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Computing Time-Consistent Equilibria: A Perturbation Approach

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Abstract

Time-consistency is a key feature of many important policy problems, such as those relating to optimal fiscal policy and optimal monetary policy. It is also important for private-sector decision-making through mechanisms such as quasi-hyperbolic discounting. These problems are generally solved using some form of projection method. The difficultly with projection methods is that their computational complexity increases rapidly with the number of state variables, limiting the sophistication of the models that can be solved. This paper develops a perturbation method for solving models with time-inconsistency that enables larger models to be more readily solved and analyzed. The method operates on a model's (generalized) Euler equations; it does not require forming a quadratic approximation to household welfare and it does not require that the model's steady state be efficient. We apply the method to several models featuring time-inconsistency and show that it exhibits good accuracy.

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Computing Time-Consistent Equilibria: A Perturbation Approach*

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Time-consistency is a key feature of many important policy problems, such as those relating to optimal fiscal policy and optimal monetary policy. It is also important for private-sector decision-making through mechanisms such as quasi-hyperbolic discounting. These problems are generally solved using some form of projection method. The difficultly with projection methods is that their computational complexity increases rapidly with the number of state variables, limiting the sophistication of the models that can be solved. This paper develops a perturbation method for solving models with time-inconsistency that enables larger models to be more readily solved and analyzed. The method operates on a model's (generalized) Euler equations; it does not require forming a quadratic approximation to household welfare and it does not require that the model's steady state be efficient. We apply the method to several models featuring time-inconsistency and show that it exhibits good accuracy.

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1 Introduction

Time-consistency features importantly in many areas of macroeconomics: fiscal policy, monetary policy, banking regulation, and sovereign lending, to name just a few. In the context of optimal monetary policy time-inconsistency emerges because some constraints on the central bank's decision problem, such as the Phillips curve, involve expectations and only bind ex ante. The central bank's well-intentioned efforts to leverage the expectations channel into inflation, guide it towards announcing policies for the future that it ultimately has no incentive to keep. In the absence of a commitment mechanism, the equilibrium of interest in such problems is usually the Markov-perfect equilibrium, which is time-consistent by construction. Time-consistency problems also feature in behavioral macroeconomics, where decision-makers may have hyperbolic or quasi-geometric preferences. The difficulty with solving models involving time-consistency is that decision-makers must recognize that the choices they make today change future behavior while simultaneously taking into account that these changes to future behavior have implications for the choices made today. To solve such models one often must make strong assumptions in order to gain analytic tractability and/or employ numerical methods whose computational intensity increases rapidly, usually exponentially, with the number of state variables. The consequence is that the models analyzed are often more stark and simplistic than one would like.

This paper presents a perturbation method to solve optimization-based dynamic macroeconomic models for first-order accurate time-consistent equilibria. The method is based on the tools used to obtain first- and second-order accurate solutions to rational expectations models, which are now widely available. When expressed in terms of first-order conditions, models that feature time-inconsistency contain what are known as generalized-Euler equations. Generalized-Euler equations differ from standard Euler equations in that they contain the levels of variables as well as their derivatives with respect to endogenous state variables. These derivatives are problematic for perturbation methods because they imply that the model's steady state cannot be solved independently of its dynamics. We overcome the challenges that these derivatives pose by applying an iterative scheme whereby a first-order accurate solution emerges from the repeated application of a second-order perturbation method.

Because our solution procedure is an iterative one that requires second-order methods to obtain a first-order accurate solution, it is slightly more demanding than a standard first-order perturbation method, but it remains significantly less demanding than projection methods.[1] Moreover, our procedure inherits the scalability of perturbation methods, allowing it to be applied to medium- to large-scale models, ones that cannot reasonably be solved accurately using projection methods, even using sparse grid technology. Further, because it is based on (generalized) Euler equations, our method does not require second-order welfare approximations and nor does it ask that the perturbation point be the model's efficient steady state.

We illustrate our approach by applying it to a range of models drawn from different areas in macroeconomics. For expositional purposes and to aid comparisons, these models are intentionally kept comparatively simple. They each contain just two state variables and they can all be solved relatively quickly using projection methods. The fact that they can be solved using projection methods without too much difficulty enables us to obtain highly accurate global solutions and allows us to more easily illustrate the accuracy of our first-order accurate solution. Furthermore, the models' simplicity allows us to provide a clearer description of how our solution method can be applied.

We are not the first to apply perturbation methods to models exhibiting time-inconsistency. In the monetary policy literature, it is common to fit a linear-quadratic (LQ) approximation to the non-linear problem (Benigno and Woodford, 2005, 2012) and compute a linear solution from the LQ approximation. However, applications of the LQ approach invariably assume that monetary policy is conducted according to the timeless perspective (Woodford, 1999) whereas our focus is on discretion. When the LQ approach is applied to problems where monetary policy is conducted with discretion, it is generally necessary to approximate about an efficient steady state with zero inflation. Although that approach allows the steady state to be obtained independently of the model's dynamics, the requirement that the steady state be efficient limits the set of models that can be analyzed. In addition, deriving a second-order accurate approximation to welfare is often analytically demanding making the method challenging to apply to even quite simple models. Dotsey and Hornstein (2003) present a numerical method to form a LQ approximation for a discretionary policy problem where the steady state is not efficient. Their method is a form of successive approximations that begins with a guess at the steady state around which to perturb the model and then iterates over the steady state until convergence is reached. Unfortunately, their approach is generally inapplicable to models whose steady state is inefficient and it is subject to the pitfalls documented in Kim and Kim (2003, 2006). A method related to the one devel-

[1] In our applications we found the linear solution to converge quickly, usually in just 4 or 5 iterations.

oped here is described in Krusell, Kuruşçu, and Smith (2002). Like ourselves, they express the problem to be solved in terms of a system containing a generalized Euler equation and solve simultaneously for the model's steady state and equilibrium dynamics. Our solution strategy extends their approach to stochastic economies and to models with multiple state variables, and it differs from their method by explicitly imposing saddle-point stability on the first-order dynamics.

The remainder of this paper is organized as follows. In the following section we outline the four models that we use to demonstrate our solution method and illustrate its accuracy. In section three we present our perturbation-based solution strategy. Section four applies our method to the four models and compares the results to those from a highly accurate projection-based solution. Section five discusses how household welfare can be recovered once the solution is obtained. Section six concludes. An appendix identifies and discusses special cases where LQ approximations can be employed to solve for time-consistent equilibria and shows why LQ methods cannot be applied generally.

2 The models

In this section we outline the four models to which we apply our solution method. The models are of varying complexity, but they are all simple enough that they can be easily described and their solution using projection methods is not too time-consuming. The latter is important because we use a highly accurate global solution as the benchmark to assess the accuracy of our first-order accurate solution. The first of the four models—the stochastic growth model—does not involve time-inconsistency. Its inclusion in the analysis establishes a benchmark for the accuracy we might hope to obtain for the models with time-inconsistency. The remaining three models all involve time-inconsistent decision-making and are drawn from various literatures: optimal fiscal policy, quasi-geometric preferences, and optimal monetary policy. We provide a brief description of each model and document its key equations; readers are referred to the original sources for complete derivations.

2.1 Model one — stochastic growth model

The stochastic growth model needs little introduction. A representative consumer/producer has capital stock, k_t, and makes decisions regarding consumption, c_t, and future capital in order to maximize expected discounted lifetime utility, which depends on the sequence of goods consumed. We assume that period-utility is of the iso-elastic form. With goods produced according to a Cobb-Douglas technology and with aggregate technology, a_t, obeying a standard stationary AR(1) process, the key equations characterizing equilibrium are:

$$a_{t+1} = \rho a_t + \varepsilon_{t+1}, \tag{1}$$

$$k_{t+1} = (1-\delta)\, k_t + e^{a_t} k_t^{\alpha} - c_t, \tag{2}$$

$$c_t^{-\sigma} = \beta E_t \left[c_{t+1}^{-\sigma} \left(1 - \delta + \alpha e^{a_{t+1}} k_{t+1}^{\alpha-1} \right) \right]. \tag{3}$$

Equation (2) is the law-of-motion for capital, which allows the capital stock to be augmented by unconsumed production and to depreciate at rate $\delta \in (0,1]$. Equation (3) is the standard consumption-Euler equation in which $\beta \in (0,1)$ is the discount factor, $\sigma \in (0,\infty)$ is the inverse of the elasticity of intertemporal substitution, and E_t is the mathematical expectations operator. When solving the model we set $\beta = 0.99$, $\alpha = 0.3$, $\delta = 0.015$ and $\rho = 0.95$. The standard deviation of the technology innovation, ε_t, is set to 0.01.

2.2 Model two — time-consistent fiscal policy

This model is taken from Ambler and Pelgrin (2010) (which draws on Klein, Krusell, and Rios-Rull, 2008), who used it as a vehicle to illustrate how to apply control methods to compute Markov-perfect policies for stochastic non-linear models. The model was further analyzed by Dennis and Kirsanova (2016) who showed that it could be solved efficiently using a projection method based on Chebyshev polynomials applied to a system of equilibrium conditions containing a generalized Euler equation.

The environment is one in which a representative consumer/producer owns the capital stock, produces using a Cobb-Douglas technology, and receives utility from consuming goods and government services. The government purchases goods, transforms them costlessly into government services, and provides them free to consumers. Government expenditure is financed through a tax levied on household-income with an allowance made for capital depreciation. The household's problem is to choose consumption and future capital to maximize expected discounted life-time utility while taking taxes and the provision of government services as given. The government's problem is to choose the level of services to provide in order to maximize household welfare, taking into account its balanced budget condition and the impact that income taxation has on households' incentives to accumulate capital. Complete descriptions of the model can be found in Ambler and Pelgrin (2010) and Dennis and Kirsanova (2016).

