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Abstract

We build a directed technical change model of the British Industrial Revolution where one intermediate goods sector uses a fixed renewable energy ("wood") quantity, and another uses coal at a fixed price. With a high enough elasticity of substitution between the two goods in producing final output, an industrial revolution, where over time the coal-using sector grows relative to the wood-using sector and its growth accelerates, is not inevitable. However, greater initial scarcity of wood and/or higher population growth puts the economy on a path to an industrial revolution. The converse slows industrialization, or even prevents it forever.

Keywords

British Industrial Revolution, directed technical change, renewable energy, coal, two-sector model, substitutability, population growth

JEL Classification

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Directed Technical Change and the British Industrial Revolution

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Abstract

We build a directed technical change model of the British Industrial Revolution where one intermediate goods sector uses a fixed renewable energy ("wood") quantity, and another uses coal at a fixed price. These resource supply conditions match the stylized facts for the British economy. With a high enough elasticity of substitution between the two goods in producing final output, an industrial revolution, where over time the coal-using sector grows relative to the wood-using sector and its growth accelerates, is not inevitable. However, greater initial scarcity of wood relative to coal, greater initial knowledge of technologies for using wood relative to technologies for using coal, and/or higher population growth puts the economy on a path to an industrial revolution. The converse slows industrialization, or even prevents it forever. The greater the elasticity of substitution and/or the smaller the output elasticity of energy is the more extensive is the set of initial conditions that lead to stagnation. Empirical calibration for the period 1560-1900 produces historically plausible results.

Keywords: Economic growth, economic history, energy, structural change

JEL Codes: N13, N73, O33, O41, Q43

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1 Introduction

Did coal play a vital role in the acceleration of British economic growth known as the Industrial Revolution? Economists and historians are divided on the importance of coal in fueling the increase in the rate of economic growth. Many researchers (e.g. Wilkinson, 1973; Wrigley, 1988, 2010; Pomeranz, 2000; Krausmann *et al*., 2008; Allen, 2009, 2012; Barbier, 2011; Gutberlet, 2012; Kander *et al*., 2013; Fernihough and O'Rourke, 2014, Gars and Olovsson, 2015) argue that innovations in the use, and growth in the quantity consumed, of coal played a crucial role in driving the Industrial Revolution. By contrast, some economic historians (e.g. Clark and Jacks, 2007; Kunnas and Myllyntaus 2009) and economists (e.g. Madsen *et al*., 2010) either argue that it was not necessary to expand the use of modern energy carriers such as coal, or do not give coal a central role (e.g. Clark, 2014). Most growth economists (e.g. Acemoglu, 2009) simply omit any role for energy in explaining economic growth. This debate matters not just for understanding the history of economic development, but also for assessing the future prospects for cutting global fossil fuel use in order to avoid dangerous climate change. We develop a model that shows both analytically and empirically how the relative scarcity of biomass energy (referred to here as "wood", which includes both firewood and charcoal) compared to coal could have directed technical change towards the development of coal-using technologies, resulting in an increase in the economic growth rate. Our baseline empirical model reproduces several stylized facts of the British Industrial Revolution. We are thus the first to show how and why the Industrial Revolution took place in a country with increasingly scarce wood and abundant coal, namely Britain.

Our model is based on Acemoglu's (2002) model of directed technical change. We use the expanding machine varieties (horizontal innovation) approach to modeling endogenous growth, which is appropriate since new types of machines and industrial processes using coal were characteristic of the Industrial Revolution. There are two intermediate goods sectors – the "Malthus" and "Solow" sectors – that produce wood-intensive and coal-intensive goods, respectively, which are then combined into final output via a high-elasticity, CES production function. Each of the intermediate sectors uses labor, an energy input – wood or coal – and sector-specific machines. Unlike previous research discussed below, we do not assume that productivity is inherently higher or faster growing in the Solow than in the Malthus sector. Instead, we assume that wood is supplied perfectly inelastically (i.e. with constant quantity), while coal is supplied perfectly elastically (i.e. at a constant price).[1] In the next section, we show that these key assumptions are consistent with the available historical data.

When the elasticity of substitution between the two intermediate goods is greater than unity, innovation activity is positively related to the relative abundance of the two sector-specific factors. Thus, an increase in the scarcity of wood relative to coal increases the level of innovation in the coal-using Solow sector relative to that in the wood-using Malthus sector. Kander and Stern (2014) show that the elasticity of substitution between biomass and fossil fuel energy was greater than unity in Sweden in the late 19th and early 20th Centuries and we assume here that that was the case for Britain too.

We show that if the elasticity of substitution in final production is high enough and wood sufficiently abundant relative to coal, with low population growth, an economy can remain trapped in a state of near-stagnation with a low rate of economic growth and increasing dominance of the Malthus, wood-using sector. We refer to this as Malthusian sluggishness. Rapid coal-driven growth does not eventually occur unless wood is relatively scarce or substitution between wood-using and coal-using goods relatively difficult. Increasing population can increase the relative scarcity of wood and drive a transition to modern economic growth by directing technological change towards the development of coal-using machines. So "necessity is the mother of invention" in our model, which is broadly the industrial equivalent of Boserup's (1981) mechanism where technical change in agrarian societies is driven mainly by rising natural resource scarcity.

