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## An Unobserved Components Model of Total Factor Productivity and the Relative Price of Investment

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This paper applies the common stochastic trends representation approach to the time series of total factor productivity and the relative price of investment to investigate the relationship between neutral technology and investment-specific technology. The permanent and transitory movements in both series are estimated efficiently via MCMC methods using band matrix algorithms. The results indicate that total factor productivity and the relative price of investment are, each, well-represented by an integrated process of order one. In addition, their time series share a common trend component that we interpret as reflecting changes in General Purpose Technology. These results suggest that (1) the traditional view of assuming that neutral technology and investment-specific technology follow independent processes is not supported by the features of the time series and (2) advances in information and communication technologies are general purpose technological progress that drive the trend in aggregate TFP in the United States.

## **Keywords**

Business cycles, Investment-specific technological change, Total Factor Productivity, Unobserved Components Model.

## **JEL Classification**

E22, E32, C32

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# An Unobserved Components Model of Total Factor Productivity and the Relative Price of Investment

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October 2020

## Abstract

This paper applies the common stochastic trends representation approach to the time series of total factor productivity and the relative price of investment to investigate the relationship between neutral technology and investment-specific technology. The permanent and transitory movements in both series are estimated efficiently via MCMC methods using band matrix algorithms. The results indicate that total factor productivity and the relative price of investment are, each, well-represented by an integrated process of order one. In addition, their time series share a common trend component that we interpret as reflecting changes in General Purpose Technology. These results suggest that (1) the traditional view of assuming that neutral technology and investment-specific technology follow independent processes is not supported by the features of the time series and (2) advances in information and communication technologies are general purpose technological progress that drive the trend in aggregate TFP in the United States.

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# 1 Introduction

Real Business Cycle (RBC) models and their modern versions in Dynamic Stochastic General Equilibrium (DSGE) models have enjoyed a great success since they were first introduced to the literature by Kydland and Prescott (1982). These models are the standard framework used by academics and policymakers to investigate macroeconomic questions and analyze the effects of monetary and fiscal policies. Despite their popularity in academic and policy circles, these models have also had their fair share of criticism. They typically have a weak amplification and internal propagation mechanism to match some crucial characteristics that we observe in the time series of macroeconomic variables. These features include, but are not limited to, the positive serial correlation in the growth of output for instance, and the bell-shaped responses of the macro variables to exogenous shocks. Early studies like Cogley and Nason (1995) extend the standard RBC model with an integrated process of technology and adjustment costs and lags, and show that the model rely largely on the exogenous process of technology to improve its fit of the time series characteristics. Modern and more advanced DSGE model include various sources of short-run rigidities as habits in consumption, investment adjustment costs, and costly factor utilization. Also, it is quite common to incorporate in these models non-stationary components identified as uncorrelated and integrated or uncorrelated and integrated smooth trend processes of neutral technology and/or investment-specific technology. While these latter additions appear rather suitable, they are quite arbitrary, driven by convenience and tractability, and most importantly not supported by existing time series evidence.

This paper aims to contribute two points. The first point is that the non-stationary component of neutral technology and investment-specific technology should be modelled each as following an integrated process of order one or ARIMA(0,1,0) process. In fact, the choice of such specification in the DSGE literature is quite dispersed. While several papers like Justiniano et al. (2011), Schmitt-Grohé and Uribe (2011), and Kaihatsu and Kurozumi (2014) adopt an ARIMA(1,1,0) specification, other notable studies as Smets and Wouters (2007) and Fisher (2006) impose either a trend stationary ARMA (1,0) process or an integrated ARIMA(0,1,0) process, respectively. Through our exploration of the literature, we find that this disconnected practice might be facilitated by the lack of guidance from time series evidence, and we hope to fill in this gap. Our second point is that these non-stationary components are not orthogonal to each other. While this orthogonality assumption is

conventional in most DSGE models, there appears to exist some evidence pointing otherwise. Basu et al. (2013) question the basis of this assumption by demonstrating that consumption-specific productivity shocks (which may be interpreted as neutral technology shocks) and investment-specific productivity shocks are correlated. Chen and Wemy (2015) arrive at the same conclusion using results from a structural VAR.

To make these two points, we use an Unobserved Component (UC) framework to decompose the time series for the Relative Price of Investment (RPI) and Total Factor Productivity (TFP) into two unobserved components: one unobserved component that reflects permanent or trend movements in the series, and the other that captures transitory movements in the series. Then, given the stock of evidence from Basu et al. (2013) and Chen and Wemy (2015) that appears to indicate that these two series might be related in the long-run, we specify that RPI and TFP share a common unobserved trend component. We label this component as General Purpose Technology (GPT), and we argue that it reflects spillover effects from innovations in Information and Communication Technologies (ICT) to aggregate productivity.

These two points are not entirely new in the literature. In particular, the relationship between neutral technology and investment-specific technology has been explored in Schmitt-Grohé and Uribe (2011) and Benati (2013) through the lenses of statistical tests of units root and co-integration. Using quarterly U.S. data over the period from 1948.I to 2004.IV, Schmitt-Grohé and Uribe (2011) find that RPI and TFP contain a non-stationary stochastic component which is common to both series. In other words, TFP and RPI are co-integrated, which implies that neutral technology and investment-specific technology should be modeled as containing a common stochastic trend. Therefore, Schmitt-Grohé and Uribe (2011) estimate a DSGE model that imposes this result and identify shocks to the common stochastic trend in neutral and investment specific productivity. They find that the shocks play a sizable role in driving business cycle fluctuations. Specifically, the common shocks explain three fourth of the variances of output and investment growth and about one third of the predicted variances of consumption growth and hours worked. However, Benati (2013) expresses some doubt about the co-integration results in Schmitt-Grohé and Uribe (2011). He claims that TFP and RPI are most likely not co-integrated, and he traces the origin of this finding of co-integration to the use of an inconsistent criteria for lag order selection in the Johansen procedure. When he uses the SIC or HQ criterion, the Johansen test points to no

co-integration. Yet, he establishes that although the two series may not be co-integrated, they may still share a common stochastic non-stationary component. Specifically, Benati (2013) uses an approach proposed by Cochrane and Sbordone (1988) that searches for a statistically significant extent of co-variation between the two series' long-horizon differences. The basis of this approach is that whatever may happen at short-to-medium run horizons for the two non-stationary series that are suspected to be related in the long-run but are not co-integrated, the two series' long-horizon differences should exhibit a statistically significant extent of co-variation. Therefore, Benati (2013) suggests that evidence points towards a common  $I(1)$  component inducing a negative co-variation between TFP and RPI at long horizons. While the UC framework that we propose in this study complements the investigation in Schmitt-Grohé and Uribe (2011) and Benati (2013), it distances itself from statistical tests and their associated issues of lag order and low power, and offers more flexibility and greater benefits. First, our framework nests all competing theories of the univariate and bivariate properties of RPI and TFP. In fact, the framework incorporates Schmitt-Grohé and Uribe (2011)'s result of co-integration and Benati (2013)'s findings of the long-run positive comovement as special cases. In that sense, we are able to evaluate the validity of all the proposed specifications of neutral technology and investment-specific technology in the DSGE literature. In addition, it is grounded on economic theory. In particular, we demonstrate that our UC framework may be derived from the neoclassical growth model used by studies in the growth accounting literature, e.g Greenwood et al. (1997), Oulton (2007), and Greenwood and Krusell (2007), to investigate the contribution of embodiment in the growth of aggregate productivity. As such, we may easily interpret the idiosyncratic unobserved component of RPI and TFP, and their potential interaction from the lenses of a well-defined economic structure.

Using the time series of the logarithm of (the inverse of) RPI and TFP from 1959.*II* to 2019.*II* in the United States, we estimate our model through Markov chain Monte Carlo (MCMC) methods developed by Chan and Jeliazkov (2009) and Grant and Chan (2017). A novel feature of this approach is that it builds upon the band and sparse matrix algorithms for state space models, which are shown to be more efficient than the conventional Kalman filter-based algorithms. The results indicate that the trend component in (the inverse of) RPI and TFP is better captured by an integrated process of order one. While the trend component is not common to both series, it appears that (the inverse of) RPI and TFP share a common stochastic trend component which

captures a positive long-run covariation between the series. We argue that the common stochastic trend component is a reflection of General Purpose Technological (GPT) progress from innovations in information and communications technologies. In fact, using industry-level and aggregate-level data, several studies, such as Cummins and Violante (2002), Basu et al. (2004), and Jorgenson et al. (2007), document that improvements in information communication technologies contributed to productivity growth in the 1990s and the 2000s in essentially every industry in the United States. Through our results, we are also able to confirm that changes in the trend of RPI have lasting impact on the long-run developments of TFP.

These findings have important implications for the modelling and estimation of DSGE models. They suggest that researchers may need to re-evaluate the approach of specifying and identifying the underlying processes of technology in DSGE models. DSGE models are typically built to explain the cyclical movements in the data. Therefore, preliminary data transformation are required before the model is estimated. As we discussed above, one alternative entails arbitrarily building into the models unit root non-cyclical components of neutral technology and investment-specific technology and filtering the raw data using these model-based transformation. Nonetheless, this requires some knowledge of the number, and nature of the exogenous processes driving the non-cyclical components. As discussed in Ferroni (2011) and Canova (2014), specification errors may plague the estimation of the structural parameters. In regards to business cycle fluctuations, a group of parameters of great interest are those related to the exogenous processes (persistence and volatility). Significant differences in their estimates imply that impulse responses, and the contribution of the structural shocks to the volatility of the observable variables, may vary and lead to erroneous conclusions about the sources of fluctuations. Through our findings on the time series characteristics of RPI and TFP, we hope to provide some clarification and direction on the specification of such model-based trend specifications.

## 2 Model

Fundamentally, there are multiple potential representations of the relationship between the trend components and the transitory components in RPI and TFP. A plausible representation is based on the concept that investment-specific technology may be interpreted as General Purpose Technology

(GPT). Simply put, GPT can be defined as a new method that leads to fundamental changes in the production process of industries using it, and it is important enough to have a protracted aggregate impact on the economy. As discussed extensively in Jovanovic and Rousseau (2005), electrification and information technology (IT) are probably the most recent GPTs so far. In fact, using industry-level data, Cummins and Violante (2002), and Basu et al. (2004) find that improvements in information communication technologies contributed to productivity growth in the 1990s in essentially every industry in the United States. Moreover, Jorgenson et al. (2007) show that much of the TFP gain in the United States in the 2000s originated in industries that are the most intensive users of information technology. Specifically, the authors look at the contribution to the growth rate of value added and aggregate TFP in the United States in 85 industries. They find that the four IT-producing industries (computer and office equipment, communication equipment, electronic components, and computer services) accounted for nearly all of the acceleration of aggregate TFP in 1995-2000. Furthermore, IT-using industries, which engaged in great IT-investment in the period 1995-2000, picked up the momentum and contributed almost half of the aggregate acceleration in 2000-2005. Overall, the authors assert that IT-related industries made significant contributions to the growth rate of TFP in the period 1960-2005. Similarly, Gordon (1990), and Cummins and Violante (2002) have argued that technological progress in areas such as equipment and software have contributed to a faster rate of decline in RPI; a fact that has also been documented in Fisher (2006) and Justiniano et al. (2011).