With the household's expected discounted lifetime utility given by:

$$U_t = E_t \left[\sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} + \mu \frac{g_t^{1-\eta}}{1-\eta} \right) \right], \tag{4}$$

$\beta \in (0,1)$, $\sigma \in (0,\infty)$, $\mu \in (0,\infty)$, $\eta \in (0,\infty)$, welfare maximization by the consumer and the government leads to the following system of constraints and first-order conditions:

$$a_{t+1} = \rho a_t + \varepsilon_{t+1}, \tag{5}$$

$$k_{t+1} = (1-\delta) k_t + e^{a_t} k_t^{\alpha} - c_t - g_t, \tag{6}$$

$$c_t^{-\sigma} = \beta E_t \left[c_{t+1}^{-\sigma} \left(1 + \left(1 - \frac{g_{t+1}}{e^{a_{t+1}} k_{t+1}^{\alpha} - \delta k_{t+1}} \right) \left(\alpha e^{a_{t+1}} k_{t+1}^{\alpha-1} - \delta \right) \right) \right], \tag{7}$$

$$\mu g_t^{-\eta} = \beta E_t \left[\left(c_{t+1}^{-\sigma} - \mu g_{t+1}^{-\eta} \right) c_k \left(a_{t+1}, k_{t+1} \right) + \mu g_{t+1}^{-\eta} \left(1 - \delta + \alpha e^{a_{t+1}} k_{t+1}^{\alpha-1} \right) \right]. \tag{8}$$

Equation (5) describes the process for aggregate technology while equation (6) summarizes the law-of-motion for capital. Relative to the stochastic growth model, equation (6) only differs in that government purchases of goods, g_t, in addition to consumption subtracts from production in determining the level of investment. The consumption-Euler equation is summarized by equation (7). In this equation it is the after-tax return on capital that matters for consumption, where the tax rate is applied to production minus depreciated capital and is determined importantly by the level of government services provided.

The final equation in the system, equation (8), is the first-order condition associated with government services. We denote the household's decision rule for consumption by $c(a_t, k_t)$. Equation (8) takes the form of a generalized Euler equation because it depends on the derivative of the household's consumption decision rule with respect to capital. This derivative enters the Euler equation because the government must account for the effect an increase in its provision of services—funded through higher income-taxation—has on household consumption via lower capital accumulation. When solving this model we take the parameterization from Ambler and Pelgrin (2010). Specifically, we set $\beta = 0.987$, $\alpha = 0.3$, $\delta = 0.05$, $\sigma = 1$, $\mu = 0.3$, $\eta = 1$, $\rho = 0.95$, and the standard deviation of the technology innovation to 0.03.

2.3 Model three — quasi-geometric discounting

This model comes from Krusell, Kuruşçu, and Smith (2002) and Maliar and Maliar (2005). Like the previous two models, we can think of this one in terms of a representative consumer/producer that owns the capital stock and that chooses consumption and future capital in order to maximize expected discounted lifetime utility. However, in this model the household/producer has quasi-geometric discounting, which is to say that its expected discounted lifetime utility is given by:

$$U_t = \frac{c_t^{1-\sigma}}{1-\sigma} + \theta E_t \left[V_{t+1}\right], \tag{9}$$

$\theta \in (0, 1]$, $\sigma \in (0, \infty)$, with:

$$V_t = \frac{c_t^{1-\sigma}}{1-\sigma} + \beta E_t \left[V_{t+1}\right], \tag{10}$$

$\beta \in (0, 1)$, where we have chosen period-utility to be of the iso-elastic form for simplicity. Together, equations (9) and (10) imply that the household discounts between today and tomorrow at rate $\beta\theta$, and between tomorrow and the next day at rate β. When θ is less than one the short-run discount rate is greater than the long-run discount rate, leading to a Strotz (1956) form of time-inconsistency.

Solving the household/producer's decision problem leads to the following key equations:

$$a_{t+1} = \rho a_t + \varepsilon_{t+1}, \tag{11}$$

$$k_{t+1} = (1-\delta) k_t + e^{a_t} k_t^{\alpha} - c_t, \tag{12}$$

$$c_t^{-\sigma} = \beta E_t \left[c_{t+1}^{-\sigma} \left(\theta \left(1 - \delta + \alpha e^{a_{t+1}} k_{t+1}^{\alpha-1}\right) + (1-\theta) k_k \left(a_{t+1}, k_{t+1}\right)\right)\right]. \tag{13}$$

Equations (11) and (12) are familiar and standard. The crucial difference between this model and the stochastic growth model lies in equation (13), which takes the form of a generalized Euler equation because it depends on the derivative of the decision rule for future capital with respect to capital, $k_k(a_t, k_t)$, in addition to the level of capital itself. This derivative enters because the current-period household can use its decision regarding future capital to alter the future state and thereby alter the decisions made by its future self. As a consequence capital accumulation has a pecuniary return in the form of the marginal product of capital and a non-pecuniary return related to the effect current-period saving has on how the future household determines its consumption. If $\theta = 1$, then this non-pecuniary return disappears, equation (13) simplifies to equation (3), and there is no time-inconsistency. When solving this model we are guided by Maliar and Maliar (2005) and set $\beta = 0.95$, $\theta = 0.95$, $\alpha = 0.36$, $\delta = 0.1$, $\sigma = 2$, $\rho = 0.95$, and the standard deviation of the technology innovation to 0.01.

2.4 Model four — time-consistent monetary policy

Our fourth model is an application of optimal discretionary monetary policy in a new Keynesian model. Our analysis of this model builds on Comincini (2020) who solved it using a value-function-iteration, but we recast it as a system of constraints and first-order conditions, one of which is a generalized Euler equation. Related models, but log-linearized and studied for an ad-hoc loss function and/or non-discretionary policy can be found in a variety of places, including Amato and Laubach (2004) and Dennis and Söderström (2006). Comincini (2020) provides a full description of the model.

The model is one in which households receive utility from consumption and dis-utility from labor, h_t, and there are external habits in consumption. We assume that expected discounted lifetime utility is additively separable in consumption and labor and takes the form:

$$U_t = E_t \left[\sum_{t=0}^{\infty} \beta^t \left(\frac{(c_t - \gamma C_{t-1})^{1-\sigma}}{1-\sigma} - \nu \frac{h_t^{1+\chi}}{1+\chi} \right) \right], \tag{14}$$

$\beta \in (0,1)$, $\sigma \in (0,\infty)$, $\nu \in (0,\infty)$, $\chi \in (0,\infty)$, and $\gamma \in (0,1)$, where C_t denotes period-t aggregate consumption. Monopolistically competitive firms employ labor and produce according to the linear production technology:

$$y_t = e^{a_t} h_t, \tag{15}$$

and set prices subject to a Rotemberg (1982) quadratic adjustment cost. The household's utility maximization leads to the labor supply equation:

$$\nu h_t^{\chi} = w_t \left(c_t - \gamma C_{t-1}\right)^{-\sigma}, \tag{16}$$

where w_t denotes the real wage, while firm's cost minimization causes real marginal costs, ω_t, to be given by:

$$\omega_t = \frac{w_t}{e^{a_t}}. \tag{17}$$

Firms are assumed to set the price for their good to maximize their expected discounted net cash flow, where the cash-flows are paid to households in the form of a dividend and valued in terms of the utility that dividend provides. From the first-order condition for price-setting we get (in a symmetric equilibrium and after aggregating across firms) the following Phillips curve for inflation, π_t:

$$\pi_t \left(1+\pi_t\right) = \frac{1-\epsilon}{\phi} + \frac{\epsilon}{\phi}\omega_t + \beta E_t \left[\frac{\left(C_{t+1} - \gamma C_t\right)^{-\sigma} e^{a_{t+1}} H_{t+1} \pi_{t+1} \left(1+\pi_{t+1}\right)}{\left(C_t - \gamma C_{t-1}\right)^{-\sigma} e^{a_t} H_t} \right], \tag{18}$$

where $\phi \in (0,\infty)$ governs the magnitude of the price-adjustment cost and $\epsilon \in (1,\infty)$ represents the price elasticity of demand. Finally, we have the resource constraint:

$$C_t = \left(1 - \frac{\phi}{2}\pi_t^2\right) e^{a_t} H_t. \tag{19}$$

We substitute equations (16) and (17) into the Phillips curve, then the central bank's decision problem is to choose π_t to maximize household welfare (equation (14)), subject to the Phillips curve and the resource constraint. The resulting system of first-order conditions is:

$$a_{t+1} = \rho a_t + \varepsilon_{t+1}, \tag{20}$$

$$C_t = \left(1 - \frac{\phi}{2}\pi_t^2\right) e^{a_t} H_t \tag{21}$$

$$\pi_t (1 + \pi_t) = \frac{1-\epsilon}{\phi} + \frac{\epsilon}{\phi}\frac{1}{e^{a_t}}\frac{\nu H_t^{\chi}}{(C_t - \gamma C_{t-1})^{-\sigma}} + \beta E_t\left[\frac{M_{t+1}}{(C_t - \gamma C_{t-1})^{-\sigma} e^{a_t} H_t}\right], \tag{22}$$

$$M_t = (C_t - \gamma C_{t-1})^{-\sigma} e^{a_t} H_t \pi_t (1 + \pi_t), \tag{23}$$

$$0 = -\phi \pi_t \lambda_{1t} - (1 + 2\pi_t)(C_t - \gamma C_{t-1})^{-\sigma} \lambda_{2t}, \tag{24}$$

$$\nu H_t^{\chi} = \left(1 - \frac{\phi}{2}\pi_t^2\right) e^{a_t}\lambda_{1t} + \left[\frac{\epsilon(1+\chi)}{\phi} H_t^{\chi} + \left(\frac{1-\epsilon}{\phi} - \pi_t(1+\pi_t)\right)(C_t - \gamma C_{t-1})^{-\sigma} e^{a_t}\right]\lambda_{2t}, \tag{25}$$

$$\begin{aligned}
\lambda_{1t} &= (C_t - \gamma C_{t-1})^{-\sigma} + \beta E_t\left[M_c(a_{t+1,}C_t)\right]\lambda_{2t} \\
&- \sigma\left(\frac{1-\epsilon}{\phi} - \pi_t(1+\pi_t)\right)(C_t - \gamma C_{t-1})^{-\sigma} e^{a_t} H_t \lambda_{2t} \\
&+ \beta E_t\left[\left(\sigma\gamma\left(\frac{1-\epsilon}{\phi} - \pi_{t+1}(1+\pi_{t+1})\right)(C_{t+1} - \gamma C_t)^{-\sigma-1} e^{a_{t+1}} H_{t+1}\right)\lambda_{2t+1}\right],
\end{aligned} \tag{26}$$

where λ_{1t} is the Lagrange multiplier on the resource constraint, λ_{2t} is the Lagrange multiplier on the Phillips curve, and M_t represents the marginal utility of the goods lost through the adjustment costs generated by inflation. Equation (26) has the form of a generalized Euler equation because it depends on the derivative $M_c(a_{t+1,}C_t)$.