Of course, this analysis abstracts from other issues such as Allen's (2009) argument that expensive labor was the reason why coal-directed innovation was profitable in Britain long before it was elsewhere, which Crafts and O'Rourke (2014) find to be a plausible explanation. We also implicitly assume that the British institutional environment was appropriate for accelerating growth to occur, for example by having the well-developed patenting system which Madsen *et al.* (2010) found to be econometrically significant.[2] Furthermore, we do not make a distinction between the usefulness of different inventions. As discussed by Crafts (2010), authors such as Mokyr (2009b) and Allen (2009) viewed "macro-inventions" like the steam engine or coke smelting as having a significant role in the Industrial Revolution. However, it can take more than a century of small improvements ("micro-inventions") for

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[1] Unlike Hanlon's (2015) study of the effect of the American civil war on British innovation, supply conditions do not change over time in our model, rather the elasticities of supply of the factor inputs are different.

[2] Though see Mokyr (2009a) on the limitations of the patent system.

technical efficiency to improve enough for a macro-invention to have a significant macro-economic impact (Allen, 2009; Clark and Jacks, 2007). Modeling technological change as deterministic and incremental, as we do here, rather than stochastic and sometimes revolutionary, therefore, arguably misses no vital feature of the Industrial Revolution. Finally, we abstract from other properties of coal relative to wood such as higher energy density per cubic meter, or per hectare of land using for energy production.

The previous research relevant to our model falls into three areas. First are "unified growth" models, which explain the takeoff from Malthusian stagnation (where any technical progress results in population rather than income growth) but do not model fossil fuels explicitly.[3] Seminal papers here are Galor and Weil (2000) and Hansen and Prescott (2002); these both include a fixed supply of land, which can be seen as a source of renewable energy. Galor and Weil have one sector with endogenous population growth and technical progress that depends on the level of population. They also assume that the return to land is zero. Hansen and Prescott have two sectors, with a land input in the agricultural, "Malthus" sector, no natural resource input to the industrial, "Solow" sector, semi-endogenous population growth, and exogenous technical progress that is assumed *a priori* to be much faster in the Solow than in the Malthus sector. Other papers in this vein include O'Rourke *et al*. (2013), who introduce directed technical change in a unified growth model, but with sectors distinguished by high or low labor skills rather than by use of land; and Kögel and Prskawetz (2001) and Strulik and Weisdorf (2008), who make assumptions about differences in productivity growth or the elasticity of consumer demand for the output from agricultural and manufacturing sectors. Lewis (1954) was, of course, the first to develop a two-sector model of the transformation of a pre-industrial economy. He assumed an infinitely elastic supply of labor in the traditional, land-based sector, and that capital was only used in the modern sector. But these assumptions about economies in the first stages of industrialization are not necessarily accurate (Gollin, 2014).

The second area of relevant literature comprises papers that do model the effect of fossil fuels on long-run growth (Tahvonen and Salo, 2001; Fröling, 2011; Gars and Olovsson, 2015; Eren and Garcia-Macia, 2013). However, like Hansen and Prescott (2002), these researchers all assume that productivity in the use of fossil fuels is higher or can increase faster than that

[3] In our review of the literature, we use the various terms each researcher uses for different energy resources such as fossil fuels, renewable energy, biomass, etc.; some of which also differ substantively from the “coal” and “wood” categories used in our model.

in the use of renewable energy. Perhaps the closest precursor of our paper is Eren and Garcia-Macia (2013) since they also explain the Industrial Revolution as a transition from using wood to using coal as the main energy source, enabled by directed technical change. But they ignore population growth, treat both coal and wood as strictly non-renewable resources, assume that energy is only and exclusively used to build machines, and assume, *a priori*, a permanently lower productivity parameter in the Malthus, wood-using sector than in the Solow, coal-using sector.[4]

The third area of relevant literature is empirical work on the historical role of coal in the Industrial Revolution. Clark and Jacks (2007) argue that an industrial revolution could still have happened in a coal-less Britain with only "modest costs to the productivity growth of the economy" (68), because the value of coal was only a modest share of British GDP, and they argue that Britain's energy supply could have been greatly expanded, albeit at about twice the cost of coal, by importing wood from the Baltic. Madsen *et al.* (2010) find that coal production in British coalmines has no econometrically significant effect on per-capita output. Both Clark and Jacks (2007) and Madsen et al. (2010) do not allow for the dynamic effects of resource scarcity on the rate of innovation. Tepper and Borowiecki (2015) also find a relatively small direct role for coal but concede that: “coal contributed to structural change in the British economy” (231), which they find was the most important factor in raising the rate of economic growth. On the other hand, Fernihough and O’Rourke (2014) and Gutberlet (2012) use geographical analysis to show the importance of access to local coal in driving industrialization and urban population growth, though Kelly *et al*. (2015) provide contradictory evidence on this point. Finally, Kander and Stern (2014) econometrically estimate a model of the transition from biomass energy (mainly wood) to fossil fuel (mainly coal) in Sweden, which shows the importance of this transition in economic growth there. However, they assume exogenous factor-augmenting technical change.