Given this empirical evidence, we propose that RPI and TFP can each be represented as the sum of a permanent component, an idiosyncratic component, and a transitory component in the following fashion:

$$z_t = \tau_t + c_{z,t}, \tag{1}$$

$$x_t = \gamma\tau_t + \tau_{x,t} + c_{x,t}, \tag{2}$$

where  $z_t$  is the logarithm of RPI,  $x_t$  is the logarithm of TFP,  $\tau_t$  is the common trend component in RPI and TFP,  $\tau_{x,t}$  is the idiosyncratic trend component in TFP, and  $c_{z,t}$  and  $c_{x,t}$  are the corresponding idiosyncratic transitory components. The parameter  $\gamma$  captures the relationship between the trends in RPI and TFP, and symbolizes the GPT argument.



We assume that the first differences of the trend components are modeled as following stationary processes,

$$\Delta\tau_t = (1 - \varphi_\mu)\zeta_1 1(t < T_B) + (1 - \varphi_\mu)\zeta_2 1(t \geq T_B) + \varphi_\mu \Delta\tau_{t-1} + \eta_t, \quad (3)$$

$$\Delta\tau_{x,t} = (1 - \varphi_{\mu_x})\zeta_{x,1} 1(t < T_B) + (1 - \varphi_{\mu_x})\zeta_{x,2} 1(t \geq t_0) + \varphi_{\mu_x} \Delta\tau_{x,t-1} + \eta_{x,t}, \quad (4)$$

where  $1(A)$  is the indicator function for the event  $A$ , and  $T_B$  is the index corresponding to the time of the break at 1982.I.

Finally, following Morley et al. (2003) and Grant and Chan (2017), the transitory components are assumed to follow AR(2) processes:

$$c_{z,t} = \phi_{z,1}c_{z,t} + \phi_{z,2}c_{z,t} + \varepsilon_{z,t}, \quad (5)$$

$$c_{x,t} = \phi_{x,1}c_{x,t} + \phi_{x,2}c_{x,t} + \varepsilon_{x,t}. \quad (6)$$

The presence of the common stochastic trend component in RPI and TFP captures the argument that innovations in information technologies have been the main driver of the trend in RPI and have also been a major source of the growth in productivity in the United States. The addition of the idiosyncratic trend component in TFP is meant to capture the potential effect of other factors, besides innovations in IT, on the growth in TFP. As mentioned in Jorgenson et al. (2007), non-IT industries (examples of such industries are farms, agricultural services and forestry, fishing) also showed a continuous acceleration in their contribution to the growth rate of aggregate TFP, from a net contribution of 0.23 percent for 1960-1995 to 0.37 percent for 1995-2000 and 0.66 percent for 2000-2005.

A complementary interpretation of the framework originates from the growth accounting literature associated with the relative importance of embodiment in the growth of technology. In particular, Greenwood et al. (1997), Greenwood and Krusell (2007) and Oulton (2007) show that the non-stationary component in TFP is a combination of the trend in neutral technology and the trend in investment specific technology. In that case, we may interpret the common component,  $\tau_t$ , as the trend in investment-specific technology,  $\tau_{x,t}$  as the trend in neutral technology, and the parameter  $\gamma$  as the current price share of investment in the value of output. In Appendix A.1, we

show that our UC framework may be derived from a simple neoclassical growth model.

We argue that this UC framework offers more flexibility as it nests all the univariate and bivariate representations of RPI and TFP in the theoretical and empirical literature. Specifically, consider the following cases:

1. If  $\gamma = 0$ , then trends in RPI and TFP are independent of each other, and this amounts to the specifications adopted in Justiniano et al. (2011), Schmitt-Grohé and Uribe (2011), and Kaihatsu and Kurozumi (2014).
2. If  $\gamma = 0$ ,  $\varphi_\mu = 0$  and  $\varphi_{\mu_x} = 0$ , then the trend components follow a random walk plus drift, and the resulting specification is equivalent to the assumptions found in Fisher (2006).
3. If  $\gamma \neq 0$ ,  $\varphi_{\mu_x} = 0$  and  $\sigma_{\eta_x} = 0$ , then RPI and TFP are co-integrated as argued in Schmitt-Grohé and Uribe (2011):
  - (a) If  $\zeta_x \neq 0$ , then the idiosyncratic trend in TFP is a linear deterministic trend, and RPI and TFP are co-integrated around a linear deterministic trend.
  - (b) If  $\zeta_x = 0$ , the RPI and TFP are co-integrated around a constant term.
4. If  $\gamma > 0$  ( $\gamma < 0$ ), then the common trend component has a positive effect on both RPI and TFP (a positive effect on RPI and a negative effect on TFP), which would imply a positive (negative) covariation between the two series. This is essentially the argument in Benati (2013)
5. If  $\varphi_\mu = 1$  and  $\varphi_{\mu_x} = 1$ . This gives rise to the following smooth evolving processes for the trend:

$$\Delta\tau_t = \zeta_1 1(t < T_B) + \zeta_2 1(t \geq T_B) + \varphi_\mu \Delta\tau_{t-1} + \eta_t, \quad (7)$$

$$\Delta\tau_{x,t} = \zeta_{x,1} 1(t < T_B) + \zeta_{x,2} 1(t \geq t_0) + \varphi_{\mu_x} \Delta\tau_{x,t-1} + \eta_{x,t}. \quad (8)$$

Through model comparisons, we are able to properly assess the validity of each of these competing assumptions and shed light on the appropriate representation between RPI and TFP.

### 3 Bayesian Estimation

In this section we provide the details of the priors and outline the Bayesian estimation of the unobserved components model in (1)–(6). In particular, we highlight how the model can be estimated efficiently using band matrix algorithms instead of conventional Kalman filter based methods.

We assume proper but relatively noninformative priors for the model parameters  $\gamma$ ,  $\boldsymbol{\phi} = (\phi_{z,1}, \phi_{z,2}, \phi_{x,1}, \phi_{x,2})'$ ,  $\boldsymbol{\varphi} = (\varphi_\mu, \varphi_{\mu_x})'$ ,  $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \zeta_{x,1}, \zeta_{x,2})'$ ,  $\boldsymbol{\sigma}^2 = (\sigma_\eta^2, \sigma_{\eta_x}^2, \sigma_z^2, \sigma_x^2)'$  and  $\boldsymbol{\tau}_0$ . In particular, we adopt a normal prior for  $\gamma$ :  $\gamma \sim \mathcal{N}(\gamma_0, V_\gamma)$  with  $\gamma_0 = 0$  and  $V_\gamma = 1$ . These values imply a weakly informative prior centered at 0. Moreover, we assume independent priors for  $\boldsymbol{\phi}$ ,  $\boldsymbol{\varphi}$ ,  $\boldsymbol{\zeta}$  and  $\boldsymbol{\tau}_0$ :

$$\boldsymbol{\phi} \sim \mathcal{N}(\boldsymbol{\phi}_0, \mathbf{V}_\boldsymbol{\phi})1(\boldsymbol{\phi} \in \mathbf{R}), \quad \boldsymbol{\varphi} \sim \mathcal{N}(\boldsymbol{\varphi}_0, \mathbf{V}_\boldsymbol{\varphi})1(\boldsymbol{\varphi} \in \mathbf{R}), \quad \boldsymbol{\zeta} \sim \mathcal{N}(\boldsymbol{\zeta}_0, \mathbf{V}_\boldsymbol{\zeta}), \quad \boldsymbol{\tau}_0 \sim \mathcal{N}(\boldsymbol{\tau}_{00}, \mathbf{V}_{\boldsymbol{\tau}_0}),$$

where  $\mathbf{R}$  denotes the stationarity region. The prior on the AR coefficients  $\boldsymbol{\phi}$  affects how persistent the cyclical components are. We assume relatively large prior variances,  $\mathbf{V}_\boldsymbol{\phi} = \mathbf{I}_4$ , so that *a priori*  $\boldsymbol{\phi}$  can take on a wide range of values. The prior mean is assumed to be  $\boldsymbol{\phi}_0 = (1.3, -0.7, 1.3, -0.7)'$ , which implies that each of the two AR(2) processes has two complex roots, and they are relatively persistent. Similarly, for the prior on  $\boldsymbol{\varphi}$ , we set  $\mathbf{V}_\boldsymbol{\varphi} = \mathbf{I}_2$  with prior mean  $\boldsymbol{\varphi}_0 = (0.9, 0.9)'$ , which implies that the two AR(1) processes are fairly persistent. Next, we assume that the priors on  $\sigma_\eta^2, \sigma_{\eta_x}^2, \sigma_z^2$  and  $\sigma_x^2$  are inverse-gamma:

$$\sigma_i^2 \sim \text{IG}(\nu_i, S_i),$$

where  $i = \eta, \eta_x, z, x$ .

Next, we outline the posterior simulator to estimate the model in (1)–(6) with the priors described above. To that end, let  $\boldsymbol{\tau} = (\tau_1, \tau_{x,1}, \tau_2, \tau_{x,2}, \dots, \tau_T, \tau_{x,T})'$  and  $\mathbf{y} = (z_1, x_1, \dots, z_T, x_T)'$ . Then, posterior draws can be obtained by sequentially sampling from the following conditional distributions:

1.  $p(\boldsymbol{\tau}, \gamma | \mathbf{y}, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0) = p(\gamma | \mathbf{y}, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0)p(\boldsymbol{\tau} | \mathbf{y}, \gamma, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0)$ ;
2.  $p(\boldsymbol{\phi} | \mathbf{y}, \boldsymbol{\tau}, \gamma, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0)$ ;

3.  $p(\boldsymbol{\varphi} | \mathbf{y}, \boldsymbol{\tau}, \gamma, \boldsymbol{\phi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0)$ ;
4.  $p(\boldsymbol{\sigma}^2 | \mathbf{y}, \boldsymbol{\tau}, \gamma, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\tau}_0)$ ;
5.  $p(\boldsymbol{\zeta}, \boldsymbol{\tau}_0 | \mathbf{y}, \boldsymbol{\tau}, \gamma, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\sigma}^2)$ ;

Below we provide details of implementing Step 1. Other steps are relatively standard and the technical details are given in Appendix A.5.