We parameterize the model by setting $\beta = 0.99$, $\sigma = 1$, $\nu = 1$, $\chi = 1$, $\gamma = 0.6$, $\epsilon = 11$, $\phi = 60$, $\rho = 0.95$, and the standard deviation of the technology innovation to 0.01. Importantly, we have not introduced a production subsidy to offset the monopolistic distortion, nor a consumption tax to offset the consumption externality—the distortions caused by monopolistic competition and external consumption habits remain.

3 A first-order perturbation solution

In this section we describe a procedure that uses perturbation to construct a first-order accurate solution to optimization-based models involving time-inconsistency. Our focus on first-order solutions stems in large part from the fact that many non-linear models are well-approximated by linear solutions. However, should it be needed, we note that the extension from first-order accuracy to second-order accuracy is both straightforward and intuitive, and employed in section 6.

Our solution method is conceptually simple and easy to implement. To illustrate it, we enlist the help of model three. Recall that for this model the key equations are:

$$a_{t+1} = \rho a_t + \varepsilon_{t+1}, \tag{27}$$

$$k_{t+1} = (1-\delta) k_t + e^{a_t} k_t^{\alpha} - c_t, \tag{28}$$

$$c_t^{-\sigma} = \beta E_t \left[c_{t+1}^{-\sigma} \left(\theta \left(1 - \delta + \alpha e^{a_{t+1}} k_{t+1}^{\alpha-1} \right) + (1-\theta) k_k (a_{t+1}, k_{t+1}) \right) \right]. \tag{29}$$

If $\theta = 1$, then the short-run discount rate and the long-run discount rate would be equal, there would be no time-inconsistency problem, equation (29) would simplify to:

$$c_t^{-\sigma} = \beta E_t \left[c_{t+1}^{-\sigma} \left(1 - \delta + \alpha e^{a_{t+1}} k_{t+1}^{\alpha-1} \right) \right], \tag{30}$$

and we could easily solve for the model's steady state (the model's zeroth-order solution). This steady state provides a point to linearize around and solving the model for its first-order accurate equilibrium dynamics becomes straightforward using a method such as Klein (2000).

The difficulty arises when $\theta \neq 1$. When $\theta \neq 1$ we cannot solve for the model's steady state without knowing the derivative of the decision rule for capital, which is part of the model's first-order solution. So in order to solve for the steady state (the zeroth-order solution) we need to know the first-order solution. Although the decision rule for capital is a non-linear function, its linear approximation has the form:

$$k(a_t, k_t) \approx k_{ss} + \psi_a (a_t - a_{ss}) + \psi_k (k_t - k_{ss}), \tag{31}$$

where ψ_a and ψ_k are derivatives, $a_{ss} = 0$ is the steady state value for (log-) technology, and k_{ss} is the unknown steady state value for capital. From equation (31), the derivative of next period's capital with respect to capital is ψ_k. Suppose we know ψ_k, then the steady state of equations (27)—(29) can be computed and is given by:

$$a_{ss} = 0, \tag{32}$$

$$k_{ss} = \left[\frac{1}{\alpha} \left(\frac{1 - \beta (1-\theta) \psi_k}{\beta \theta} \right) - 1 + \delta \right]^{\frac{1}{\alpha - 1}}, \tag{33}$$

$$c_{ss} = k_{ss}^{\alpha} - \delta k_{ss}, \tag{34}$$

which provides a point around which equations (27)—(29) can be linearized.

Unfortunately, we cannot produce a first-order accurate solution from an iterative procedure that begins by guessing ψ_k, solving the resulting linearized rational expectations model, extracting an update of ψ_k from the solution, and iterating to convergence. The reason this approach is incorrect is that in order to have a first-order accurate solution we require that the derivative $k_k(a_{t+1}, k_{t+1})$ in equation (29) be approximated to first-order accuracy, and this requires that the decision rule for capital itself be approximated to second-order accuracy. Therefore, we require the model's first-order accurate solution to obtain its zeroth-order solution, we require its second-order accurate solution to obtain its first-order solution, we require its third-order accurate solution to obtain its second-order solution, etc.

Acknowledging this inconvenient recursion our solution procedure is as follows. Rather than assume that the model's third-order accurate solution is known, we simply assume that the terms in the third-order (and higher) accurate solution are sufficiently small that they can be safely ignored. Ignoring terms higher than second-order, the solution for next period's capital has the approximation:

$$\begin{aligned} k(a_t, k_t) \approx k_{ss} + \psi_a (a_t - a_{ss}) + \psi_k (k_t - k_{ss}) + \frac{\psi_{aa}}{2}(a_t - a_{ss})^2 \\ + \psi_{ak}(a_t - a_{ss})(k_t - k_{ss}) + \frac{\psi_{kk}}{2}(k_t - k_{ss})^2, \end{aligned} \tag{35}$$

and the partial derivative of future capital with respect to k_t is:

$$k_k(a_t, k_t) \approx \psi_k + \psi_{ak}(a_t - a_{ss}) + \psi_{kk}(k_t - k_{ss}). \tag{36}$$

With the building blocks established, our method for computing a first-order accurate solution to the model is as follows:

1. Set the loop-counter to zero, $i = 0$, set the convergence tolerance, *tol*, and initialize values for ψ_k^0, ψ_{ak}^0, ψ_{kk}^0 (that will be stored in the vector ψ^0).

2. With the derivative approximated by equation (36), solve equations (27)—(29) using a second-order perturbation method. This solution delivers an estimate of the steady state and the first- and second-order equilibrium dynamics.

3. Increment the loop-counter, $i = i + 1$, and extract ψ_k^i, ψ_{ak}^i, ψ_{kk}^i (that will be stored in ψ^i) from the second-order solution for next period's capital, which takes the form of equation (35).

4. While $\left\|\psi^{i}-\psi^{i-1}\right\| > tol$, return to step 2.

5. Exit.

Having exited the algorithm we discard all second-order terms, retaining the steady state and the linear terms, which provide a solution that is first-order accurate. In our application of this method to the four models above, we use the Gomme and Klein (2011) second-order perturbation method to solve each model and to impose saddle-point stability on the first order dynamics, and we linearize the model rather than log-linearize it. We linearize the models because the resulting solutions can be compared more directly to those obtained using the projection method. It is straightforward to modify the procedure to get a first-order accurate log-solution.

3.1 A general environment

The solution method is described above in the context of a specific model, but its general application should be readily apparent. Suppose the model is described by the general non-linear form:

$$E_t\left[\widetilde{\mathbf{f}}\left(\mathbf{y}_{t+1},\mathbf{x}_{t+1},\frac{\partial \mathbf{y}_{t+1}}{\partial \mathbf{x}_{t+1}},\mathbf{y}_t,\mathbf{x}_t,\varepsilon_{t+1}\right)\right]=\mathbf{0}, \tag{37}$$

where $\mathbf{x}_t$ is an $n_x \times 1$ vector of predetermined variables, $\mathbf{y}_t$ is an $n_y \times 1$ vector of non-predetermined variables, ε_t is an $s \times 1$ vector of innovations, and $\widetilde{\mathbf{f}}$ is an $(n_y + n_x) \times 1$ vector of functions, whose solution takes the form:

$$\mathbf{x}_{t+1}=\mathbf{h}\left(\mathbf{x}_t,\sigma\right)+\sigma\varepsilon_{t+1}, \tag{38}$$

$$\mathbf{y}_t=\mathbf{g}\left(\mathbf{x}_t,\sigma\right), \tag{39}$$

where σ is a perturbation parameter. Equation (37) differs from the class of models considered in Schmitt-Grohé and Uribe (2004), Gomme and Klein (2011), Binning (2013) only through the presence of the Jacobian term: $\frac{\partial \mathbf{y}_{t+1}}{\partial \mathbf{x}_{t+1}}$.

Our iterative solution procedure estimates the Jacobian $\frac{\partial \mathbf{y}_{t+1}}{\partial \mathbf{x}_{t+1}}$ from a conjecture at equation (39) and substitutes the estimated Jacobian into equation (37). After this substitution, equation (37) can be written as:

$$E_t\left[\mathbf{f}\left(\mathbf{y}_{t+1},\mathbf{x}_{t+1},\mathbf{y}_t,\mathbf{x}_t,\varepsilon_{t+1}\right)\right]=\mathbf{0}, \tag{40}$$

to which standard second-order perturbation methods can be applied. The second-order solution for the non-predetermined variables that is produced is then differentiated with respect to $\mathbf{x}_t$ to obtain a revised estimate of the Jacobian, and we then iterate to convergence. We note that it is possible to extend this procedure to obtain solutions with higher-order accuracy. For example, by assuming that it is the fourth-order terms that can be safely ignored, conjecturing a second-order expression for the derivative, $\mathbf{g}_{\mathbf{x}}(\mathbf{x}_t)$, and using a third-order perturbation solver such as Binning (2013) to solve the model; updating the conjecture and iterating to convergence.