The outline of the paper is as follows. In the second section, we examine the available data on economic growth, energy use and energy prices in the period of the Industrial Revolution, and thus explain our choice of stylized facts that we wish to reproduce in our model. In the third section, we present our model. In the fourth, we analyze theoretically the

[4] There are many other papers that look at the role of resources in endogenous growth models but assume there is only one type of resource. For example, Peretto and Valente (2015) model final output as a high elasticity of substitution CES aggregate of a continuum of intermediates that are each produced using a CES production function in land and labor. Schäfer (2014) assumes machines are made from a non-renewable resource and produce two intermediate goods using either skilled or unskilled labor.

factors affecting the direction of technical change and predictions for the evolution of pre-industrial economies, which either undergo or do not undergo a transition to modern economic growth. In the fifth, we present our baseline empirical simulation of British history, together with counterfactual simulation scenarios that support the popular view that plentiful, cheap coal was indeed a necessary, though not necessarily sufficient, condition for the Industrial Revolution to happen in Britain in the 18th and 19th centuries. The final section concludes.

2 Stylized Facts

Figures 1 and 2 show the evolution of GDP per capita and its growth rate over 20-year periods from 1540-1900.[5] Up to 1660, GDP per capita was flat or declining, after which it grew at an accelerating rate, though the growth rate was quite erratic and in the second half of the 19th Century ranged from 0.8% to 1.9% p.a., which is low by 20th or 21st Century standards.[6]

Figure 3 shows the real prices of coal and charcoal in London and the Western Britain (Allen, 2009). The price of charcoal rose steeply from the beginning of the 17th Century to the late 18th Century after which it appears to level off and possibly fall (Fouquet, 2011). The price of coal though is relatively stable over time in both regions. Clark and Jacks (2007) explain that throughout this period innovation overcame the effects of depletion resulting in the long-run supply of coal being highly elastic. Figure 4 shows the energy content of firewood (including charcoal) and coal consumed in England and Wales (Warde, 2007, Appendix). Firewood provided about 80% of total fuel in 1560, declining to about 25% by 1700 and to zero by 1850. The quantity of firewood used was fairly constant from about 1560 until 1800. Though timber was increasingly imported to Britain, especially in the 19th Century (Iriarte-Goñi and Ayuda, 2012), there does not seem to have been significant international trade in firewood (Thomas, 1986; Warde, 2007). Coal use increased 700-fold over the period. Though the quantity of firewood used eventually fell to zero during the 19th Century, for simplicity our model will assume that wood use for energy (including charcoal)

[5] For 1870 to 1900 we use the Composite GDP (E) measure of real GDP at 2006 prices from Hills *et al*. (2010). From 1540 to 1870 we used the growth rates from Broadberry *et al*.'s (2015) estimate of GDP for Great Britain in constant prices of 1700 to project real GDP back to 1540.

[6] Though an acceleration of the rate of economic growth was a defining feature of the Industrial Revolution, the time path of income (per capita) over the last millennium is still deeply disputed among economic historians (Fouquet and Broadberry, 2015). For example, Clark (2013) notes that while he estimates English income to have changed very little between pre-industrial times and 1800, the data now published in Broadberry *et al*. (2015) estimate that income nearly tripled between 1270 and 1800.

was constant throughout.

Gentvilaite *et al*. (2015) calculate that the energy cost share declined from approximately 25% of total costs in 1800 to 10% today in the United Kingdom. Energy intensity in Britain increased till the end of the 19th Century after which it declined (Kander *et al*., 2013). From 1720 to 1900 it roughly doubled, but prior to the mid-18th Century it was fairly constant (Figure 5). However, if one includes only coal and wood in the energy aggregate then intensity also rose since the early 17th Century and quadrupled by 1900. Given the data shown here, it seems that the cost share of energy may have risen till the late 17th Century as the price of wood rose, before beginning a slow decline as cheaper coal became an increasingly large share of total energy use. Given these facts, we do not need to be able to model a rapid decline in energy intensity or in the energy cost share over time − which would not be the case if we were modeling 19th Century Sweden (Kander and Stern, 2014) − so it is reasonably consistent with history that our model will assume a constant energy cost share.