**Step 1.** Since  $\boldsymbol{\tau}$  and  $\gamma$  enter the likelihood multiplicatively, we sample them jointly to improve efficiency of the posterior sampler. In particular, we first sample  $\gamma$  marginally of  $\boldsymbol{\tau}$ , followed by drawing  $\boldsymbol{\tau}$  conditional on the sampled  $\gamma$ . The latter step is straightforward as the model specified in (1)–(6) defines a linear state space model for  $\boldsymbol{\tau}$ . In what follows we derive the full conditional distribution of  $p(\boldsymbol{\tau} | \mathbf{y}, \gamma, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0)$ . Then, we outline a Metropolis-Hastings algorithm to sample  $\gamma$  marginally of  $\boldsymbol{\tau}$ .

To derive the conditional distribution  $p(\boldsymbol{\tau} | \mathbf{y}, \gamma, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0)$ , note that by Bayes' theorem we have

$$p(\boldsymbol{\tau} | \mathbf{y}, \gamma, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0) \propto p(\mathbf{y} | \boldsymbol{\tau}, \gamma, \boldsymbol{\phi}, \boldsymbol{\sigma}^2) p(\boldsymbol{\tau} | \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0),$$

where the conditional likelihood  $p(\mathbf{y} | \boldsymbol{\tau}, \gamma, \boldsymbol{\phi}, \boldsymbol{\sigma}^2)$  and the prior  $p(\boldsymbol{\tau} | \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0)$  are, respectively, defined by the observation equations (1), (2), (5) and (6) and the state equations (3)–(4).

First, we derive the conditional likelihood  $p(\mathbf{y} | \boldsymbol{\tau}, \gamma, \boldsymbol{\phi}, \boldsymbol{\sigma}^2)$ . Letting  $\mathbf{c} = (c_{z,1}, c_{x,1}, \dots, c_{z,T}, c_{x,T})'$  we can stack the observation equations (1)–(2) over  $t = 1, \dots, T$  to get

$$\mathbf{y} = \mathbf{\Pi}_\gamma \boldsymbol{\tau} + \mathbf{c},$$

where  $\mathbf{\Pi}_\gamma = \mathbf{I}_T \otimes \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$ . Here  $\mathbf{I}_T$  is the  $T$ -dimensional identity matrix and  $\otimes$  is the Kronecker product. Next, we stack (5)–(6) over  $t = 1, \dots, T$  to get

$$\mathbf{H}_\phi \mathbf{c} = \boldsymbol{\epsilon},$$

where  $\boldsymbol{\epsilon} = (\epsilon_{z,1}, \epsilon_{x,1}, \dots, \epsilon_{z,T}, \epsilon_{x,T})'$  and

$$\mathbf{H}_\phi = \begin{pmatrix} \mathbf{I}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{A}_1 & \mathbf{I}_2 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{I}_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{I}_2 & \cdots & \mathbf{0} \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{I}_2 \end{pmatrix}$$

with  $\mathbf{A}_1 = \begin{pmatrix} -\phi_{z,1} & 0 \\ 0 & -\phi_{x,1} \end{pmatrix}$  and  $\mathbf{A}_2 = \begin{pmatrix} -\phi_{z,2} & 0 \\ 0 & -\phi_{x,2} \end{pmatrix}$ . Since the determinant of  $\mathbf{H}_\phi$  is one for any  $\phi$ , it is invertible. It follows that  $(\mathbf{c} | \phi, \boldsymbol{\sigma}^2) \sim \mathcal{N}(\mathbf{0}, (\mathbf{H}'_\phi \boldsymbol{\Sigma}_\epsilon^{-1} \mathbf{H}_\phi)^{-1})$ , where  $\boldsymbol{\Sigma}_\epsilon = \text{diag}(\sigma_z^2, \sigma_x^2, \dots, \sigma_z^2, \sigma_x^2)$ .

It then follows that

$$(\mathbf{y} | \boldsymbol{\tau}, \gamma, \phi, \boldsymbol{\sigma}^2) \sim \mathcal{N}(\boldsymbol{\Pi}_\gamma \boldsymbol{\tau}, (\mathbf{H}'_\phi \boldsymbol{\Sigma}_\epsilon^{-1} \mathbf{H}_\phi)^{-1}). \quad (9)$$

Next, we derive the prior  $p(\boldsymbol{\tau} | \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0)$ . To that end, construct the  $T \times 1$  vector of indicators  $\mathbf{d}_0 = (1(1 < T_B), 1(2 < T_B), \dots, 1(T < T_B))'$ , and similarly define  $\mathbf{d}_1$ . Moreover, let

$$\tilde{\boldsymbol{\mu}}_\tau = \mathbf{d}_0 \otimes \begin{pmatrix} (1 - \varphi_\mu) \zeta_1 \\ (1 - \varphi_{\mu_x}) \zeta_{x,1} \end{pmatrix} + \mathbf{d}_1 \otimes \begin{pmatrix} (1 - \varphi_\mu) \zeta_2 \\ (1 - \varphi_{\mu_x}) \zeta_{x,2} \end{pmatrix} + \begin{pmatrix} (1 + \varphi_\mu) \tau_0 - \varphi_\mu \tau_{-1} \\ (1 + \varphi_{\mu_x}) \tau_{x,0} - \varphi_{\mu_x} \tau_{x,-1} \\ -\varphi_\mu \tau_0 \\ -\varphi_{\mu_x} \tau_{x,0} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then, we can stack the state equations (A.5)–(A.6) over  $t = 1, \dots, T$  to get

$$\mathbf{H}_\varphi \boldsymbol{\tau} = \tilde{\boldsymbol{\mu}}_\tau + \boldsymbol{\eta},$$

where  $\boldsymbol{\eta} = (\eta_1, \eta_{x,1}, \dots, \eta_T, \eta_{x,T})'$  and

$$\mathbf{H}_\varphi = \begin{pmatrix} \mathbf{I}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{B}_1 & \mathbf{I}_2 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{B}_2 & \mathbf{B}_1 & \mathbf{I}_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 & \mathbf{B}_1 & \mathbf{I}_2 & \cdots & \mathbf{0} \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{B}_2 & \mathbf{B}_1 & \mathbf{I}_2 \end{pmatrix}$$

with  $\mathbf{B}_1 = \begin{pmatrix} -(1 + \varphi_\mu) & 0 \\ 0 & -(1 + \varphi_{\mu_x}) \end{pmatrix}$  and  $\mathbf{B}_2 = \begin{pmatrix} \varphi_\mu & 0 \\ 0 & \varphi_{\mu_x} \end{pmatrix}$ . Since the determinant of  $\mathbf{H}_\varphi$  is one for any  $\varphi$ , it is invertible. It then follows that

$$(\boldsymbol{\tau} \mid \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0) \sim \mathcal{N}(\boldsymbol{\mu}_\tau, (\mathbf{H}'_\varphi \boldsymbol{\Sigma}_\eta^{-1} \mathbf{H}_\varphi)^{-1}), \quad (10)$$

where  $\boldsymbol{\mu}_\tau = \mathbf{H}_\varphi^{-1} \tilde{\boldsymbol{\mu}}_\tau$  and  $\boldsymbol{\Sigma}_\eta = \text{diag}(\sigma_\eta^2, \sigma_{\eta_x}^2, \dots, \sigma_\eta^2, \sigma_{\eta_x}^2)$ . Combining (9) and (10) and using standard regression results (see, e.g., Chan et al. 2019, pp.217-219), we have

$$(\boldsymbol{\tau} \mid \mathbf{y}, \gamma, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0) \sim \mathcal{N}(\hat{\boldsymbol{\tau}}, \mathbf{K}_\tau^{-1}),$$

where

$$\mathbf{K}_\tau = \mathbf{H}'_\varphi \boldsymbol{\Sigma}_\eta^{-1} \mathbf{H}_\varphi + \boldsymbol{\Pi}'_\gamma \mathbf{H}'_\phi \boldsymbol{\Sigma}_\epsilon^{-1} \mathbf{H}_\phi \boldsymbol{\Pi}_\gamma, \quad \hat{\boldsymbol{\tau}} = \mathbf{K}_\tau^{-1} (\mathbf{H}'_\varphi \boldsymbol{\Sigma}_\eta^{-1} \mathbf{H}_\varphi \boldsymbol{\mu}_\tau + \boldsymbol{\Pi}'_\gamma \mathbf{H}'_\phi \boldsymbol{\Sigma}_\epsilon^{-1} \mathbf{H}_\phi \mathbf{y}).$$

Since the precision matrix  $\mathbf{K}_\tau$  is banded — i.e., it is sparse and its nonzero elements are arranged along the main diagonal — one can sample from  $p(\boldsymbol{\tau} \mid \mathbf{y}, \gamma, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0)$  efficiently using the precision sampler in Chan and Jeliazkov (2009).

Next, we outline a Metropolis-Hastings algorithm to sample  $\gamma$  marginally of  $\boldsymbol{\tau}$ . For that we need to evaluate the integrated likelihood

$$p(\mathbf{y} \mid \gamma, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0) = \int_{\mathbb{R}^{2T}} p(\mathbf{y} \mid \boldsymbol{\tau}, \gamma, \boldsymbol{\phi}, \boldsymbol{\sigma}^2) p(\boldsymbol{\tau} \mid \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0) d\boldsymbol{\tau}.$$

Traditionally, this is done by using the Kalman filter. However, it turns out that we can obtain an analytical expression of the integrated likelihood and evaluate it efficiently using band matrix routines. Using a similar derivation in Chan and Grant (2016), one can show that

$$\begin{aligned}
 & p(\mathbf{y} \mid \gamma, \phi, \varphi, \zeta, \sigma^2, \tau_0) \\
 &= (2\pi)^{-T} |\Sigma_\epsilon|^{-\frac{1}{2}} |\Sigma_\eta|^{-\frac{1}{2}} |\mathbf{K}_\tau|^{-\frac{1}{2}} e^{-\frac{1}{2}(\mathbf{y}'\mathbf{H}'_\phi \Sigma_\epsilon^{-1} \mathbf{H}_\phi \mathbf{y} + \mu'_\tau \mathbf{H}'_\varphi \Sigma_\eta^{-1} \mathbf{H}_\varphi \mu_\tau - \hat{\tau}' \mathbf{K}_\tau \hat{\tau})}.
 \end{aligned} \tag{11}$$

The above expression involves a few large matrices, but they are all banded. Consequently, it can be evaluated efficiently using band matrix algorithms; see Chan and Grant (2016) for technical details.

Finally, we implement a Metropolis-Hastings step to sample  $\gamma$  with the Gaussian proposal  $\mathcal{N}(\hat{\gamma}, K_\gamma^{-1})$ , where  $\hat{\gamma}$  is the mode of  $\log p(\mathbf{y} \mid \gamma, \phi, \varphi, \zeta, \sigma^2, \tau_0)$  and  $K_\gamma$  is the negative Hessian evaluated at the mode.

We refer the readers to Appendix A.5 for implementation detail of other steps of the posterior sampler.

