat{s}_t^n)(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{\tau_H} - \frac{\hat{p}_{t-1} - \varepsilon_t}{\tau_H}\right) + \theta(1-s-s\hat{s}_t^n)\frac{1}{2} \\
&= (1-s)\frac{1}{2} + s(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{\tau_H}\right) \\
&- (1-\theta)(1-s)(\hat{p}_{t-1} - p_t^* - \varepsilon_t)\,\mathbb{E}\left(\frac{1}{\tau}\right) + \theta s\hat{s}_t^n(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{\tau_H}\right) + s(1-\alpha)\left(-\frac{\hat{p}_{t-1} - \varepsilon_t}{\tau_H}\right) - \theta s\hat{s}_t^n\frac{1}{2}.
\end{aligned}
$$

Thus, we have

$$\log P_t - \log M_t = (1-s)(\log p_H) \cdot \left\{ (1-s)\frac{1}{2} + s(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{\tau_H}\right)\right\}$$
$$+ s(\log p_L) \cdot \left\{ (1-s)\left(\alpha + (1-\alpha)\left(\frac{1}{2} - \frac{\log p_L - \log p_H}{\tau_H}\right)\right) + s\frac{1}{2}\right\}$$
$$+ \theta(1-s)(-\varepsilon_t) \cdot \left\{ (1-s)\frac{1}{2} + s(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{\tau_H}\right)\right\}$$
$$+ (1-\theta)(1-s)p_t^* \cdot \left\{ (1-s)\frac{1}{2} + s(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{\tau_H}\right)\right\}$$
$$+ \theta s \hat{s}_t^n (\log p_L) \cdot \left\{ (1-s)\left(\alpha + (1-\alpha)\left(\frac{1}{2} - \frac{\log p_L - \log p_H}{\tau_H}\right)\right) + s\frac{1}{2}\right\}$$
$$- \theta s \hat{s}_t^n (\log p_H) \cdot \left\{ (1-s)\frac{1}{2} + s(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{\tau_H}\right)\right\}$$
$$+ \theta(1-s)\hat{p}_{t-1} \cdot \left\{ (1-s)\frac{1}{2} + s(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{\tau_H}\right)\right\}$$
$$+ (1-\theta)(1-s)(\log p_H)$$
$$\cdot \left\{ \theta s \hat{s}_t^n (1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{\tau_H}\right) - \theta s \hat{s}_t^n \frac{1}{2} - s(1-\alpha)\frac{p_t^*}{\tau_H} + \theta(1-s)(-p_t^* + \hat{p}_{t-1} - \varepsilon_t)\,\mathbb{E}\left(\frac{1}{\tau}\right)\right\}$$
$$+ s(\log p_L)\left\{ (1-\theta)(1-s)(1-\alpha)\frac{p_t^*}{\tau_H} + \theta s \hat{s}_t^n \frac{1}{2} + \theta(-s\hat{s}_t^n)\left(\alpha + (1-\alpha)\left(\frac{1}{2} - \frac{\log p_L - \log p_H}{\tau_H}\right)\right)\right\}$$
$$+ s(\log p_L)\left\{ \theta(1-s)(1-\alpha)\left(\frac{\hat{p}_{t-1} - \varepsilon_t}{\tau_H}\right)\right\}$$
$$+ \theta(1-s)(\log p_H)\left\{ -(1-\theta)(1-s)(\hat{p}_{t-1} - p_t^* - \varepsilon_t)\,\mathbb{E}\left(\frac{1}{\tau}\right) + \theta s \hat{s}_t^n (1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{\tau_H}\right)\right\}$$
$$+ \theta(1-s)(\log p_H)\left\{ s(1-\alpha)\left(-\frac{\hat{p}_{t-1} - \varepsilon_t}{\tau_H}\right) - \theta s \hat{s}_t^n \frac{1}{2}\right\}$$

$$\log P_t - \log M_t = \hat{P}_t = -\hat{Y}_t = (1-s)(\log p_H) \cdot \left\{ (1-s)\frac{1}{2} + s(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{\tau_H}\right)\right\}$$
$$+ s(\log p_L) \cdot \left\{ (1-s)\left(\alpha + (1-\alpha)\left(\frac{1}{2} - \frac{\log p_L - \log p_H}{\tau_H}\right)\right) + s\frac{1}{2}\right\}$$
$$+ \theta(1-s)(-\varepsilon_t) \cdot \left\{ (1-s)\frac{1}{2} + s(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{\tau_H}\right)\right\}$$
$$+ \{(1-\theta)(1-s)p_t^* - \theta s \hat{s}_t^n (\log p_H) + \theta(1-s)\hat{p}_{t-1}\} \cdot \left\{ (1-s)\frac{1}{2} + s(1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{\tau_H}\right)\right\}$$
$$+ \theta s \hat{s}_t^n (\log p_L) \cdot \left\{ (1-s)\left(\alpha + (1-\alpha)\left(\frac{1}{2} - \frac{\log p_L - \log p_H}{\tau_H}\right)\right) + s\frac{1}{2}\right\}$$
$$+ (1-\theta)(1-s)(\log p_H)$$
$$\cdot \left\{ \theta s \hat{s}_t^n (1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{\tau_H}\right) - \theta s \hat{s}_t^n \frac{1}{2} - s(1-\alpha)\frac{p_t^*}{\tau_H} + \theta(1-s)(-p_t^* + \hat{p}_{t-1} - \varepsilon_t)\,\mathbb{E}\left(\frac{1}{\tau}\right)\right\}$$
$$+ s(\log p_L)\left\{ (1-\theta)(1-s)(1-\alpha)\frac{p_t^*}{\tau_H} + \theta s \hat{s}_t^n \frac{1}{2} + \theta(-s\hat{s}_t^n)\left(\alpha + (1-\alpha)\left(\frac{1}{2} - \frac{\log p_L - \log p_H}{\tau_H}\right)\right)\right\}$$
$$+ s(\log p_L)\left\{ \theta(1-s)(1-\alpha)\left(\frac{\hat{p}_{t-1} - \varepsilon_t}{\tau_H}\right)\right\}$$
$$+ \theta(1-s)(\log p_H)\left\{ -(1-\theta)(1-s)(\hat{p}_{t-1} - p_t^* - \varepsilon_t)\,\mathbb{E}\left(\frac{1}{\tau}\right) + \theta s \hat{s}_t^n (1-\alpha)\left(\frac{1}{2} - \frac{\log p_H - \log p_L}{\tau_H}\right)\right\}$$
$$+ \theta(1-s)(\log p_H)\left\{ s(1-\alpha)\left(-\frac{\hat{p}_{t-1} - \varepsilon_t}{\tau_H}\right) - \theta s \hat{s}_t^n \frac{1}{2}\right\}, \tag{57}$$

where

$$\hat{p}_t = \theta(\hat{p}_{t-1} - \varepsilon_t) + (1-\theta)p_t^*. \tag{58}$$