3 The Model

We assume there are two energy sources – coal and wood – which are good substitutes for each other, and can both be augmented by technological change. In common with Acemoglu (2002), technical change is modeled as an expansion of machine varieties, but as in Acemoglu *et al*. (2012), in addition to intermediate machines and labor, natural resources contribute to production. While only one sector has a resource input in Acemoglu *et al*. (2012), in our model each sector has a resource input – "wood" or coal. We model only the industrial sectors of the economy, not any resource extraction sectors, so we treat the resource inputs as being effectively "imported" into the economy. Therefore, we do not need to consider the non-renewable nature of coal – or the renewable nature of wood – explicitly. Following our discussion of Figures 3 and 4, we assume the wood quantity and coal price are exogenously fixed. Except in the Constant Population scenario in Section 5, we assume population, and hence the labor force, grow exogenously, so that the available wood quantity per worker falls. As in Acemoglu *et al*. (2012), we use discrete time and assume that a patent for any variety of machine only lasts one period, here 20 years.[7] We assume that at the

[7] If innovators are granted perpetual patents then they need to consider the net present value of the stream of future profits when deciding how much to invest in innovation activities. As explained by Acemoglu (2002), this decision is then complicated because not only might the interest rate vary over time off a balanced growth path – and in our model a balanced growth path is highly unlikely due to the fixed wood supply – but also the relative prices of the two goods will change over time. This would lead to a complicated dynamic programming

beginning of each period, patents for all existing machine varieties are re-issued at random, meaning that all varieties (new and existing) are produced by monopolistic firms, which maximize only current period profits.[8] The 20-year period also is a convenient time step for the assumption that all machines depreciate fully within one period. As a result, the consumer plays no active role in our model: profit maximization ensures that consumption is maximized and there is no intertemporal investment decision, which greatly simplifies the model. We use a hybrid of Acemoglu's (2002) lab equipment and knowledge-based R&D models, with production of new varieties depending on both existing knowledge and R&D expenditure.

3.1 Production

Final output, Y, is produced competitively from two intermediate goods, Y_M and Y_S, via a constant elasticity of substitution production function:

$$Y_t = \left[\gamma Y_{M,t}^{\frac{\sigma-1}{\sigma}} + (1-\gamma) Y_{S,t}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \tag{1}$$

where $\sigma > 1$ is the elasticity of substitution, $0 < \gamma < 1$ is the distribution parameter, and t indicates the (discrete) time period.[9] The two intermediate goods are produced competitively using the following Cobb-Douglas technologies:

$$Y_{M,t} = \frac{1}{\beta}\left(\int_0^{N_{M,t}} x_{M,t}(j)^\beta dj\right) \bar{E}_M^\alpha L_{M,t}^{1-\alpha-\beta} \tag{2}$$

$$Y_{S,t} = \frac{1}{\beta}\left(\int_0^{N_{S,t}} x_{S,t}(j)^\beta dj\right) E_{S,t}^\alpha L_{S,t}^{1-\alpha-\beta} \tag{3}$$

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problem, which is why Acemoglu (2002) focuses on deviations from a steady state. 20 years is the current length of a UK patent. The 1624 Statute of Monopolies set a 14 year period (Khan and Sokoloff, 2004).

[8] This is similar to the assumption in Acemoglu *et al*. (2012). We could instead assume that when the patent expires each machine variety is produced competitively in all following periods, so that its price equals marginal cost, so that newly developed machine varieties will be priced higher than older varieties and used in smaller amounts (see Gancia and Zilibotti (2005) and Appendix B9 of Acemoglu *et al*. (2012) for similar models). This is what is seen in the real world, where new technologies are expensive and sold in smaller quantities but later become commodified. However, this assumption complicates our analytical model without changing our qualitative results or adding any useful insights.

[9] Kander and Stern (2014) estimate that the elasticity of substitution between biomass energy and fossil fuels was much greater than one in Sweden in the 19th and early 20th Centuries. This implies that the elasticity of substitution between biomass-intensive and coal-intensive goods was also greater than unity. Intuitively, consumers do not care very much whether products are made using coal or wood as the energy source.

where $0 < \alpha, \beta, \alpha + \beta < 1$.[10] Subscript *M* (Malthus) indicates the sector using the fixed wood supply, $\bar{E}_M$, and a range $N_{M,t}$ of varieties of wood-using machines as inputs, with each variety *j* used in amount $x_{M,t}(j)$. Subscript *S* (Solow) indicates the sector using an indefinitely expandable coal supply, $E_{S,t}$, and a range $N_{S,t}$ of varieties of coal-using machines as inputs, with each variety used in amount $x_{S,t}(j)$.[11] The initial ranges of machine varieties that can be used with wood and coal, respectively $N_{M,0} > 0$ and $N_{S,0} > 0$, are given as parameters. $L_{M,t}$ and $L_{S,t}$ are the labor levels used in each sector, the sum of which, L_t, is assumed to be exogenous and equal to the level of population:

$$L_{M,t} + L_{S,t} = L_t \tag{4}$$

In our baseline simulation, population L_t closely matched to British history, as described in Section 5 below. We use final output, *Y*, as the numeraire, normalizing its price to 1. The prices of the two goods inputs are thus related as follows:

$$\gamma^{\sigma} p_{M,t}^{1-\sigma} + (1-\gamma)^{\sigma} p_{S,t}^{1-\sigma} = 1 \tag{5}$$