## 4 Empirical Results

In this section we report parameter estimates of the bivariate unobserved components model defined in (1)–(6). The data set consists of the time-series of the logarithm of (the inverse of) RPI and TFP from 1959.II to 2019.II. RPI is computed as the investment deflator divided by the consumption deflator, and it is easily accessible from the Federal Reserve Economic Database (FRED). While the complete description of the deflators along with the accompanying details of the computation of the series are found in DiCecio (2009), it is worth emphasizing that the investment deflator corresponds to the quality-adjusted investment deflator calculated following the approaches in Gordon (1990), Cummins and Violante (2002), and Fisher (2006). On the other hand, we compute TFP based on the aggregate TFP growth, which is measured as the growth rate of the business-sector TFP corrected for capital utilization. The capital utilization adjusted aggregate TFP growth series is produced by Fernald (2014), and is widely regarded as the best available measure of neutral technology.<sup>1</sup>

First, we perform statistical break tests to verify, as has been established in the empirical liter-

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<sup>1</sup>The data is updated on John Fernald's webpage <http://www.frbsf.org/economic-research/economists/john-fernal/>

ature, that RPI experienced a break in its trend around the early 1982's. Following the recommendations in Bai and Perron (2003), we find a break date at 1982.*I* in the mean of the log difference of RPI as documented in Fisher (2006), Justiniano et al. (2011), and Benati (2013). In addition, a consequence of our specification is that RPI and TFP must share a common structural break. Consequently, we follow the methodology outlined in Qu and Perron (2007) to test whether or not the trends in the series are orthogonal<sup>2</sup>. The estimation results suggest that RPI and TFP might share a common structural break at 1980.*I*. This break date falls within the confidence interval of the estimated break date between the same two time series, [1973.*I*, 1982.*III*], documented in Benati (2013). Therefore, the evidence from structural break tests does not rule out the presence of a single break at 1982.*I* in the common stochastic trend component of RPI and TFP.

To jumpstart the discussion of the estimation results, we provide a graphical representation of the fit our model as illustrated by the fitted values of the two series in Figure 1. It is clear from the graph that the bivariate unobserved components model is able to fit both series quite well.

[Figure 1 about here.]

We report the estimates of model parameters in Table 1 and organize the discussion of our findings around the following two points: (1) the cross-series relationship, and (2) the within-series relationship.

#### 4.1 The cross-series relationship between (the inverse of) RPI and TFP

We can evaluate the cross-series relationship through the estimated value of the parameter that captures the extent of the relationship between the trends in RPI and TFP, namely,  $\gamma$ .

The estimated value of the parameter  $\gamma$ , ( $\hat{\gamma} = 0.592$ ), is significantly different from zero. This suggests that (the inverse of) RPI and TFP share a common stochastic component, and the traditional approach in macroeconomic theory of assuming that neutral and investment specific technology follow independent processes is clearly invalid. In addition, the size of  $\hat{\gamma}$  is quite large, and it indicates that trends in (the inverse of) RPI and TFP are closely related such that permanent

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<sup>2</sup>First, we test for the presence of a structural break in TFP following the recommendations in Bai and Perron (2003). We find that the mean of the log difference in TFP exhibits a break at 1968.*I*; a date that is similar to that documented in Benati (2013) who estimates the break date at 1968.*II*. Next, following the approach in Qu and Perron (2007), we regress a vector of both, a series of RPI and a series of TFP, on a vector of constant, linear trends and random errors. We use a trimming value  $\epsilon = 0.20$  and allow up to 3 breaks.



changes in RPI have a strong long-run effect on TFP. These results are consistent with the view that permanent changes in RPI might be representation of innovations in General Purpose Technology. In fact, in the context of a DSGE framework, Schmitt-Grohé and Uribe (2011) identify two common trend shocks via the assumption that RPI and TFP are co-integrated, and use impulse response functions analysis to argue that one of the shocks possesses one of the crucial characteristics of GPT: technological diffusion. Specifically, they find that the shocks generate an increase in both investment-specific productivity (which is exactly identified with the inverse of RPI) and TFP in the long-run, but a decline in both types of productivity during the initial transition. A similar argument is made in Chen and Wemy (2015) who identify investment-specific technological shocks and neutral technological shocks sequentially, and show that both shocks (1) induce almost identical dynamics to macro variables, (2) are highly correlated, (3) explain more than 50 percent of the forecast error variance in TFP, and (4) generate a small initial decrease in TFP but a permanent increase in TFP in the long-run. Hence, they interpret the response of TFP as an indication that capital-embodied technology is a form on General purpose technology. Last, but not least, the fact that the estimated value is positive implies a positive long-run co-variation between (the inverse of) RPI and TFP as argued in Benati (2013).

## 4.2 The within-series relationship in RPI and TFP

The within-series relationship is concerned with the relative importance of the permanent component and the transitory component in (the inverse of) RPI and TFP. This relationship is captured via the estimated values of (i) the drift parameters,  $\zeta$  and  $\zeta_x$ , (ii) the autoregressive parameters of the permanent components in RPI and TFP,  $\varphi_\mu$  and  $\varphi_{\mu_x}$ , (iii) the autoregressive parameters of the transitory components in RPI and TFP,  $\phi_{z,1}$ ,  $\phi_{z,2}$ ,  $\phi_{x,1}$  and  $\phi_{x,2}$ , (iv) and the standard deviation of the permanent and transitory components,  $\sigma_\eta$ ,  $\sigma_{\eta_x}$ ,  $\sigma_z$ , and  $\sigma_x$ .

First, it is evident from Figure 2 that the estimated trend in RPI,  $\tau_t$ , is strongly trending upward, especially after the break at 1982.I. In fact, the growth rate of  $\tau_t$  has more than doubled after 1982.I: as reported in Table 1, the posterior means of  $\zeta_1$  and  $\zeta_2$  are 0.004 and 0.009, respectively. This is consistent with the narrative that the decline in the mean growth of RPI has accelerated in the period after 1982.I, and this acceleration has been facilitated by the rapid price decline in information processing equipment and software. In contrast, the estimated trend  $\tau_{x,t}$  shows gradual

decline after 1982.*I* — its growth rate decreases from 0.001 before the break to  $-0.003$  after the break, although both figures are not statistically different from zero. Overall, it appears that trends in both series are captured completely by the stochastic trend component in RPI, and this piece of evidence provides additional support that GPT might be an important driver of the growth in TFP in the United States.

Moving on to the estimated autoregressive parameters of the permanent component,  $\widehat{\varphi}_\mu = 0.109$  and  $\widehat{\varphi}_{\mu_x} = -0.053$ , we note that growths in RPI and TFP do not appear to be as serially correlated as reported in the empirical literature. For instance, Justiniano et al. (2011) report a posterior median value of 0.287 for the investment-specific technological process and 0.163 for neutral technology. With regards to the AR(2) processes that describes the transitory components in RPI and TFP, the estimated autoregressive parameters indicate that these components are relatively more persistent than the growth components of the series.

Furthermore, the estimated values of the variance of the innovations lead to some interesting observations. The variance of the idiosyncratic growth rate in TFP,  $\sigma_{\eta_x}^2 = 5.04 \times 10^{-5}$ , is larger than its counterpart for the transitory component,  $\sigma_x^2 = 7.84 \times 10^{-6}$ . On the other hand, the variance of the growth rate in RPI,  $\sigma_\eta^2 = 5.34 \times 10^{-6}$ , is smaller than the variance of its transitory component,  $\sigma_z^2 = 1.79 \times 10^{-5}$ .

Overall, these results seem to indicate that (i) transitory shocks are more persistent than growth shocks, (ii) TFP growth shocks generate more variability than shocks to the growth rate of RPI, and (iii) shocks to the transitory component of RPI are more volatile than shocks to the growth rate of RPI. While these findings are qualitatively similar to results from estimated DSGE models, the magnitude of the estimates differ quite significantly. Justiniano et al. (2010) is one notable study which considers both growth and stationary neutral technology shocks and investment shocks, and the authors report a standard deviation for the IST growth shock and the stationary marginal efficiency of investment (MEI) shock of 0.6229 and 5.786, respectively. Since the autocorrelation parameters of the underlying processes of technology and the standard deviation of their innovations play a crucial role in the persistence and volatility of economic fluctuations, larger than necessary estimated values may mistakenly attribute a relatively important portion of such fluctuations to these shocks.

[Figure 2 about here.]

[Table 1 about here.]

## 5 Model Comparison

In this section, we explore the fit of our bivariate unobserved components model defined in (1)–(6) to alternative restricted specifications that encompass the various assumptions in the theoretical and empirical literature. Therefore, our model constitutes a complete laboratory that may be used to evaluate the plausibility of competing specifications.

We adopt the Bayesian model comparison framework to compare various specifications via the Bayes factor. More specifically, suppose we wish to compare model  $M_0$  against model  $M_1$ . Each model  $M_i, i = 0, 1$ , is formally defined by a likelihood function  $p(\mathbf{y} | \boldsymbol{\theta}_i, M_i)$  and a prior distribution on the model-specific parameter vector  $\boldsymbol{\theta}_i$  denoted by  $p(\boldsymbol{\theta}_i | M_i)$ . Then, the Bayes factor in favor of  $M_0$  against  $M_1$  is defined as

$$\text{BF}_{01} = \frac{p(\mathbf{y} | M_0)}{p(\mathbf{y} | M_1)},$$

where  $p(\mathbf{y} | M_i) = \int p(\mathbf{y} | \boldsymbol{\theta}_i, M_i)p(\boldsymbol{\theta}_i | M_i)d\boldsymbol{\theta}_i$  is the marginal likelihood under model  $M_i, i = 0, 1$ . Note that the marginal likelihood is the marginal data density (unconditional on the prior distribution) implied by model  $M_i$  evaluated at the observed data  $\mathbf{y}$ . Since the marginal likelihood can be interpreted as a joint density forecast evaluated at the observed data, it has a built-in penalty for model complexity. If the observed data is likely under the model, the associated marginal likelihood would be “large” and vice versa. It follows that  $\text{BF}_{01} > 1$  indicates evidence in favor of model  $M_0$  against  $M_1$ , and the weight of evidence is proportional to the value of the Bayes factor. For a textbook treatment of the Bayes factor and the computation of the marginal likelihood, see, Chan et al. (2019).

### 5.1 Testing $\gamma = 0$

In the first modified model, we impose the restriction that  $\gamma = 0$ , which essentially amounts to testing the GPT theory. If the restricted model were preferred over the unrestricted model, then the trends in RPI and TFP would be orthogonal, and the traditional view of specifying neutral

technology and investment-specific technology would be well founded.

The posterior mean of  $\gamma$  is estimated to be about 0.59, and the posterior standard deviation is 0.5. Most of the mass of the posterior distribution is on positive values — the posterior probability that  $\gamma > 0$  is 0.9. To formally test if  $\gamma = 0$ , we compute the Bayes factor in favor of the baseline model in (1)–(6) against the unrestricted version with  $\gamma = 0$  imposed. In this case the Bayes factor can be obtained by using the Savage-Dickey density ratio  $p(\gamma = 0)/p(\gamma = 0 | \mathbf{y})$ .