4 Results

In this section we solve each of the four models described in section 2 and present the results. First, we solve the models using a projection method based on Chebyshev polynomials and Gauss-Hermite quadrature. We obtain a highly accurate solution from this projection method that we then use as the benchmark against which to assess the properties and accuracy of the first-order accurate solution procedure. For the projection method, details regarding the number of solution nodes, the order of the polynomials, the domain for each state variable, etc, are summarized in Table 1.

Table 1: Projection method details

	Model one	Model two	Model three	Model four
Quadrature nodes	21	21	21	21
Solution nodes $(a_t, k_t/c_{t-1})$	$(21, 51)$	$(21, 51)$	$(21, 51)$	$(21, 51)$
Polynomial orders $(a_t, k_t/c_{t-1})$	$(6, 9)$	$(6, 14)$	$(6, 9)$	$(5, 6)$
Technology domain	$[\pm 0.096]$	$[\pm 0.288]$	$[\pm 0.096]$	$[\pm 0.096]$
Endog. state domain	$[27, 42]$	$[4, 17]$	$[2.6, 4.7]$	$[1.3, 1.8]$
$log_{10}(\Vert\text{Euler error}\Vert_\infty)$	-11.8	-7.9	-10.0	-8.2

For all of the models the domain for the technology shock was chosen to be plus/minus three unconditional standard deviations while that for the endogenous state variable (capital for models one—three, lagged consumption for model four) was chosen to cover that variable's stationary distribution determined through a stochastic simulation (one million periods). We used 21 nodes for technology and 51 nodes for the endogenous state variable, with the order of the polynomial for these two variables varying according to the model. The nodes are constructed from the roots of the corresponding Chebyshev polynomial. The last row of Table 1 reports on the Euler-equations errors for each model and speaks to the accuracy of the solution. These errors are based on the consumption-Euler equation for models one—three[2] and on the Phillips curve for model four. To compute the errors we used a uniform

[2]The $log_{10}(\Vert\text{Euler error}\Vert_\infty)$ from the government spending Euler equation in model two is -8.7.

grid with 101 points for each state variable.

Turning to the results, in addition to the solution itself, we compute and show for each model aspects of the solution that researchers and policy-makers are typically interested in: decision rules, stationary distributions, and impulse response functions.

Beginning with the stochastic growth model, the linear solution expressed as deviations from steady state is:

$$\begin{bmatrix} \hat{a}_{t+1} \\ \hat{k}_{t+1} \\ \hat{c}_t \\ \hat{y}_t \end{bmatrix} = \begin{bmatrix} 0.950 & 0.000 \\ 2.216 & 0.971 \\ 0.680 & 0.039 \\ 2.896 & 0.025 \end{bmatrix} \begin{bmatrix} \hat{a}_t \\ \hat{k}_t \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \left[\epsilon_{t+1}\right], \tag{41}$$

with the steady state given by: $a_{ss} = 0$, $k_{ss} = 34.609$, $c_{ss} = 2.377$, and $y_{ss} = 2.896$. Investment is computed from the solutions for output and consumption. From this solution we plot decision rules, stationary distributions and impulse response function, see Figure 1.

The panels in the top row of Figure 1 display the solution for capital, consumption, output, and investment, respectively, as a function of capital, holding technology at its steady state value. The panels in the second row of Figure one are similar, expect the solutions are shown as functions of technology, holding capital at its steady state value. Stationary distributions for these variables are shown in the third row while the panels in the final row display response functions for a positive one standard deviation technology shock.

Recall that this model is not one in which decision-makers face a time-inconsistency problem. In this sense it represents a benchmark for accuracy against which the remaining three models can be compared. What is clear from Figure 1 is that the first-order approximation is really quite accurate over most of the domain for capital and technology that was used to obtain the projection solution. Ideally, the solutions to the models with time-inconsistency will exhibit a similar level of accuracy.

The solution results for model two, the model of time-consistent fiscal policy, are shown in Figure 2. As previously, decision rules are shown in the top two rows of panels, stationary distributions are shown in the third row, and impulse response functions are shown in the fourth row.

Looking at the decision rules first, the panels in the top two rows of Figure 2 show that the first-order solution is very accurate in the vicinity of the steady state, but that the accuracy deteriorates when capital (first row) and technology (second row) is far from steady state. This decrease in accuracy is to be expected from a first-order accurate perturbation

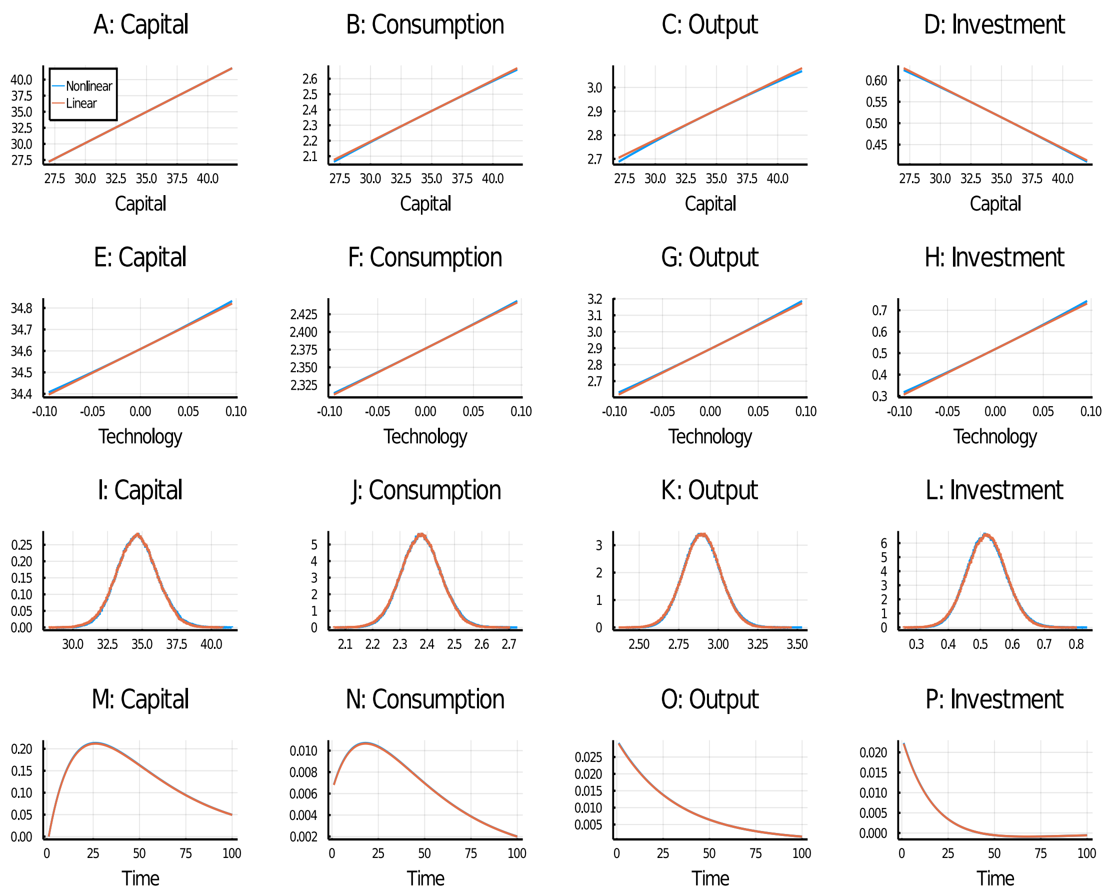

Figure 1: Results for stochastic growth model

solution. The panels in the third and fourth rows of Figure 2 reveal that the stationary distributions and the impulses response functions (in particular) are accurately approximated, a consequence of the fact that the model spends very little time in the regions of the domain where the decision rules have less accuracy.

The first-order solution for model two is:

$$\begin{bmatrix} \hat{a}_{t+1} \\ \hat{k}_{t+1} \\ \hat{c}_t \\ \hat{g}_t \\ \hat{y}_t \end{bmatrix} = \begin{bmatrix} 0.950 & 0.000 \\ 1.206 & 0.929 \\ 0.538 & 0.066 \\ 0.158 & 0.022 \\ 1.902 & 0.067 \end{bmatrix} \begin{bmatrix} \hat{a}_t \\ \hat{k}_t \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \left[\epsilon_{t+1} \right]. \tag{42}$$

For this model investment is calculated by subtracting consumption and government spending from output. The steady state is: $a_{ss} = 0$, $k_{ss} = 8.531$, $c_{ss} = 1.150$, $g_{ss} = 0.326$, and $y_{ss} = 1.902$.