The goods price ratio is given in competitive equilibrium by:[12]

$$p_t \equiv \frac{p_{M,t}}{p_{S,t}} = \frac{\gamma}{1-\gamma}\left(\frac{Y_{M,t}}{Y_{S,t}}\right)^{-\frac{1}{\sigma}} = \Gamma {y_t}^{-\frac{1}{\sigma}}, \text{where } \Gamma \equiv \frac{\gamma}{1-\gamma} \text{ and } y_t \equiv \frac{Y_{M,t}}{Y_{S,t}} \tag{6}$$

which we use in later working to replace p_t by y_t or *vice versa*. The marginal value products and hence prices of wood and coal are respectively given by:

$$e_{M,t} = p_{M,t}\frac{\alpha}{\beta}\left(\int_0^{N_{M,t}} x_{M,t}(j)^{\beta} dj\right) \bar{E}_M^{\alpha-1} L_{M,t}^{1-\alpha-\beta} = \alpha p_{M,t}\frac{Y_{M,t}}{\bar{E}_M} \tag{7}$$

$$\bar{e}_S = p_{S,t}\frac{\alpha}{\beta}\left(\int_0^{N_{S,t}} x_{S,t}(j)^{\beta} dj\right) E_{S,t}^{\alpha-1} L_{S,t}^{1-\alpha-\beta} = \alpha p_{S,t}\frac{Y_{S,t}}{E_{S,t}} \tag{8}$$

where the coal price, $\bar{e}_S$, is assumed to be constant, as noted above. The common wage rate equals similar expressions for the marginal value product of labor:

$$w_t = p_{M,t}(1-\alpha-\beta)\frac{Y_{M,t}}{L_{M,t}} = p_{S,t}(1-\alpha-\beta)\frac{Y_{S,t}}{L_{S,t}} \tag{9}$$

[10] For reasons of analytical tractability we use this Cobb-Douglas form, which departs from the more realistic assumption that the elasticity of substitution between energy and machines is less than 1, as used in previous research (Stern and Kander, 2012; Kander and Stern, 2014). Numerical simulations show that an elasticity less than 1 gives results not much different from those in this paper.

[11] As is standard in this literature, we use an integral rather than a summation over machine varieties for computational tractability (Aghion and Howitt, 2009, p71; Acemoglu, 2009, p425).

[12] Because we choose this definition instead of $p_t \equiv p_{S,t}/p_{M,t}$, the energy price ratio is defined as wood/coal not coal/wood, and thus rises during the transitional stages of an industrial revolution.

3.2 Market for Machines

Given the above, the first order conditions for profit maximization by competitive manufacturers of each intermediate good $Y_i, i = M, S$, imply that the amount of each variety of machine that they demand is:

$$x_{i,t}(j) = \left(\frac{p_{i,t}E_{i,t}^{\alpha}L_{i,t}^{1-\alpha-\beta}}{\chi_{i,t}(j)}\right)^{\frac{1}{1-\beta}} \tag{10}$$

Following Acemoglu (2002), we set the marginal cost of manufacturing a machine at a common constant, ψ. Given our assumption that all machines are produced under a single-period patent, each machine variety is supplied by a monopolist that maximizes profit, which for variety j is given by:

$$\pi_{i,t}(j) = \left(\chi_{i,t}(j) - \psi\right)x_{i,t}(j) = p_{i,t}E_{i,t}^{\alpha}L_{i,t}^{1-\alpha-\beta}[x_{i,t}(j)]^{\beta} - \psi x_{i,t}(j) \tag{11}$$

Maximizing profit then results in a (privately) optimal machine price of $\chi_{i,t}^{*}(j) = \frac{\psi}{\beta}$. Following Acemoglu (2002), we set marginal cost $\psi = \beta$ so that $\chi_{i,t}^{*}(j) = 1$. Then, from (10), the optimal amount of each machine variety sold by each monopolist is given by:

$$x_{i,t}^{*}(j) = (p_{i,t}E_{i,t}^{\alpha}L_{i,t}^{1-\alpha-\beta})^{\frac{1}{1-\beta}} \tag{12}$$

and profit per new variety is therefore:

$$\pi_{i,t}(j) = [\chi_i^{*}(j) - \psi]x_{i,t}^{*}(j) = (1-\beta)\left(p_{i,t}E_{i,t}^{\alpha}L_{i,t}^{1-\alpha-\beta}\right)^{\frac{1}{1-\beta}} \tag{13}$$