The Bayes factor in favor of the baseline model is about 1.2, suggesting that even though there is some evidence against the hypothesis  $\gamma = 0$ , the evidence is not strong. To better understand this result, we plot the prior and posterior distributions of  $\gamma$  in Figure 3. As is clear from the figure, the prior and posterior densities at  $\gamma = 0$  have similar values. However, it is also apparent that the data moves the prior distribution to the right, making larger values of  $\gamma$  more likely under the posterior distribution. Hence, there seems to be some support for the hypothesis that  $\gamma > 0$ .

[Figure 3 about here.]

## 5.2 Testing $\varphi_\mu = \varphi_{\mu_x} = 0$

Next, we test the joint hypothesis that  $\varphi_\mu = \varphi_{\mu_x} = 0$ . In this case, the restricted model would allow the trend component to follow a random walk plus drift process such that the growths in RPI and TFP are constant. This would capture the assumptions in Fisher (2006).

Again the Bayes factor in favor of the baseline model against the restricted model with  $\varphi_\mu = \varphi_{\mu_x} = 0$  imposed can be obtained by computing the Savage-Dickey density ratio  $p(\varphi_\mu = 0, \varphi_{\mu_x} = 0)/p(\varphi_\mu = 0, \varphi_{\mu_x} = 0 | \mathbf{y})$ . The Bayes factor in favor of the restricted model with  $\varphi_\mu = \varphi_{\mu_x} = 0$  is about 14. This indicates that there is strong evidence in favor the hypothesis  $\varphi_\mu = \varphi_{\mu_x} = 0$ . This is consistent with the estimation results reported in Table 1 — the estimates of  $\varphi_\mu$  and  $\varphi_{\mu_x}$  are both small in magnitude with relatively large posterior standard deviations. This is an example where the Bayes factor favors a simpler, more restrictive model.

Despite restricting  $\varphi_\mu = \varphi_{\mu_x} = 0$ , this restricted model is able to fit the data very well, as shown in Figure 4.

[Figure 4 about here.]

### 5.3 Testing $\varphi_{\mu_x} = \sigma_{\eta_x}^2 = 0$

Now, we test the joint hypothesis that  $\varphi_{\mu_x} = \sigma_{\eta_x}^2 = 0$ . This restriction goes to the heart of the debate between Schmitt-Grohé and Uribè (2011) and Benati (2013). If the restricted model held true, then RPI and TFP would be co-integrated as argued by Schmitt-Grohé and Uribè (2011). On the other hand, if the restrictions were rejected, we would end with a situation where RPI and TFP are not co-integrated, but still share a common component as shown in Benati (2013).

Since zero is at the boundary of the parameter space of  $\sigma_{\eta_x}^2$ , the relevant Bayes factor cannot be computed using the Savage-Dickey density ratio. Instead, we compute the log marginal likelihoods of the baseline model and the restricted version with  $\varphi_{\mu_x} = \sigma_{\eta_x}^2 = 0$ . The marginal likelihoods of the two models are obtained by using the adaptive importance sampling estimator known as the cross-entropy method proposed in Chan and Eisenstat (2015).

The marginal likelihood of the baseline model is 1714, compared to 1532 of the restricted version. This means that the Bayes factor in favor of the baseline model is 182 (in log scale), suggesting overwhelming support for the unrestricted model. This shows that we can reject the joint hypothesis  $\varphi_{\mu_x} = \sigma_{\eta_x}^2 = 0$ . This result along with the fact that  $\gamma > 0$  implies a positive covariation between RPI and TFP. While this conclusion is identical to the result in Benati (2013), our approach sets itself apart in the sense that it provides a quantitative measure of the relationship between the trends in RPI and TFP, and it offers an interpretation that may be easily traced to economic theory.

### 5.4 Testing $\varphi_{\mu} = \varphi_{\mu_x} = 1$

Lastly, we test the joint hypothesis that  $\varphi_{\mu} = \varphi_{\mu_x} = 1$ . Since the value 1 is at the boundary of the parameter space of both  $\varphi_{\mu}$  and  $\varphi_{\mu_x}$ , the relevant Bayes factor cannot be computed using the Savage-Dickey density ratio. We instead compute the log marginal likelihoods of the baseline model and the restricted version with  $\varphi_{\mu} = \varphi_{\mu_x}$ , again using the cross-entropy method in Chan and Eisenstat (2015). The former value is 1714 and the latter is 1347, suggesting overwhelming support for the baseline model. Hence, we can reject the joint hypothesis  $\varphi_{\mu} = \varphi_{\mu_x} = 1$ . These are consistent with the estimation results reported in Table 1. Specifically, under the baseline model,  $\varphi_{\mu}$  and  $\varphi_{\mu_x}$  are estimated to be, respectively, 0.109 and  $-0.053$ , and both values are far from 1.

## 6 Discussion and Implications

The estimation of our bivariate unobserved components model defined in (1)–(6) provides us with a set of results that we summarize as follows:

1. The trends in RPI and TFP appear to be better represented by integrated processes of order one.
2. RPI and TFP are not co-integrated but they share a common stochastic trend component.
3. The common stochastic trend component has a positive effect on both RPI and TFP, which suggests a positive covariation between the two series.
4. The common stochastic trend component may be interpreted as General Purpose Technological progress emanating from advances in information and communication technology.
5. The GPT progress appears to have long-run effect on the growth rate of TFP in the United States.

These results have important ramifications for theoretical and empirical models of economic fluctuations, and we devote the rest of this section to the discussion of such implications. Starting with theoretical models of business cycles, it is apparent from our results that the univariate processes of neutral technology and investment-specific technology should be modelled as integrated processes. Now, the question is “Should the process be an  $ARIMA(0,1,0)$  or an  $ARIMA(1,1,0)$ ?”. Based on studies in the literature, the answer is not very clear. Cogley and Nason (1995) demonstrate that Real Business Cycles (RBC) models with integrated exogenous (technological) processes tend to fit the data better, but subsequent studies have differed in their choice of specifications. For instance, Fisher (2006) imposes a unit process for neutral technology and investment-specific technology shocks, and, in more recent papers like Justiniano et al. (2011), it is common to encounter an  $ARIMA(1,1,0)$  process. As we point out in the introduction, the choice of such specifications is driven, among other things, by the desire to match the autocorrelation function for output growth and the hump-shaped responses of macro variables to exogenous shocks. If output growth is positively correlated as documented in the literature, the  $ARIMA(1,1,0)$  process will be the obvious choice since a unit root process would produce zero autocorrelation. Cogley and Nason (1995)

validates this interpretation and show that an RBC model with an ARIMA (1,1,0) process for neutral technology shock generates an autocorrelation function that is similar to its data counterpart. However, the authors also assert that models with either unit root technology shocks or ARIMA (1,1,0) shocks have great success at matching the permanent impulse response function of output, but perform poorly in terms of generating the hump-shaped transitory impulse responses. Based on the results from our framework, we make the following suggestion: if the econometrician is attempting to match the autocorrelation function in macro variables, then technology shocks should be modelled as following an ARIMA (1,1,0) process. On the other hand, if the econometrician wants the theoretical technology shocks to capture as close as possible the characteristics of their empirical counterparts, the an ARIMA (0,1,0) process would be preferable.

Furthermore, our findings clearly indicate that the conventional approach of assuming independent trends processes of neutral technology and investment-specific technology is not supported by the characteristics of the time series. The way we incorporate such trends in business cycle models have great implications for the identification of the sources of macroeconomic fluctuations. First, the choice of the built-in non-stationary component naturally dictates the way we transform the raw data prior to the estimation of the structural parameters. For instance, in Justiniano et al. (2011), the DSGE model features unit root processes for neutral technology and investment-specific technology, such that the observed variables are the log-difference of output, consumption, investment and real wages (along with the level of hours, inflation and the Federal Funds rate). This approach implicitly imposes a rigid equivalence between the model assumption and the data. Such equivalence could potentially lead to specifications errors in the estimation of the structural parameters which depend on the time series characteristics of the non-stationary component. Furthermore, we observe that when the trend components of both technological processes are assumed to be co-integrated as Schmitt-Grohé and Uribè (2011) argue, then shocks to the common stochastic trend in neutral and investment specific productivity play a sizable role in driving business cycle fluctuations: three fourth of the variances of output and investment growth and about one third of the predicted variances of consumption growth and hours. On the other hand, when the processes are assumed to be independent as in Justiniano et al. (2011), shocks to the marginal efficiency of investment are the single most important source of macroeconomic fluctuations as they explain between 60 and 85 percent of the variance of output, hours and investment at business cycle frequencies.

These two issues are closely related to the views expressed in Ferroni (2011) and Canova (2014) about the specification and estimation of DSGE models. Both authors recommend an alternative one-step estimation approach that allows to specify a reduced-form representation of the trend component, which is ultimately combined to the DSGE model for estimation. Our results about the time series characteristics of RPI and TFP should provide some guidance on the specification of such reduced-form models.

With regards to empirical models, our findings highlight some of the econometric shortcomings of long-run VAR models of technology. Beginning with Blanchard and Quah (1989), many subsequent studies have identified technology shocks via the assumption that “the unit root in productivity arises exclusively from neutral technology shocks”. The estimation of our model reveals that there are at least two technology-related sources of the unit root in TFP; therefore, we must rethink identification strategies based on such long-run restrictions.

## 7 Conclusion

The aim of the paper is to evaluate the long-run relationship between the relative price of investment (RPI) and total factor productivity (TFP). Using the unobserved component decomposition that allows us to separate a trend component from a cyclical component in a time series, we specify a model that features a common component between the trends component in RPI and TFP. The main result of the analysis is that RPI and TFP share a common stochastic trend component that may drive the mean growth rate of aggregate productivity in the United States. As documented in many studies, we view this common stochastic trend component as capturing permanent changes in General Purpose Technology from innovations in information and communication technologies. In addition, our findings reveal that neutral technology and investment-specific technology should not be modelled as being generated by two separate and independent exogenous processes as it is typically performed in the DSGE literature.



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Figure 1: Fitted values of  $\hat{z}_t = \tau_t$  and  $\hat{x}_t = \gamma\tau_t + \tau_{x,t}$ .