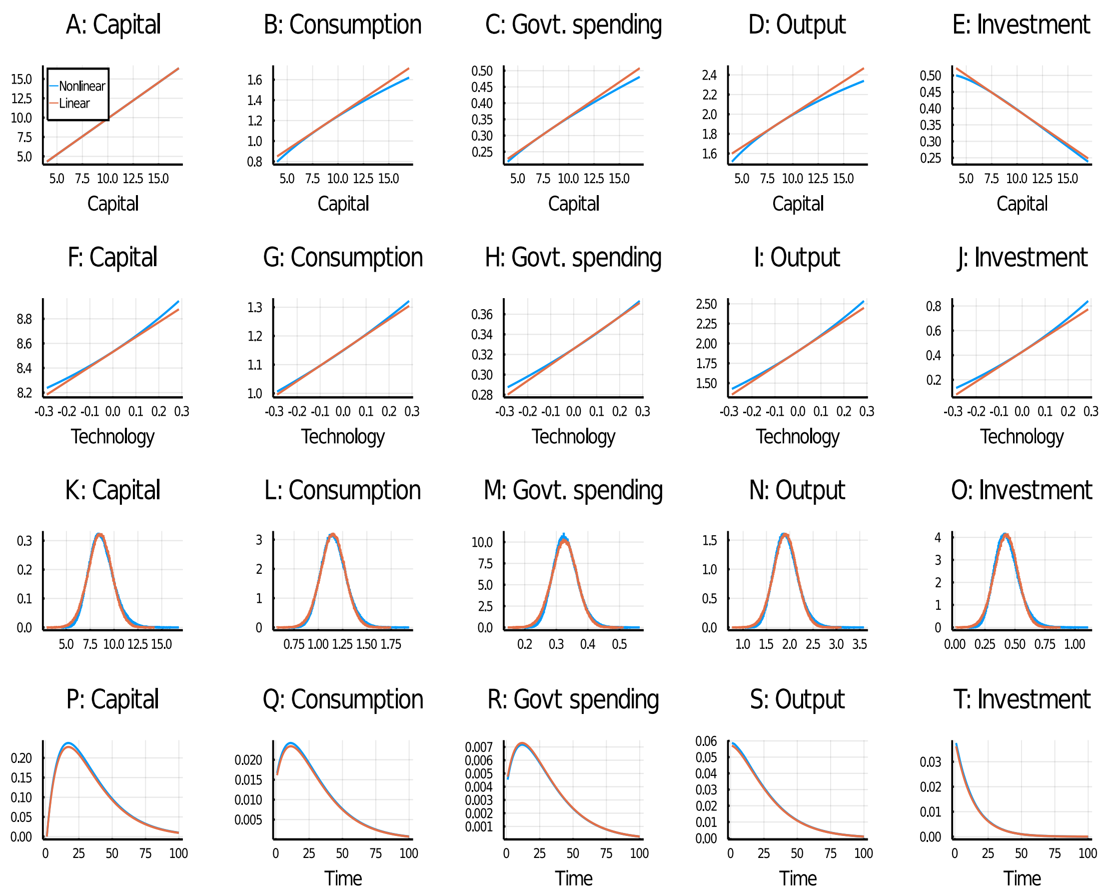

Figure 2: Results for time-consistent fiscal policy

Turning to the third model with quasi-geometric discounting. The linear solution to this model is given by:

$$\begin{bmatrix} \hat{a}_{t+1} \\ \hat{k}_{t+1} \\ \hat{c}_t \\ \hat{y}_t \end{bmatrix} = \begin{bmatrix} 0.950 & 0.000 \\ 0.755 & 0.906 \\ 0.821 & 0.154 \\ 1.576 & 0.160 \end{bmatrix} \begin{bmatrix} \hat{a}_t \\ \hat{k}_t \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \left[\epsilon_{t+1}\right]. \tag{43}$$

Again, investment is the residual between output and consumption. The steady state is: $a_{ss} = 0$, $k_{ss} = 3.538$, $c_{ss} = 1.222$, and $y_{ss} = 1.576$.

The results for the behavioral macro-model with quasi-geometric discounting are shown in Figure 3, following the same layout as the two previous figures. From an accuracy perspective, the results are similar to model two. The accuracy of the first-order approximation is clearly good, especially in the vicinity of the steady state. When capital is far from steady state the linear decision rules tend to overstate consumption and output, which leads investment to be overstated too, but this inaccuracy is in the tail-region of the domain where the model spends very little time.

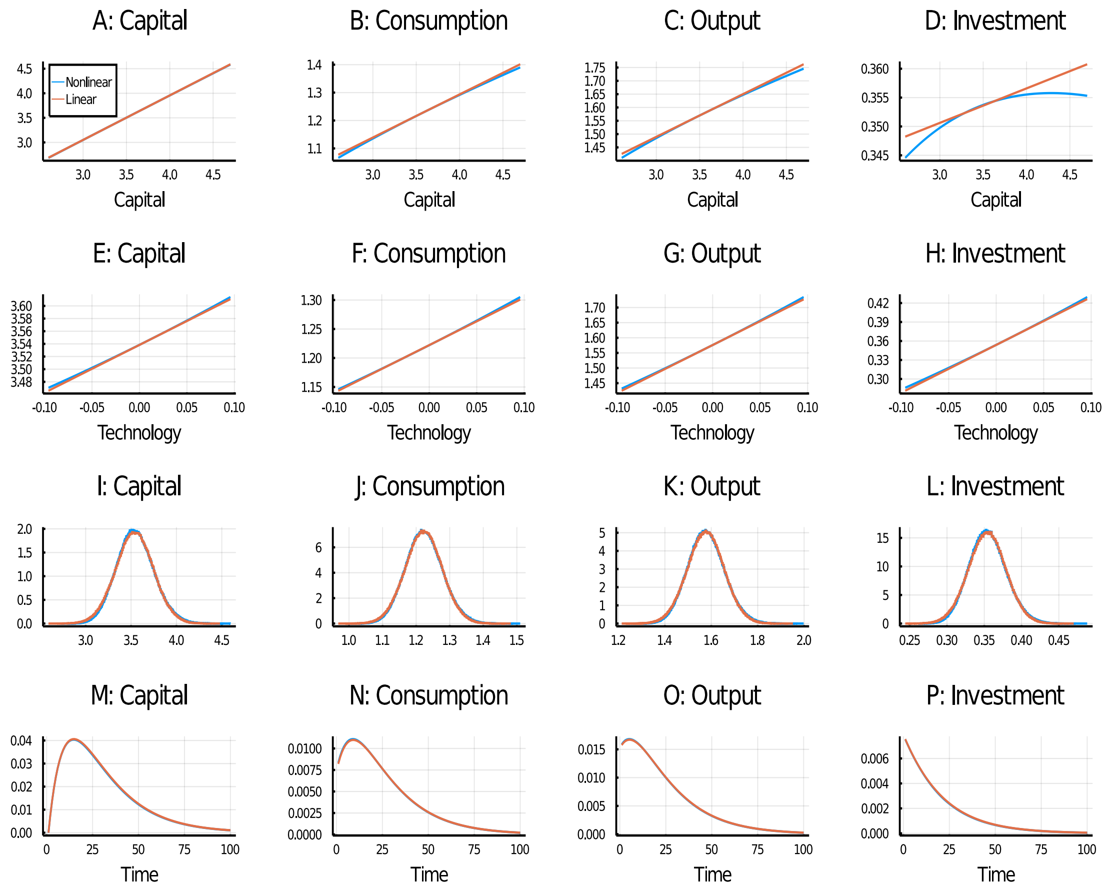

Figure 3: Results for quasi-geometric discounting

The panels in the third row of Figure 3 show that the stationary distributions are well-approximated even though the linear solution omits precautionary effects from uncertainty. The impulse responses displayed in the final row reveal high accuracy, even in the early periods of the responses when the effects of the shock are largest.

Our final set of results relate to the model with time-consistent monetary policy. Among the three models with time-consistency considered in this paper, this is the most complicated. Its solution requires solving a system with 12 non-predetermined variables, although only a few—consumption, labor, and inflation—are of primary interest.

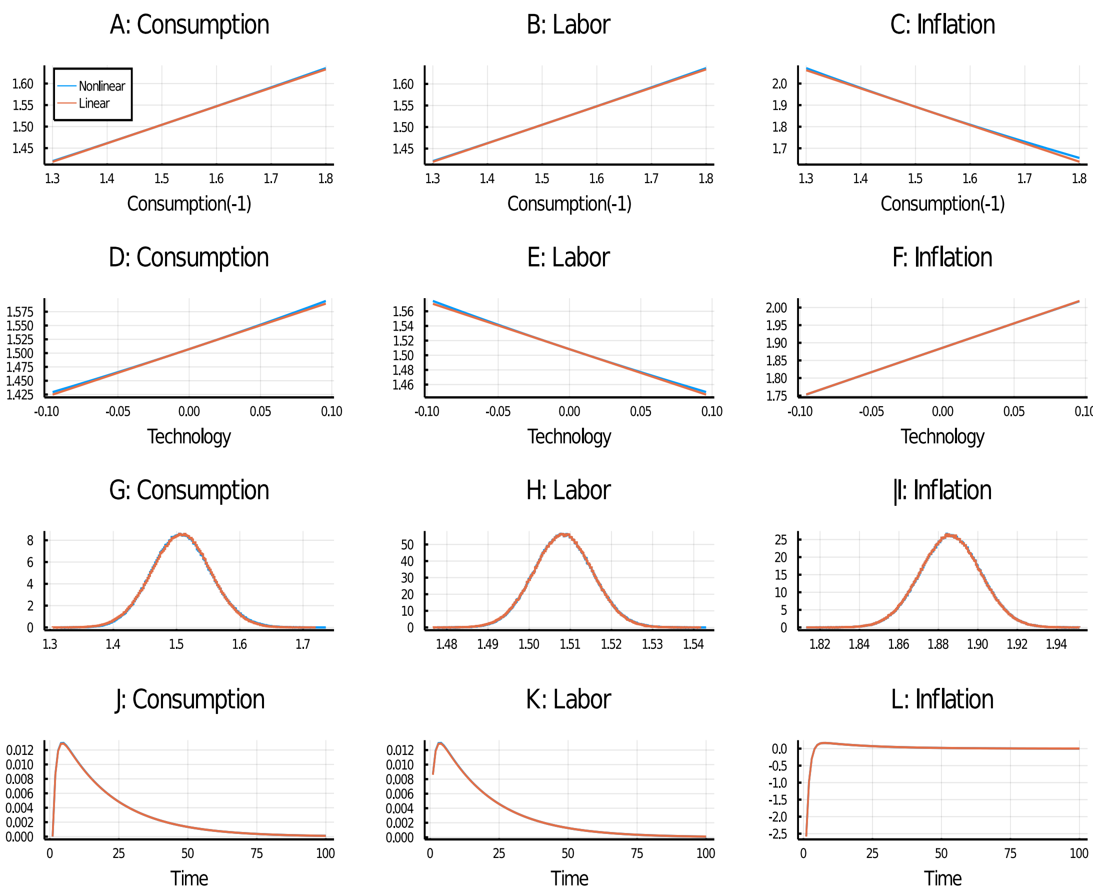

Figure 4: Results for time-consistent monetary policy

The model's linear solution is:

$$\begin{bmatrix} \hat{a}_{t+1} \\ \hat{c}_t \\ \hat{h}_t \\ \hat{\pi}_t \end{bmatrix} = \begin{bmatrix} 0.950 & 0.000 \\ 0.860 & 0.430 \\ -0.647 & 0.429 \\ 1.384 & -0.849 \end{bmatrix} \begin{bmatrix} \hat{a}_t \\ \hat{c}_{t-1} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \left[\epsilon_{t+1} \right], \tag{44}$$

while the steady state equal to: $a_{ss} = 0$, $c_{ss} = 1.507$, $h_{ss} = 1.508$, and $\pi_{ss} = 1.886$.