So the relative profitability, $\pi_{M,t}(j)/\pi_{S,t}(j)$, of innovating in the two sectors depends on the effects of two ratios: the ratio of the intermediate goods prices $\left(p_{M,t}/p_{S,t}\right)$, and the ratio of the market sizes $\left(\bar{E}_M^{\alpha}L_{M,t}^{1-\alpha-\beta}/E_{S,t}^{\alpha}L_{S,t}^{1-\alpha-\beta}\right)$, which depends on both the number of workers in each sector and the relative scarcity of the two energy inputs. As shown in Appendix A, substituting $x_{i,t}^{*}(j)$ from (12) into the production functions (2) and (3) gives these intermediate outputs:

$$Y_{M,t}(p_t, N_{M,t}) = \frac{1}{\beta}N_{M,t}p_{M,t}^{\frac{\beta}{1-\beta}}(p_t)\bar{E}_M^{\frac{\alpha}{1-\beta}}L_{M,t}^{\frac{1-\alpha-\beta}{1-\beta}}(p_t) \tag{14}$$

$$Y_{S,t}(p_t, N_{S,t}) = \frac{1}{\beta}N_{S,t}p_{S,t}^{\frac{\beta}{1-\beta}}(p_t)E_{S,t}^{\frac{\alpha}{1-\beta}}(p_t, N_{S,t})L_{S,t}^{\frac{1-\alpha-\beta}{1-\beta}}(p_t) \tag{15}$$

and this expression for the optimal quantity of coal use:

$$E_{S,t}\left(p_t, N_{S,t}\right) = \left(\frac{\alpha N_{S,t}}{\beta\bar{e}_S}\right)^{\frac{1-\beta}{1-\alpha-\beta}} p_{S,t}^{\frac{1}{1-\alpha-\beta}}(p_t)\; L_{S,t}(p_t) \tag{16}$$

3.3 Technology Innovation

Our most general innovation assumption is that new machine varieties generated in sector *i* and period *t*, $\Delta N_{i,t} \equiv N_{i,t} - N_{i,t-1}$, (with Δ similarly defined for all other time-dependent variables), are a function of the range of varieties in the previous period in the same sector, $N_{i,t}$, and R&D expenditure in that sector, $R_{i,t}$:

$$\Delta N_{i,t} = \eta N_{i,t-1}^{\mu} R_{i,t}^{\nu};\ \eta > 0, 0 < \ \mu, \nu < 1 \tag{17}$$

We thus assume diminishing returns in knowledge production in each sector, both to prior knowledge within that sector ($\mu < 1$),[13] and to research expenditure as more innovating firms enter the sector and spend on R&D ($\nu < 1$), which is necessary to obtain an equilibrium. We rearrange (17) to give the total cost of producing new varieties in the sector in a given period:

$$R_{i,t} = \left(\frac{\Delta N_{i,t}}{\eta N_{i,t-1}^{\mu}}\right)^{\frac{1}{\nu}} \tag{18}$$

The free entry condition (Acemoglu and Zilibotti, 2001) means that the profit from the last variety, $\pi_{i,t}(j)$ from (13), will equal the marginal cost of producing a new variety in a sector in a given period, $\partial R_{i,t}/\partial\left(\Delta N_{i,t}\right)$ calculated from (18):[14]

$$(1-\beta)\left(p_{i,t}E_{i,t}^{\alpha}L_{i,t}^{1-\alpha-\beta}\right)^{\frac{1}{1-\beta}} = \frac{1}{\nu}\left(\frac{1}{\eta N_{i,t-1}^{\mu}}\right)^{\frac{1}{\nu}}\left(\Delta N_{i,t}\right)^{\frac{1-\nu}{\nu}} \tag{19}$$

Rearranging (19) and defining $h \ \equiv \frac{1-\nu}{\nu}$ then gives:

$$\frac{\Delta N_{i,t}}{N_{i,t-1}} = N_{i,t-1}^{\frac{\mu+\nu-1}{h\nu}}\left(\nu\eta^{\frac{1}{\nu}}(1-\beta)\right)^{\frac{1}{h}}\left(p_{i,t}E_{i,t}^{\alpha}L_{i,t}^{1-\alpha-\beta}\right)^{\frac{1}{h(1-\beta)}} \tag{20}$$

However, the $N_{i,t-1}^{\frac{\mu+\nu-1}{h\nu}}$ term in (20) makes our model analytically intractable, so almost everywhere we impose the restriction of constant returns to scale in knowledge production in (17): $\nu = 1 - \mu$, hereafter referred to as *CRS innovation*. We will show later (Proposition 6 in Section 4) that under this and one other parameter restriction, an industrial revolution, where production becomes ever more concentrated in the Solow, coal-using sector, must entail accelerating economic growth, as observed historically in Britain. However, Proposition 5 will show that if $\nu < \frac{1-\alpha-\beta}{1-\beta}(1-\mu)$, growth of machine varieties in an industrial revolution

[13] We are assuming what Acemoglu (2002) calls "extreme state dependence", where there are no spillovers between the sectors, so that $\Delta N_{M,t}$ is unaffected by $N_{S,t-1}$ and *vice versa*.

[14] Because of diminishing returns, this is an equality rather than the usual inequality, so there will always be innovation in both sectors as long as both intermediate goods are produced.

could be at an accelerating, decelerating, or momentarily constant rate. This means that the accelerating growth shown in Proposition 6 is as much a result of our assumption of CRS innovation as a result of our model.