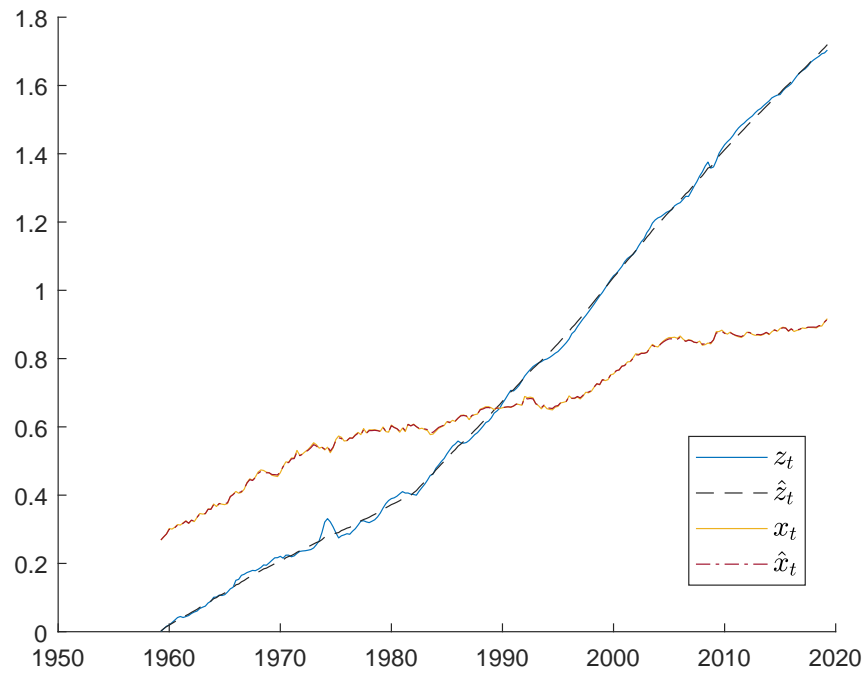


Figure 2: Posterior means of  $\tau_t$  and  $\tau_{x,t}$ .

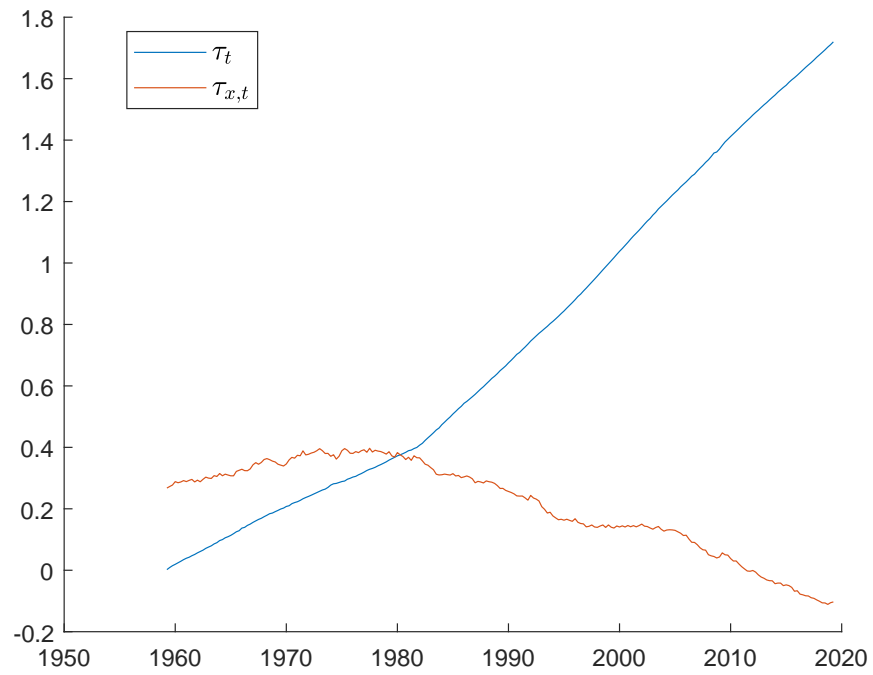


Figure 3: Prior and posterior distributions of  $\gamma$ .

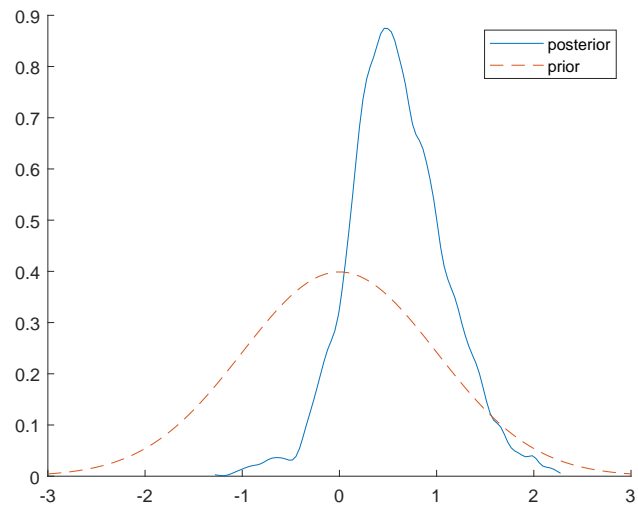


Figure 4: Fitted values of  $\hat{z}_t = \tau_t$  and  $\hat{x}_t = \gamma\tau_t + \tau_{x,t}$  of the restricted model with  $\varphi_\mu = \varphi_{\mu_x} = 0$ .

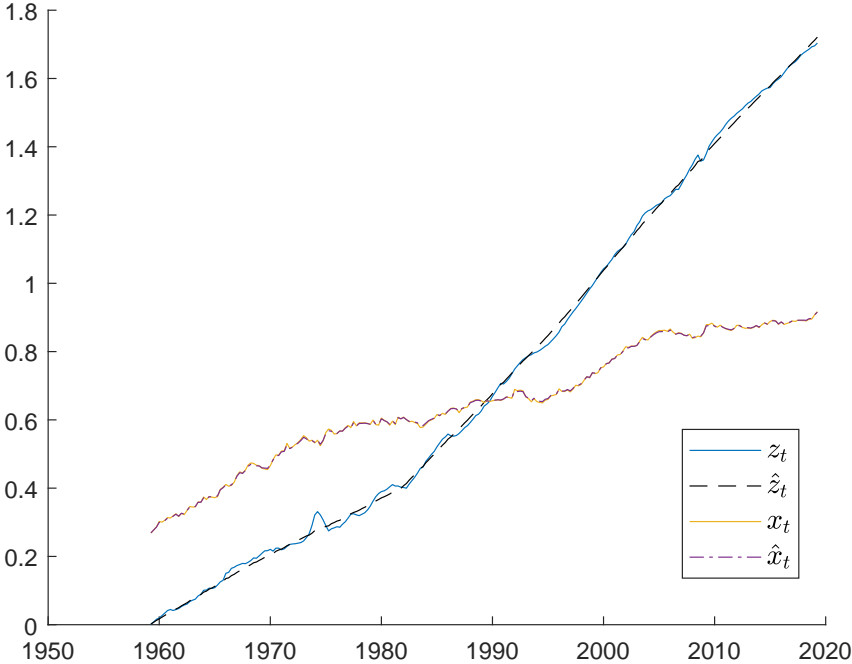




Table 1: Posterior means and standard deviations of model parameters

	posterior mean	posterior std. dev.	$\mathbb{P}(\cdot > 0)$
$\gamma$	0.592	0.500	0.90
$\zeta_1$	0.004	0.000	1.00
$\zeta_2$	0.009	0.000	1.00
$\zeta_{x,1}$	0.001	0.002	0.70
$\zeta_{x,2}$	-0.003	0.004	0.23
$\varphi_\mu$	0.109	0.168	0.78
$\varphi_{\mu_x}$	-0.053	0.075	0.22
$\phi_{z,1}$	1.456	0.074	1.00
$\phi_{z,2}$	-0.568	0.075	0.00
$\phi_{x,1}$	0.914	0.355	0.99
$\phi_{x,2}$	-0.243	0.266	0.20
$\sigma_z^2$	$1.79 \times 10^{-5}$	$3.38 \times 10^{-6}$	1.00
$\sigma_x^2$	$7.84 \times 10^{-6}$	$3.83 \times 10^{-6}$	1.00
$\sigma_\eta^2$	$5.34 \times 10^{-6}$	$2.35 \times 10^{-6}$	1.00
$\sigma_{\eta_x}^2$	$5.04 \times 10^{-5}$	$7.42 \times 10^{-6}$	1.00

# A Appendix

## A.1 Model economy

In this section, we use a neoclassical model similar to the structure in Greenwood et al. (1997), Oulton (2007), and Moura (2020) to derive our UC framework in equations (1)-(6). Consider the following model

$$Y_t = C_t + I_t = z_t K_t^\alpha (X_t^z h_t)^{1-\alpha}, \quad (\text{A.1})$$

$$I_t^* = a_t X_t^a I_t \quad (\text{A.2})$$

$$K_{t+1} = (1 - \delta)K_t + I_t^*. \quad (\text{A.3})$$

Equation (A.1) is the aggregate production function where output, ( $Y_t$ ), which can be used for either consumption, ( $C_t$ ), or gross investment, ( $I_t$ ), is produced with capital, ( $K_t$ ), and labor, ( $h_t$ ). The production function is subjected to a stationary neutral technological shock, ( $a_t$ ), and a non-stationary neutral technological shock, ( $X_t^z$ ). Equation (A.2) relates investment in efficiency units, ( $I_t^*$ ), to gross investment. The terms  $a_t$  and  $X_t^a$  denote, respectively, stationary and non-stationary investment-specific technology. Finally, equation (A.3) shows the evolution of the capital stock. It is important to note that output and gross investment are both measured in units of consumption goods.

In this model economy, Total Factor Productivity (TFP) and the Relative Price of Investment (RPI), denoted as  $P_t^I$ , are given, respectively, by

$$TFP_t = \frac{Y_t}{K_t^\alpha h_t^{1-\alpha}} = z_t (X_t^z)^{1-\alpha} \quad (\text{A.4})$$

$$P_t^I = \frac{1}{a_t X_t^a}. \quad (\text{A.5})$$

Along a balanced growth path, the stochastic trend in  $Y_t$  and  $C_t$  is given by  $X_t^z (X_t^a)^{\frac{\alpha}{1-\alpha}}$  while the stochastic trend in investment measured in efficiency units  $I_t^*$  is  $X_t^z (X_t^a)^{\frac{1}{1-\alpha}}$ .