Figure 4 shows that the decision rules are approximated with a high-level of accuracy. Recall that we did not introduce a production subsidy to offset the effects of monopolistic competition and nor did we introduce a tax to offset the consumption externality. For this reason, the model exhibits a discretionary inflation bias, which is shown in panel I, where the unconditional mean of inflation is just under 1.9 percent per annum for the model's chosen parameterization. The first-order solution accurately captures this inflation bias, and indeed captures the model's stationary distribution very well. The impulse responses shown in the final row of Figure 4 well-capture the decline in inflation that arises from a rise in technology and the transition dynamics that occur as the model returns to steady state.

One final note regarding model four is that if we introduce an optimal subsidy to offset the monopolistic distortion and an optimal consumption tax to offset the consumption externality, then the steady state is efficient, there is no discretionary inflation bias, and the steady state inflation rate is zero. In that case it is possible to form a LQ approximation around the zero-inflation efficient steady state (because the Phillips curve is not needed to derive a valid second-order approximation to conditional welfare). In this special case, then, the model can be solved using a LQ solution technology, and the solution obtained would be identical to the first-order solution from our perturbation approach.

4.1 Summary

The take-away message from Figures 1—4 is that the first-order accurate perturbation solution delivers solutions that are very similar to the nonlinear solutions for these models. For all three aspects of the solution that we looked at: decision rules, stationary distributions, and impulse response functions, the first-order perturbation method produces solutions with properties that conform closely to those from the projection method. Because the perturbation method scales well with the size of the system whereas projections methods become exponentially more time-consuming, this provides good reason to expect that our projection method can be a powerful and accurate alternative to projection-methods for medium- to large-scale macro-models in which time-inconsistency matters.

5 Welfare

Our solution method works with a model's first-order conditions and does not compute conditional welfare as part of the solution process. In this section we discuss how conditional welfare can be computed.

We first note that when a model's non-linear first-order conditions are being solved using a projection method that computing conditional welfare is straightforward. One simply augments the system of equations being solved with a recursive representation of conditional welfare and then solves the expanded system of equations. To give a concrete example, for model two that involves time-consistent fiscal policy, one can obtain conditional welfare, U_t, by solving the expanded system:

$$a_{t+1} = \rho a_t + \varepsilon_{t+1}, \tag{45}$$

$$k_{t+1} = (1-\delta) k_t + e^{a_t} k_t^{\alpha} - c_t - g_t, \tag{46}$$

$$c_t^{-\sigma} = \beta E_t \left[c_{t+1}^{-\sigma} \left(1 + \left(1 - \frac{g_{t+1}}{e^{a_{t+1}} k_{t+1}^{\alpha} - \delta k_{t+1}} \right) \left(\alpha e^{a_{t+1}} k_{t+1}^{\alpha-1} - \delta \right) \right) \right], \tag{47}$$

$$\mu g_t^{-\eta} = \beta E_t \left[\left(c_{t+1}^{-\sigma} - \mu g_{t+1}^{-\eta} \right) c_k \left(a_{t+1}, k_{t+1} \right) + \mu g_{t+1}^{-\eta} \left(1 - \delta + \alpha e^{a_{t+1}} k_{t+1}^{\alpha-1} \right) \right], \tag{48}$$

$$U_t = \frac{c_t^{1-\sigma}}{1-\sigma} + \mu \frac{g_t^{1-\eta}}{1-\eta} + \beta E_t \left[U_{t+1} \right]. \tag{49}$$

If we were using our perturbation procedure to obtain a solution that was second-order accurate or higher, then we could mimic this approach and solve an expanded system, but with a first-order accurate solution we cannot. Nor can we take a direct quadratic approximation to conditional welfare, which gives the following for the fiscal policy model:

$$U_t \approx E_t \left[\sum_{t=0}^{\infty} \beta^t u_t \right], \tag{50}$$

where

$$u_t = u_{ss} + c_{ss}^{-\sigma} \left(c_t - c_{ss} \right) + \mu g_{ss}^{-\eta} \left(g_t - g_{ss} \right) - \frac{\sigma c_{ss}^{-\sigma-1}}{2} \left(c_t - c_{ss} \right)^2 - \frac{\mu \eta g_{ss}^{-\eta-1}}{2} \left(g_t - g_{ss} \right)^2, \tag{51}$$

and evaluate it using the model's first order solution. This fails because equation (51) contains linear terms.

As Kim and Kim (2006) emphasize, computing conditional welfare correctly to second-order requires that the model be solved to second-order accuracy. Accordingly, if one is interested in computing the conditional welfare associated with the time-consistent policy, then the solution procedure used would require using a third-order perturbation method to enable the derivative in the generalized-Euler equation to be approximated to second-order accuracy. The approach parallels that described in section 3, but with a third-order perturbation method used and with iteration occurring over an expanded set of coefficients. Upon convergence, and after discarding the third-order terms, the resulting solution will be second-order accurate and can be used to compute conditional welfare using a quadratic approximation that retains linear terms (like equations (50)—(51) above), (Kim and Kim, 2006).

We implement the method described above and apply it to each of the four models. For this purpose we use the third-order perturbation method developed in Binning (2013). The

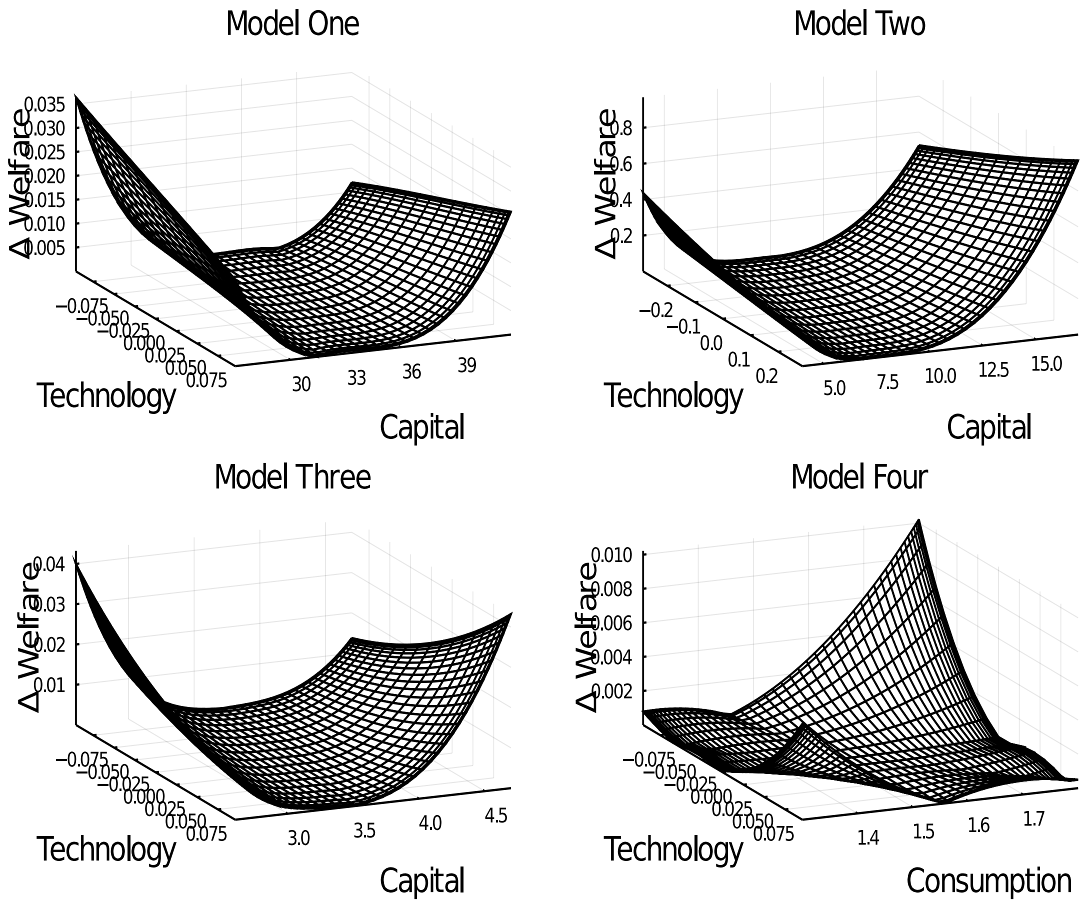

Figure 5: Absolute differences in computed welfare

absolute differences between the welfare computed using this second-order accurate solution and that computed using the projection method are displayed in figure 5.

Figure 5 shows that welfare is well-approximated in the vicinity of the steady state for all four models, with the absolute differences in welfare close to zero. Further, with the exception of model two, the perturbation solution delivers an estimate of welfare that is accurate over the entire domain shown (which is the domain used for the projection method). For model two, welfare looks to be less well-approximated, most noticeably when capital is far from steady state.