3.4 Household

Each household supplies a unit of labor inelastically. Consumers' income consists of the profits from the sale of machines and wages. Total consumption is given by $C_t = Y_t - I_t - \sum_i R_{i,t} - \sum_i e_{i,t} E_{i,t}$, where I is total expenditure on producing machines. As already noted, the consumer is only a passive consumer of final output, so we need not specify consumption any further than this. Population is set exogenously as explained in Section 5.

3.5 Equilibrium

The model yields a system of three simultaneous equations for three unknowns in any period *t*: the intermediate good price ratio $p_t \equiv \frac{p_{M,t}}{p_{S,t}}$, already seen in (6), and the numbers of Malthus sector (wood-using) and Solow sector (coal-using) machine varieties, $N_{M,t}$ and $N_{S,t}$, as given by (20) after substituting in the relevant functions, and $\nu = 1 - \mu$ and $h = \frac{\mu}{1-\mu}$:

$$p_t = \frac{\gamma}{1-\gamma}\left(\frac{Y_{M,t}(p_t, N_{M,t})}{Y_{S,t}(p_t, N_{S,t})}\right)^{-\frac{1}{\sigma}} \tag{21}$$

$$\frac{N_{M,t} - N_{M,t-1}}{N_{M,t-1}} = \left[(1-\mu)\eta^{\frac{1}{1-\mu}}(1-\beta)\right]^{\frac{1-\mu}{\mu}} \left[p_{M,t}(p_t)\bar{E}_M^{\alpha} L_{M,t}^{1-\alpha-\beta}(p_t)\right]^{\frac{1-\mu}{\mu(1-\beta)}} \tag{22}$$

$$\begin{gathered}\frac{N_{S,t} - N_{S,t-1}}{N_{S,t-1}} = \\ \left[(1-\mu)\eta^{\frac{1}{1-\mu}}(1-\beta)\right]^{\frac{1-\mu}{\mu}} \left[p_{S,t}(p_t)E_{S,t}^{\alpha}(p_t, N_{S,t}) L_{S,t}^{1-\alpha-\beta}(p_t)\right]^{\frac{1-\mu}{\mu(1-\beta)}}\end{gathered} \tag{23}$$

Appendix A gives the explicit functional forms needed here for $p_{M,t}(p_t)$ and $L_{M,t}(p_t)$ (and hence $Y_{M,t}(p_t, N_{M,t})$ via (14)), and for $p_{S,t}(p_t)$ and $L_{S,t}(p_t)$ (and hence for $Y_{S,t}(p_t, N_{S,t})$ via (16) and (15)). Given all these functional forms and the model parameters at the start of period *t*, namely $\bar{E}_M$, $\bar{e}_S$, $N_{M,t-1}$, $N_{S,t-1}$, α, β, γ, σ, μ, η and L_t, we establish the following:

DEFINITION 1. *An equilibrium is given by the sequences of wages* (w_t), *intermediate output prices* $(p_{M,t}, p_{S,t})$, *wood prices* $(e_{M,t})$, *coal demands* $(E_{S,t})$, *labor demands* $(L_{M,t}, L_{S,t})$, *machine demands* $(x_{M,t}, x_{S,t})$, *and expenditures on innovation* $(R_{M,t}, R_{S,t})$ *such that in each period t:* p_t *is given by (21) and* $N_{M,t}$ *and* $N_{S,t}$ *are given by (22) and (23), respectively.*

4 Analytical Results

4.1 Introduction

Given the historically representative asymmetry of our model's key sectoral assumptions − a constant wood quantity, $\bar{E}_M$, in the Malthus sector and a constant coal price, $\bar{e}_S$, in the Solow sector − a balanced growth path *à la* Acemoglu (2009), where the intermediate good price ratio $p_t \equiv \frac{p_{M,t}}{p_{S,t}}$ is constant, is not relevant here. Such a path is possible only in an economy that does not undergo an industrial revolution, and then only for highly specific parameter values. If the economy is industrializing, the output ratio of the two intermediate goods will be falling (falling because we define this ratio as $y_t \equiv \frac{Y_{M,t}}{Y_{S,t}}$ not $\frac{Y_{S,t}}{Y_{M,t}}$), and their relative price ratio, p_t, will be rising. Instead, we derive several key analytical results for non-balanced growth paths.

Many of these are illustrated by Figures 7a-b and 8a-b. Each pair of Figures shows phase diagrams for the goods ratio, y_t, against the machine varieties ratio, $N_t \equiv N_{M,t}/N_{S,t}$, and for the relative wood/coal price ratio or energy price ratio, $e_t \equiv e_{M,t}/\bar{e}_S$, against N_t. Figures 7a-b show that if the elasticity of substitution in final production, σ, is *high* (to be defined shortly), "divergent development" occurs: depending on the economy's starting point, the economy either stays in *Malthusian sluggishness* (MS), where total output grows but becomes ever more concentrated in the Malthus sector (so the goods ratio y_t rises forever, as in Fig. 7a); or it undergoes an *industrial revolution* (IR, i.e. y_t forever falling towards zero) and also with eventually a "modern economic growth" phase where the energy price ratio, e_t, falls forever (as in Fig. 7b). By contrast, Figures 8a-b shows that with less than a *high* elasticity of substitution, an IR must eventually happen, and will entail a forever-rising energy price ratio, whatever the economy's starting point.