Using the standard Divisia definition of aggregate output, output,  $Y_t^D$ , is well-approximated by

a share weighted index<sup>3</sup>

$$Y_t^D = C_t^{1-\gamma} I_t^{*\gamma} \quad (\text{A.6})$$

where  $\gamma$  is the current price share of investment in the value of output. Under this definition of output, TFP is defined as

$$TFP_t^D = \frac{Y_t^D}{K_t^\alpha h_t^{1-\alpha}} \quad (\text{A.7})$$

Therefore, expressions (A.4) and (A.7) may be combined to yield

$$TFP_t^D = z_t (X_t^z)^{1-\alpha} \frac{Y_t^D}{Y_t}, \quad (\text{A.8})$$

To determine the stochastic balanced growth path in  $TFP_t^D$ , we start with expression (A.8) and apply the logarithm on both sides,

$$\log(TFP_t^D) = \log(z_t) + (1-\alpha)\log(X_t^z) + \log(Y_t^D) - \log(Y_t),$$

then we take the first difference, and use the definition in A.6 to get

$$\begin{aligned} \log(TFP_t^D) - \log(TFP_{t-1}^D) &= (1-\alpha)\left(\log X_t^z - \log X_{t-1}^z\right) + (1-\gamma)\left(\log C_t - \log C_{t-1}\right) \\ &\quad + \gamma\left(\log I_t^* - \log I_{t-1}^*\right) - \left(\log Y_t - \log Y_{t-1}\right) \\ &= (1-\alpha)\left(\log X_t^z - \log X_{t-1}^z\right) \\ &\quad + (1-\gamma)\left[\log\left(X_t^z (X_t^a)^{\frac{\alpha}{1-\alpha}}\right) - \log\left(X_{t-1}^z (X_{t-1}^a)^{\frac{\alpha}{1-\alpha}}\right)\right] \\ &\quad + \gamma\left[\log\left(X_t^z (X_t^a)^{\frac{1}{1-\alpha}}\right) - \log\left(X_{t-1}^z (X_{t-1}^a)^{\frac{1}{1-\alpha}}\right)\right] \\ &\quad - \left[\log\left(X_t^z (X_t^a)^{\frac{\alpha}{1-\alpha}}\right) - \log\left(X_{t-1}^z (X_{t-1}^a)^{\frac{\alpha}{1-\alpha}}\right)\right] \\ &= (1-\alpha)\left(\log X_t^z - \log X_{t-1}^z\right) + \gamma\left[\log X_t^a - \log X_{t-1}^a\right] \end{aligned}$$

where we use the fact that, along a balanced growth path,  $Y_t$  and  $C_t$  grow at the same rate as

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<sup>3</sup>The Divisia index for output is  $\Delta \log Y_t^D = (1-\gamma)\Delta \log C_t + \gamma\Delta \log I_t^*$ . Without loss of generality, we may normalize the levels of  $\log Y_t^D$ ,  $\log C_t$ , and  $\log I_t^*$  at period 0 to be zero, and rearrange to obtain the expression for  $Y_t^D$ . We use the superscript ‘‘D’’ to emphasize the dependence of output on the Divisia definition. Also,  $Y_t^D$  differs from  $Y_t$  in the sense that the latter is measured in consumption units.

$X_t^z (X_t^a)^{\frac{1}{1-\alpha}}$  while  $I_t^*$  grows at the faster rate  $X_t^z (X_t^a)^{\frac{1}{1-\alpha}}$ . Then, using a simple change of variable to specify technology so that it is Hicks neutral instead of Harrod neutral, i.e.  $\tilde{X}_t^z = (X_t^z)^{1-\alpha}$ , we obtain

$$\begin{aligned} \log(TFP_t^D) - \log(TFP_{t-1}^D) &= \left( \log \tilde{X}_t^z - \log \tilde{X}_{t-1}^z \right) + \gamma \left[ \log X_t^a - \log X_{t-1}^a \right] \\ \log(TFP_t^D) - \log \tilde{X}_t^z - \gamma \log X_t^a &= \log(TFP_{t-1}^D) - \log \tilde{X}_{t-1}^z - \gamma \log X_{t-1}^a \\ \frac{TFP_t^D}{\tilde{X}_t^z (X_t^a)^\gamma} &= \frac{TFP_{t-1}^D}{\tilde{X}_{t-1}^z (X_{t-1}^a)^\gamma}, \end{aligned}$$

or in other words, along the balanced growth path,  $\frac{TFP_t^D}{\tilde{X}_t^z (X_t^a)^\gamma}$  is stationary. We may apply the same logic to the relative price of investment,  $P_t^I$ , to show that  $\frac{P_t^I}{X_t^a}$  is also stationary along the stochastic balanced growth path. Let  $tfp_t^D$  and  $p_t^I$  denote stationary TFP and stationary RPI, respectively; therefore, applying the logarithm and rearranging yields the system

$$\log P_t^I = \log X_t^a + \log p_t^I \tag{A.9}$$

$$\log TFP_t^D = \gamma \log X_t^a + \log \tilde{X}_t^z + \log tfp_t^D \tag{A.10}$$

Suppose that we further assume that the growth rate of the non-stationary neutral technology, and the non-stationary investment-specific technology, denoted as  $\mu_t^z \equiv \Delta \log \tilde{X}_t^z$  and  $\mu_t^a \equiv \Delta \log X_t^a$ , respectively, follow stationary  $AR(1)$  processes, and the stationary components of TFP and RPI,  $\log tfp_t^D$  and  $\log p_t^I$ , follow  $AR(1)$  processes as we specify in the paper. Therefore, equations (A.9), (A.10) along with the assumed specifications constitute the full characterization of our UC framework illustrated in equations (1)-(6) within the manuscript.

## A.2 Common Trend in RPI and TFP

In this section, I empirically revisit two issues that have been explored in both Schmitt-Grohé and Uribe (2011) and Benati (2013). The first is to determine whether RPI and TFP possess each a stochastic non-stationary component, and the second is to assess whether these non-stationary components are related. Therefore, I perform unit root tests and co-integration tests using U.S. data over the period 1959 : Q2 to 2019.Q2 using the logarithm of RPI and the logarithm of TFP.

### A.3 Unit root tests

I carry out Augmented Dickey Fuller (ADF) tests that examine the null hypothesis that the logarithms of RPI and TFP have a unit root. The lag order is chosen based on the Schwartz and Hannan-Quinn criteria (SIC and HQ, respectively).<sup>4</sup> The results of the tests are presented in Table 2, and they clearly indicate that the tests fail to reject the null hypothesis of the presence of a unit root in the series at the standard 5% confidence level.

An alternative to the ADF test is the Kwiatkowski, Phillips, Schmidt, and Shin (KPSS) test that evaluates the null hypothesis that the time series is stationary in levels. The lag length is still selected according to the SIC and HQ criteria, and I allow for the possibility of a time trend in the series. The results are illustrated in Table 3, and they are consistent with those obtained from ADF tests: the KPSS rejects the null hypothesis of stationarity in the logarithm of the relative price of investment and total factor productivity.

Overall, stationarity tests are unequivocal in terms of the univariate properties of RPI and TFP: both time series contain a stochastic non-stationary component.

[Table 2 about here.]

[Table 3 about here.]

### A.4 Co-integration between RPI and TFP

The results from the previous section clearly indicate that the time series of both RPI and TFP contain a non-stationary stochastic component. With that information at hand, I assess the extent to which these two non-stochastic components might be co-integrated. Specifically, I perform the Johansen's trace test that evaluates the null hypothesis that there is no co-integration relationship between the two series. A rejection of this hypothesis would indicate that RPI and TFP are driven by a single stochastic component. Consistent with the previous section, I still select the lag length according to the SIC and HQ criteria, and the result points to 1 as the optimal lag order. As discussed in Benati (2013), the lag order selection for VARs containing integrated variables in the Johansen's procedure may greatly affect the results of the test. Therefore, I also consider lag orders

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<sup>4</sup>I use JMulti to determine the optimal lag order for each variable. I set 10 as the maximum number of endogenous variables. The results indicate that 1 or 2 might be optimal. I obtained the same results using codes provided by Benati (2013).

of 7 and 3 as alternatives, as in Schmitt-Grohé and Uribè (2011) and Benati (2013), respectively. A final and crucial component of the test is the specification of the data generating process (DGP) for the co-integrated model as this step has great importance on the results of the test. In other words, should a deterministic term, constant or linear term, be included in the DGP? Such decisions are usually guided by the underlying process of the variables which may or may not contain a drift term. Both the inverse of RPI and TFP appear to be trending upward, hence I consider the addition of a constant and linear term in the DGP. The results of the Johansen’s trace tests are shown in Table 4.

[Table 4 about here.]

The co-integration results are inconclusive. When the lag order is 7, the null hypothesis of a zero co-integrating vector is rejected at the standard 5% confidence level, a result that echoes those in Benati (2013) and Schmitt-Grohé and Uribè (2011). However, the test fails to reject the null hypothesis of a zero co-integrating vector when a lag order of 1 or 3 is considered. Therefore, it appears that it is impossible to claim with certainty that RPI and TFP are driven by a single stochastic trend component.

## A.5 Estimation Details

In this appendix we provide the estimation details of the bivariate unobserved components model specified in the main text. For convenience we reproduce the UC model below:

$$z_t = \tau_t + c_{z,t}, \tag{A.1}$$

$$x_t = \gamma\tau_t + \tau_{x,t} + c_{x,t}, \tag{A.2}$$

where  $\tau_t$  and  $\tau_{x,t}$  are the trend components, whereas  $c_{z,t}$  and  $c_{x,t}$  are the transitory components. The transitory components are assumed to follow the AR(2) processes below:

$$c_{z,t} = \phi_{z,1}c_{z,t-1} + \phi_{z,2}c_{z,t-2} + \epsilon_{z,t}, \tag{A.3}$$

$$c_{x,t} = \phi_{x,1}c_{x,t-1} + \phi_{x,2}c_{x,t-2} + \epsilon_{x,t}, \tag{A.4}$$

where  $\epsilon_{z,t} \sim \mathcal{N}(0, \sigma_z^2)$  and  $\epsilon_{x,t} \sim \mathcal{N}(0, \sigma_x^2)$ , and the initial conditions  $c_{z,0}, c_{x,0}, c_{z,-1}$  and  $c_{x,-1}$  are assumed to be zero. For the trend components, we model the first differences of  $\Delta\tau_t$  and  $\Delta\tau_{x,t}$  as stationary processes, each with a break at  $t = T_B$  with a different unconditional mean. More specifically, consider:

$$\Delta\tau_t = (1 - \varphi_\mu)\zeta_1 1(t < T_B) + (1 - \varphi_\mu)\zeta_2 1(t \geq T_B) + \varphi_\mu \Delta\tau_{t-1} + \eta_t, \quad (\text{A.5})$$

$$\Delta\tau_{x,t} = (1 - \varphi_{\mu_x})\zeta_{x,1} 1(t < T_B) + (1 - \varphi_{\mu_x})\zeta_{x,2} 1(t \geq T_B) + \varphi_{\mu_x} \Delta\tau_{x,t-1} + \eta_{x,t}, \quad (\text{A.6})$$

where  $1(\cdot)$  denotes the indicator function,  $\eta_t \sim \mathcal{N}(0, \sigma_\eta^2)$  and  $\eta_{x,t} \sim \mathcal{N}(0, \sigma_{\eta_x}^2)$  are independent of each other at all leads and lags. The initial conditions  $\boldsymbol{\tau}_0 = (\tau_0, \tau_{-1}, \tau_{x,0}, \tau_{x,-1})'$  are treated as unknown parameters.

Section 3 in the main text outlines a 5-block posterior simulator to estimate the above bivariate unobserved components model, and provide details of the first step. Below we describe the implementation details of the remaining steps.