It is important to recognize that the welfare approximation that the perturbation method employs achieves second-order accuracy through the iterative use of a third-order perturbation solution to the model. Because a third-order solution is used, the resulting welfare approximation requires derivatives that are higher than second-order, at least for the models containing a generalized Euler equation (models two—four). As a consequence, the accuracy shown here is achieved through the use of first-, second-, and third-derivatives of the utility function and the constraints. As attractive as LQ methods are it seems difficult to apply them to models with time-inconsistency other than in special cases. One notable special case is that of discretionary monetary policy when the steady state is efficient and is characterized by zero inflation. Another special case is where the model does not contain any endogenous state variables (see appendices A.1 and A.2).

6 Conclusions

This paper has developed and illustrated a procedure for solving dynamic stochastic models containing generalized-Euler equations—models exhibiting time-inconsistency—using perturbation methods. The approach does not require the optimization problem being solved to be reformulated in terms of an approximate LQ problem, but works, instead, with systems of first-order conditions. For this reason, the procedure does not involve forming a second-order approximation to household welfare and it can be applied without modification to models where the steady state is inefficient. Because it utilizes a perturbation around the steady state, the procedure inherits all the strengths and weaknesses of perturbation methods. In particular, the method scales well and can be applied to medium- to large-scale models, but loses accuracy in regions away from the steady state. We note how the solution method can be used to take log-linear approximations, as opposed to linear approximations, and illustrate in the context of computing conditional welfare how it can be adapted to construct second- or higher-order accurate approximations.

To demonstrate the procedure and to assess its accuracy, we apply it to four different models, three of which involve time-consistent decision-making: the stochastic growth model, time-consistent fiscal policy, quasi-geometric preferences, and time-consistent monetary policy. To asses its accuracy, we compare the results from the perturbation method to those from a projection-based solution and show that the method is accurate except in regions far from the steady state, regions where the model spends little time. Lastly, we show how perturbation can be employed to compute welfare and note that the approximation requires third-order derivatives, which undermines the general applicability of LQ methods to models with time-inconsistency.

Appendix A: LQ approximations

In this Appendix we consider two cases where a valid LQ approximation to a problem involving time-inconsistency can be formed. In Appendix A1 we show that a valid LQ approximation is possible when the model's steady state is efficient; in Appendix A2 we show that a valid LQ approximation is possible when the model does not contain any endogenous state variables. In Appendix A3 we turn to the general case where the model's steady state is not efficient and the model contains endogenous state variables. For this general case we show that a valid LQ approximation is not possible because such an approximation requires knowing the equilibrium decision rules for the choice variables, along with their first- and second-derivatives with respect to the endogenous state variables.

Appendix A1: Efficient steady state

Here we briefly analyze the case where the model's steady state is efficient and show that it can be solved using LQ methods. Typically, but not always, the decision-maker facing the time-inconsistency problem will be a policy maker, such as the fiscal authority (model two) or the central bank (model four), but it could also be the representative household (model three). Before introducing the problem, we note that the case considered here gains traction when considering optimal discretionary monetary policy because in those models the steady state can often be rendered efficient through the simple introduction of a production subsidy or an employment subsidy to offset the distortionary effects of monopolistic competition. For other applications where time-inconsistency matters, such as fiscal policy or quasi-geometric preferences, this case is of less interest.

Efficient steady state

Let $\mathbf{z}_t$ be an $s \times 1$ vector of shocks with innovations $\varepsilon_t \sim i.i.d[\mathbf{0}, \mathbf{\Omega}]$ and $\mathbf{y}_t$ be an $n \times 1$ vector of non-predetermined/choice variables. We suppose that the constraints on the planner's problem are:

$$\mathbf{z}_{t+1} - \mathbf{\Gamma}\mathbf{z}_t - \varepsilon_{t+1} = \mathbf{0}, \tag{A1}$$

$$\mathbf{p}\left(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{z}_t\right) = \mathbf{0}, \tag{A2}$$

and that the representative household's expected discounted lifetime utility takes the form:

$$U_t = E_0\left[\sum_{t=0}^{\infty} \beta^t u\left(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{z}_t\right)\right]. \tag{A3}$$

The functions, $\mathbf{p}$ and u, are each assumed to be concave and smooth. Equations (A1) and (A2) distinguish between equations for the shocks processes, equation (A1), and equations that bind on the endogenous variables, equation (A2). We assume that there are s shocks and less than n equations in $\mathbf{p}$. We further assume that $\mathbf{\Gamma}$ has spectral radius less than one and therefore that the steady state for $\mathbf{z}_t$ found from equation (A1) is the zero vector, $\mathbf{0}$.

To determine the efficient steady state for $\mathbf{y}_t$ we formulate the planner's problem, which is to choose $\{\mathbf{y}_t, \lambda_t\}_{t=0}^{\infty}$ to extremize the Lagrangian:

$$\mathcal{L} = E_0 \left[\sum_{t=0}^{\infty} \beta^t \left(u\left(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{z}_t\right) + \lambda_t^T \mathbf{p}\left(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{z}_t\right) \right) \right], \tag{A4}$$

leading to the first-order conditions:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{y}_t} : \mathbf{u}_2 + \lambda^T \mathbf{p}_2 + \beta E_t \left[\mathbf{u}_1^{'} + \left(\lambda^{'}\right)^T \mathbf{p}_1^{'} \right] = \mathbf{0}, \tag{A5}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} : \mathbf{p} = \mathbf{0}. \tag{A6}$$

where we have suppressed the function arguments and time subscripts, introduced "primes" to signify next-period values, and used function-subscripts to denote the derivative with respect to the numbered argument. We denote by $\overline{\mathbf{y}}$ and $\overline{\lambda}$ the (efficient) steady state values that satisfy jointly equations (A5) and (A6).

Time-consistent decision problem

Let the first-order conditions and constraints aggregated across households and firms that are not already included in equation (A2) be denoted:

$$E_t \left[\mathbf{f}\left(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{y}_{t+1}, \mathbf{z}_t\right) \right] = \mathbf{0}. \tag{A7}$$

Because the model's steady state is efficient, equation (A7) holds at the values for $\overline{\mathbf{y}}$ and $\overline{\mathbf{z}}$ determined from the planner's problem. The problem for the decision-maker facing the time-consistency problem is to choose $\{\mathbf{y}_t, \lambda_t, \mu_t\}$ to solve the Bellman equation:

$$V\left(\mathbf{y}_{t-1}, \mathbf{z}_t\right) = \max_{\{\mathbf{y}_t, \lambda_t, \mu_t\}} \left[\begin{array}{c} u\left(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{z}_t\right) + \lambda_t^T \left[\mathbf{p}\left(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{z}_t\right)\right] \\ +\mu_t^T E_t \left[\mathbf{f}\left(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{g}\left(\mathbf{y}_t, \mathbf{z}_{t+1}\right), \mathbf{z}_t\right)\right] + \beta E_t \left[V\left(\mathbf{y}_t, \mathbf{z}_{t+1}\right)\right] \end{array} \right], \tag{A8}$$

where $\mathbf{y}_t = \mathbf{g}\left(\mathbf{y}_{t-1}, \mathbf{z}_t\right)$ is the unknown equilibrium law-of-motion relating $\mathbf{y}_t$ to the state variables. From the Bellman equation, we recover equations (A2) and (A7) and obtain the following first-order and envelope conditions:

$$\mathbf{0} = \mathbf{u}_2 + \lambda^T \mathbf{p}_2 + \mu^T (\mathbf{f}_3 \mathbf{g}_1 + \mathbf{f}_2) + \beta E_t \left[\mathbf{V}_1^{'}\right], \tag{A9}$$

$$\mathbf{V}_1 = \mathbf{u}_1 + \lambda^T \mathbf{p}_1 + \mu^T \mathbf{f}_1, \tag{A10}$$

$$\mathbf{V}_2 = \mathbf{u}_3 + \lambda^T \mathbf{p}_3 + \mu^T \mathbf{f}_4, \tag{A11}$$

respectively; again we have suppressed the function arguments and time-subscripts for clarity. Combining equations (A9) and (A10) we get:

$$\mathbf{0} = \mathbf{u}_2 + \lambda^T \mathbf{p}_2 + \mu^T (\mathbf{f}_3 \mathbf{g}_1 + \mathbf{f}_2) + \beta E_t \left[\mathbf{u}_1^{'} + \left(\lambda^{'}\right)^T \mathbf{p}_1^{'} + \left(\mu^{'}\right)^T \mathbf{f}_1^{'}\right], \tag{A12}$$

which is in the form of a generalized Euler equation (note equation (A12) can be simplified using equation (A5)).