**Step 2.** To sample  $\boldsymbol{\phi}$ , we write (A.3)–(A.4) as a regression with coefficient vector  $\boldsymbol{\phi}$ :

$$\mathbf{c} = \mathbf{X}_\phi \boldsymbol{\phi} + \boldsymbol{\epsilon},$$

where  $\mathbf{c} = (c_{z,1}, c_{x,1}, \dots, c_{z,T}, c_{x,T})'$  and  $\mathbf{X}_\phi$  is a  $2T \times 4$  matrix consisting of lagged values of  $(c_{z,t}, c_{x,t})$ . Then, by standard regression results, we have

$$(\boldsymbol{\phi} | \mathbf{y}, \boldsymbol{\tau}, \gamma, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0) \sim \mathcal{N}(\hat{\boldsymbol{\phi}}, \mathbf{K}_\phi^{-1}) 1(\boldsymbol{\phi} \in \mathbf{R}),$$

where

$$\mathbf{K}_\phi = \mathbf{V}_\phi^{-1} + \mathbf{X}'_\phi \boldsymbol{\Sigma}_\epsilon^{-1} \mathbf{X}_\phi, \quad \hat{\boldsymbol{\phi}} = \mathbf{K}_\phi^{-1} \left( \mathbf{V}_\phi^{-1} \boldsymbol{\phi}_0 + \mathbf{X}'_\phi \boldsymbol{\Sigma}_\epsilon^{-1} \mathbf{c} \right).$$

A draw from this truncated normal distribution can be obtained by the acceptance-rejection method, i.e., keep sampling from  $\mathcal{N}(\hat{\boldsymbol{\phi}}, \mathbf{K}_\phi^{-1})$  until  $\boldsymbol{\phi} \in \mathbf{R}$ .

**Step 3.** Next, we simulate from  $p(\boldsymbol{\varphi} | \mathbf{y}, \boldsymbol{\tau}, \gamma, \boldsymbol{\phi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0)$ . As in Step 2, we first write (A.5)–

(A.6) as a regression with coefficient vector  $\varphi$ :

$$\Delta\boldsymbol{\tau} = \boldsymbol{\mu}_\varphi + \mathbf{X}_\varphi\boldsymbol{\varphi} + \boldsymbol{\eta},$$

where  $\Delta\boldsymbol{\tau} = (\Delta\tau_1, \Delta\tau_{x,1}, \dots, \Delta\tau_T, \Delta\tau_{x,T})'$ ,  $\boldsymbol{\mu}_\varphi = (\zeta_1 1(1 < T_B) + \zeta_2 1(1 \geq T_B), \dots, \zeta_1 1(T < T_B) + \zeta_2 1(T \geq T_B))'$  and

$$\mathbf{X}_\varphi = \begin{pmatrix} \Delta\tau_0 - \zeta_1 1(1 < T_B) - \zeta_2 1(1 \geq T_B) & 0 \\ 0 & \Delta\tau_{x,0} - \zeta_{x,1} 1(1 < T_B) - \zeta_{x,2} 1(1 \geq T_B) \\ \vdots & \vdots \\ \Delta\tau_{T-1} - \zeta_1 1(T < T_B) - \zeta_2 1(T \geq T_B) & 0 \\ 0 & \Delta\tau_{x,T-1} - \zeta_{x,1} 1(T < T_B) - \zeta_{x,2} 1(T \geq T_B) \end{pmatrix}.$$

Again, by standard regression results, we have

$$(\boldsymbol{\varphi} | \mathbf{y}, \boldsymbol{\tau}, \gamma, \boldsymbol{\phi}, \boldsymbol{\zeta}, \boldsymbol{\sigma}^2, \boldsymbol{\tau}_0) \sim \mathcal{N}(\widehat{\boldsymbol{\varphi}}, \mathbf{K}_\varphi^{-1}) 1(\boldsymbol{\varphi} \in \mathbf{R}),$$

where

$$\mathbf{K}_\varphi = \mathbf{V}_\varphi^{-1} + \mathbf{X}'_\varphi \boldsymbol{\Sigma}_\eta^{-1} \mathbf{X}_\varphi, \quad \widehat{\boldsymbol{\varphi}} = \mathbf{K}_\varphi^{-1} (\mathbf{V}_\varphi^{-1} \boldsymbol{\varphi}_0 + \mathbf{X}'_\varphi \boldsymbol{\Sigma}_\eta^{-1} (\Delta\boldsymbol{\tau} - \boldsymbol{\mu}_\varphi)).$$

A draw from this truncated normal distribution can be obtained by the acceptance-rejection method.

**Step 4.** To implement Step 4, note that  $\sigma_\eta^2, \sigma_{\eta_x}^2, \sigma_z^2, \sigma_x^2$  are conditionally independent given  $\boldsymbol{\tau}$  and other parameters. In fact, they have the following inverse-gamma distributions:

$$\begin{aligned} (\sigma_\eta^2 | \mathbf{y}, \boldsymbol{\tau}, \gamma, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\tau}_0) &\sim \mathcal{IG} \left( \nu_\eta + \frac{T}{2}, S_\eta + \frac{1}{2} \sum_{t=1}^T \eta_t^2 \right), \\ (\sigma_{\eta_x}^2 | \mathbf{y}, \boldsymbol{\tau}, \gamma, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\tau}_0) &\sim \mathcal{IG} \left( \nu_{\eta_x} + \frac{T}{2}, S_{\eta_x} + \frac{1}{2} \sum_{t=1}^T \eta_{x,t}^2 \right), \\ (\sigma_z^2 | \mathbf{y}, \boldsymbol{\tau}, \gamma, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\tau}_0) &\sim \mathcal{IG} \left( \nu_z + \frac{T}{2}, S_z + \frac{1}{2} \sum_{t=1}^T \epsilon_{z,t}^2 \right), \\ (\sigma_x^2 | \mathbf{y}, \boldsymbol{\tau}, \gamma, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\zeta}, \boldsymbol{\tau}_0) &\sim \mathcal{IG} \left( \nu_x + \frac{T}{2}, S_x + \frac{1}{2} \sum_{t=1}^T \epsilon_{x,t}^2 \right). \end{aligned}$$

**Step 5.** Next, we jointly sample  $\boldsymbol{\delta} = (\boldsymbol{\zeta}', \boldsymbol{\tau}_0')$  from its full conditional distribution. To that end,



we write (A.5)– (A.6) as a regression with coefficient vector  $\boldsymbol{\delta}$ :

$$\mathbf{H}_\varphi \boldsymbol{\tau} = \mathbf{X}_\delta \boldsymbol{\delta} + \boldsymbol{\eta},$$

where

$$\mathbf{X}_\delta = \left( \begin{array}{cc} & \begin{matrix} 1 + \varphi_\mu & -\varphi_\mu & 0 & 0 \\ 0 & 0 & 1 + \varphi_{\mu_x} & -\varphi_{\mu_x} \\ -\varphi_\mu & 0 & 0 & 0 \\ 0 & 0 & -\varphi_{\mu_x} & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{matrix} \\ \mathbf{d}_0 \otimes \begin{pmatrix} 1 - \varphi_\mu & 0 \\ 0 & 1 - \varphi_{\mu_x} \end{pmatrix} & \mathbf{d}_1 \otimes \begin{pmatrix} 1 - \varphi_\mu & 0 \\ 0 & 1 - \varphi_{\mu_x} \end{pmatrix} \end{array} \right).$$

Then, by standard regression results, we have

$$(\boldsymbol{\delta} \mid \mathbf{y}, \boldsymbol{\tau}, \gamma, \boldsymbol{\phi}, \boldsymbol{\varphi}, \sigma^2) \sim \mathcal{N}(\widehat{\boldsymbol{\delta}}, \mathbf{K}_\delta^{-1}),$$

where

$$\mathbf{K}_\delta = \mathbf{V}_\delta^{-1} + \mathbf{X}'_\delta \boldsymbol{\Sigma}_\eta^{-1} \mathbf{X}_\delta, \quad \widehat{\boldsymbol{\delta}} = \mathbf{K}_\delta^{-1} (\mathbf{V}_\delta^{-1} \boldsymbol{\delta}_0 + \mathbf{X}'_\delta \boldsymbol{\Sigma}_\eta^{-1} \mathbf{H}_\varphi \boldsymbol{\tau}),$$

where  $\mathbf{V}_\delta = \text{diag}(\mathbf{V}_\zeta, \mathbf{V}_{\tau_0})$  and  $\boldsymbol{\delta}_0 = (\boldsymbol{\zeta}'_0, \boldsymbol{\tau}'_{00})'$ .

Table 2: ADF test: testing the null hypothesis of the presence of a unit root.

Test	Variable	Lags	Test Statistic	Critical Value	Reject Null
ADF	Log RPI	1	-1.6893	-3.4310	No
ADF	Log RPI	2	-1.7358	-3.4310	No
ADF	Log RPI	5	-1.8142	-3.4313	No
ADF	Log TFP	1	-2.5138	-3.4310	No
ADF	Log TFP	2	-2.4844	-3.4310	No
ADF	Log TFP	5	-2.3264	-3.4313	No

*Note:* ADF stands for Augmented Dickey Fuller test. RPI and TFP correspond to the relative price of investment and total factor productivity, respectively. In all cases, the model includes a constant and a time trend. The data series span from 1959.II to 2019.II.. The lag order is selected according to the SIC and HQ criteria.

Table 3: KPSS test: testing the null hypothesis of stationarity.

Test	Variable	Lags	Test Statistic	Critical Value	Reject Null
KPSS	Log RPI	1	2.9071	0.1460	Yes
KPSS	Log RPI	2	1.9482	0.1460	Yes
KPSS	Log RPI	5	0.9899	0.1460	Yes
KPSS	Log TFP	1	0.9144	0.1460	Yes
KPSS	Log TFP	2	0.6196	0.1460	Yes
KPSS	Log TFP	5	0.3238	0.1460	Yes

*Note:* RPI and TFP correspond to the relative price of investment and total factor productivity, respectively. In all cases, the model includes a constant and time trend. The data series span from 1959.II to 2019.II. The lag order is selected according to the SIC and HQ criteria.

Table 4: Johansen's trace test for co-integration between RPI and TFP.

Null	Alternative	Deterministic trend	Lags	p-value
$r = 0$	$r > 0$	Yes	1	0.0845
$r = 0$	$r > 0$	Yes	3	0.0975
$r = 0$	$r > 0$	Yes	7	0.0236

*Note:* The co-integration tests are performed on the logarithms of the relative price of investment and total factor productivity. The sample period is 1959.II to 2019.II. The variable  $r$  denotes the number of co-integrating vectors. RPI and TFP correspond to the relative price of investment and total factor productivity, respectively, and the model includes a constant and time trend. The lag order is selected according to the SIC and HQ criteria.