The next step is to form a second-order approximation to the Bellman equation around the efficient steady state. Because the steady state is efficient, this approximation employs a second-order approximation to the constraints in equation (A2), but only a first-order approximation to the equations in equation (A7). The second-order approximation to the Bellman equation is:

$$\begin{aligned}\begin{bmatrix} \widehat{\mathbf{y}}_{t-1}^T & \widehat{\mathbf{z}}_t^T \end{bmatrix} \begin{bmatrix} \overline{\mathbf{V}}_{11} & \overline{\mathbf{V}}_{12} \\ \overline{\mathbf{V}}_{21} & \overline{\mathbf{V}}_{22} \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{y}}_{t-1} \\ \widehat{\mathbf{z}}_t \end{bmatrix} + v = {} & f.o.t. + s.o.t. \\ & + \lambda_t^T \left(\overline{\mathbf{p}}_1 \widehat{\mathbf{y}}_{t-1} + \overline{\mathbf{p}}_2 \widehat{\mathbf{y}}_t + \overline{\mathbf{p}}_3 \widehat{\mathbf{z}}_t\right) \\ & + \mu_t^T \left(\overline{\mathbf{f}}_1 \widehat{\mathbf{y}}_{t-1} + \left(\overline{\mathbf{f}}_2 + \overline{\mathbf{f}}_3 \overline{\mathbf{g}}_1\right) \widehat{\mathbf{y}}_t + \overline{\mathbf{f}}_4 \widehat{\mathbf{z}}_t\right) \\ & + \beta E_t \left[\begin{bmatrix} \widehat{\mathbf{y}}_t^T & \widehat{\mathbf{z}}_{t+1}^T \end{bmatrix} \begin{bmatrix} \overline{\mathbf{V}}_{11} & \overline{\mathbf{V}}_{12} \\ \overline{\mathbf{V}}_{21} & \overline{\mathbf{V}}_{22} \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{y}}_t \\ \widehat{\mathbf{z}}_{t+1} \end{bmatrix} + v\right]\end{aligned} \tag{A13}$$

where $f.o.t.$ and $s.o.t.$ refer to first order terms and second order terms, respectively, and the first order terms are:

$$\begin{aligned} f.o.t. = {} & \left(\overline{\mathbf{u}}_1 + \overline{\lambda}^T \overline{\mathbf{p}}_1 + \overline{\mu}^T \overline{\mathbf{f}}_1 - \overline{\mathbf{V}}_1\right) \widehat{\mathbf{y}}_{t-1} + \left(\overline{\mathbf{u}}_2 + \overline{\lambda}^T \overline{\mathbf{p}}_2 + \overline{\mu}^T \left(\overline{\mathbf{f}}_3 \overline{\mathbf{g}}_1 + \overline{\mathbf{f}}_2 + \beta \overline{\mathbf{V}}_1\right)\right) \widehat{\mathbf{y}}_t \\ & + \left(\overline{\mathbf{u}}_3 + \overline{\lambda}^T \overline{\mathbf{p}}_3 + \overline{\mu}^T \overline{\mathbf{f}}_4 - \overline{\mathbf{V}}_2\right) \widehat{\mathbf{z}}_t, \end{aligned} \tag{A14}$$

Looking at the expression for the first-order terms in the welfare approximation, equations (A9)—(A11) imply that the first order terms equal zero, that the welfare approximation contains only second-order terms and is a quadratic, and that the approximated Bellman equation is LQ.

Appendix A2: No endogenous state variables

In this appendix we treat that case where the underlying model does not have any endogenous state variables. We show that it is possible in this case to formulate the time-inconsistent decision problem in terms of an approximate LQ problem.

Let the model be described by the system:

$$\mathbf{z}_{t+1} - \mathbf{\Gamma}\mathbf{z}_t - \varepsilon_{t+1} = \mathbf{0}, \tag{B1}$$

$$E_t\mathbf{h}\left(\mathbf{y}_t, \mathbf{y}_{t+1}, \mathbf{z}_t\right) = \mathbf{0}, \tag{B2}$$

where equation (B2) represents the constraints and first-order conditions associated with the decision-maker that does not face a time-inconsistency problem, and let the representative household's expected discounted lifetime utility take the form:

$$U_t = E_0\left[\sum_{t=0}^{\infty}\beta^t u\left(\mathbf{y}_t, \mathbf{z}_t\right)\right]. \tag{B4}$$

The problem for the decision-maker facing time-inconsistency is to choose $\{\mathbf{y}_t, \lambda_t\}$ to solve the Bellman equation:

$$V\left(\mathbf{z}_t\right) = \max_{\{\mathbf{y}_t,\lambda_t\}}\left[u\left(\mathbf{y}_t, \mathbf{z}_t\right) + \lambda_t^T E_t\left[\mathbf{h}\left(\mathbf{y}_t, \mathbf{g}\left(\mathbf{z}_{t+1}\right), \mathbf{z}_t\right)\right] + \beta E_t\left[V\left(\mathbf{z}_{t+1}\right)\right]\right], \tag{B5}$$

where $\mathbf{y}_{t+1} = \mathbf{g}\left(\mathbf{z}_{t+1}\right)$ is the unknown equilibrium law-of-motion relating $\mathbf{y}_{t+1}$ to the state variables. From the Bellman equation, the first-order conditions are:

$$\frac{\partial V\left(\mathbf{z}\right)}{\partial \mathbf{y}} : \mathbf{u}_1 + \lambda^T\mathbf{h}_1 = \mathbf{0}, \tag{B6}$$

along with equation (B2), while the derivative with respect to $\mathbf{z}_t$ (which will be useful later) is:

$$\frac{\partial V\left(\mathbf{z}\right)}{\partial \mathbf{z}} : \mathbf{u}_2 + \lambda^T\mathbf{h}_3 = \mathbf{0}. \tag{B7}$$

From equations (B1)—(B2) and (B6) we can determine the steady state values for $\overline{\mathbf{z}}$, $\overline{\mathbf{y}}$, and $\overline{\lambda}$, which need not be efficient.

The next step is to form a second-order approximation to the Bellman equation around the steady state. Using $s.o.t$ to denote second order terms the approximation gives:

$$\widehat{\mathbf{z}}_t^T\mathbf{V}\widehat{\mathbf{z}}_t + v = \max\left[f.o.t. + s.o.t + \lambda_t^T\left(\overline{\mathbf{h}}_1\widehat{\mathbf{y}}_t + \overline{\mathbf{h}}_3\widehat{\mathbf{z}}_t\right) + \beta E_t\left[\widehat{\mathbf{z}}_{t+1}^T\mathbf{V}\widehat{\mathbf{z}}_{t+1} + v\right]\right], \tag{B8}$$

where the linear terms, $f.o.t.$, are given by:

$$f.o.t = \left(\overline{\mathbf{u}}_1 + \overline{\lambda}^T\overline{\mathbf{h}}_1\right)\widehat{\mathbf{y}}_t + \left(\overline{\mathbf{u}}_2 + \overline{\lambda}^T\overline{\mathbf{h}}_3\right)\widehat{\mathbf{z}}_t. \tag{B9}$$

Looking at the expression for *f.o.t.*, equations (B6) and (B7) imply that the first order terms equal zero, that the welfare approximation is a quadratic, and that the approximated Bellman equation is LQ.

Appendix A3: Endogenous state variables and distorted steady state

Here we consider models for which the steady state is not efficient and in which there are endogenous state variables. Models two, three, and four from the main text fall into this category. For this category of models we illustrate the difficulties associated with trying to form a valid LQ approximation to the problem and show why such an approximation is not possible.

Let the model be described by the system:

$$\mathbf{z}_{t+1} - \mathbf{\Gamma}\mathbf{z}_t - \varepsilon_{t+1} = \mathbf{0}, \tag{C1}$$

$$E_t \left[\mathbf{h}\left(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{y}_{t+1}, \mathbf{z}_t\right)\right] = \mathbf{0}, \tag{C2}$$

with the representative household's expected discounted lifetime utility given by:

$$U_t = E_0 \left[\sum_{t=0}^{\infty} \beta^t u\left(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{z}_t\right)\right]. \tag{C3}$$

The problem for the decision-maker in the decentralized model is to choose $\{\mathbf{y}_t, \lambda_t\}$ to solve the Bellman equation:

$$V\left(\mathbf{y}_{t-1}, \mathbf{z}_t\right) = \max_{\{\mathbf{y}_t, \lambda_t\}} \left[u\left(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{z}_t\right) + \lambda_t^T \left[E_t\left[\mathbf{h}\left(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{g}\left(\mathbf{y}_t, \mathbf{z}_{t+1}\right), \mathbf{z}_t\right)\right]\right] + \beta E_t\left[V\left(\mathbf{y}_t, \mathbf{z}_{t+1}\right)\right]\right], \tag{C4}$$

where $\mathbf{y}_t = \mathbf{g}\left(\mathbf{y}_{t-1}, \mathbf{z}_t\right)$ is the unknown equilibrium law-of-motion relating $\mathbf{y}_t$ to the state variables. From the Bellman equation, the first-order conditions and the envelope conditions are:

$$\mathbf{0} = \mathbf{u}_2 + \lambda^T\left(\mathbf{h}_3\mathbf{g}_1 + \mathbf{h}_2\right) + \beta E_t\left[\mathbf{V}_1^{'}\right], \tag{C5}$$

$$\mathbf{V}_1 = \mathbf{u}_1 + \lambda^T \mathbf{h}_1, \tag{C6}$$

$$\mathbf{V}_2 = \mathbf{u}_3 + \lambda^T \mathbf{h}_4, \tag{C7}$$

along with equation (C2). Equations (C5) and (C6) can be combined to give:

$$\mathbf{0} = \mathbf{u}_2 + \lambda^T\left(\mathbf{h}_3\mathbf{g}_1 + \mathbf{h}_2\right) + \beta E_t\left[\mathbf{u}_1^{'} + \left(\lambda^{'}\right)^T \mathbf{h}_1^{'},\right]. \tag{C8}$$

Ordinarily, equations (C1), (C2) and (C8) would be solved to obtain the steady state values for $\overline{\mathbf{z}}$,$\overline{\mathbf{y}}$, and $\overline{\lambda}$. However, in this case, this is not possible because equation (C8) is a generalized Euler equation that contains the unknown derivative, $\mathbf{g}_1$.

Suppose that the steady state values $\overline{\mathbf{z}}$,$\overline{\mathbf{y}}$, and $\overline{\lambda}$ were somehow known. To produce a valid LQ approximation of the Bellman equation we need to form a second-order Taylor approximation to both period-t utility, $u(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{z}_t)$, and equation (C2). Approximating the former is not difficult. However, equation (C2) contains $\mathbf{g}$ so a second-order approximation to equation (C2) requires taking first- and second-derivatives of $\mathbf{g}$, but $\mathbf{g}$ is unknown. Even if one were to guess values for these derivatives in an attempt to implement at an iterative solution (successive approximation) the approach would fail because the linear solution obtained would not allow a correct update of $\mathbf{g}$'s second derivatives.

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