Analytic proofs of this divergence result in the *high* substitutability case are available only for the ahistorical counterfactual where population is constant, but by continuity they must hold for some degree of population growth, and numerical simulations confirm these properties for empirically relevant population growth. Under *high* substitutability, we will also show analytically (and without assuming constant population) that economic growth accelerates over time on an IR path, and must eventually become faster than on an MS path; though as noted earlier, this specific result requires the CRS innovation assumption, $\nu = 1 - \mu$, which we make throughout, except in Propositions 5, 7 and 8. Proposition 5 explores the different effects on sectoral growth rates of different assumptions about ν; while quite general comparative static analysis in Propositions 7 and 8, not requiring either constant population or *high* substitutability assumptions, shows the effect of key parameters on the economy's state of development.

4.2 Notation

We first establish notation for several variable ratios, parameter values, and terms, some of which have already appeared:

$$e_t \equiv \frac{e_{M,t}}{\bar{e}_S};\ E_t \equiv \frac{\bar{E}_M}{E_{S,t}};\ l_t \equiv \frac{L_{M,t}}{L_{S,t}};\ p_t \equiv \frac{p_{M,t}}{p_{S,t}};\ y_t \equiv \frac{Y_{M,t}}{Y_{S,t}} \tag{24}$$

$$N_t \equiv \frac{N_{M,t}}{N_{S,t}};\ n_{M,t} \equiv \frac{\Delta N_{M,t}}{N_{M,t-1}} \equiv \frac{N_{M,t}-N_{M,t-1}}{N_{M,t-1}};\ n_{S,t} \equiv \frac{\Delta N_{S,t}}{N_{S,t-1}};\ n_t \equiv \frac{n_{M,t}}{n_{S,t}} \tag{25}$$

$$m \equiv \frac{\mu}{1-\mu};\ h \equiv \frac{1-\nu}{\nu};\ \Gamma \equiv \frac{\gamma}{1-\gamma};\ \tilde{\sigma} \equiv 1+\frac{1}{1-\beta} < \sigma^{\dagger} \equiv 1+\frac{1}{1-\alpha-\beta} \tag{26}$$

Note from (24)-(25) that industrial development means falling ratios of machine varieties, N_t, energy quantities, E_t, and, as noted above, of intermediate goods, y_t; so N_t is best thought of as a measure of *non*-development. Also note this relationship between N_t and n_t:

$$\Delta N_t \equiv \frac{N_{M,t}}{N_{S,t}} - \frac{N_{M,t-1}}{N_{S,t-1}} \gtreqless 0 \Leftrightarrow \frac{\frac{N_{M,t}-N_{M,t-1}}{N_{M,t-1}}}{\frac{N_{S,t}-N_{S,t-1}}{N_{S,t-1}}} \equiv n_t \gtreqless 1 \tag{27}$$

and $n_t > 0$ always, since from (17), machine varieties always grow ($\Delta N_{i,t} > 0$).

4.3 Definitions of Degrees of Substitutability, Industrial Revolution, and Malthusian Sluggishness

The elasticity of substitution, σ, is *low* if $1 < \sigma < \tilde{\sigma}$, *medium* if $\tilde{\sigma} < \sigma < \sigma^{\dagger}$, and *high* if $\sigma > \sigma^{\dagger}$.[15] We define a development path of the model to undergo an IR if $\Delta N_t < 0$ and $\Delta y_t < 0$ forever after some *t* on the path, so that Solow machine varieties and goods output are rising relative to Malthus varieties and output, with $N_t \to 0$ and $y_t \to 0$ as $t \to \infty$. We define MS to occur on a path if $\Delta N_t > 0, \Delta y_t > 0$ initially and forever, with $N_t \to \infty$ and $y_t \to \infty$ as $t \to 0$, so that it never undergoes an IR. Lastly, we define an IR development path to have a *modern economic growth* phase after some time *t* if e_t rises before *t* and falls forever after *t*.

[15] We ignore the theoretically degenerate cases $\sigma = \tilde{\sigma}$ and $\sigma = \sigma^{\dagger}$ and the empirically uninteresting range $\sigma \leq 1$.

Throughout, our analysis treats what are formally differences in discrete time ($\Delta y, \Delta N$, etc) as differentials in continuous time (dy, dN, etc); our many simulations (mostly not reported in Section 5) have confirmed that the analytic results thus found here hold true numerically.

4.4 General Results for the (y.N) and (e.N) Phase Diagrams

We now prove the phase-diagram properties shown in Figs 7a-8b in several steps. We start with the following equations that determine the direction of technical change n_t in (y, N)-space and (e, N)-space, whose derivations are given in Appendix B1:

$$n_t = \Gamma^{\frac{1}{m}} y_t^{\frac{\sigma-1}{m\sigma}} N_t^{\frac{-1}{m}} \tag{28}$$

and

$$y_t = \Gamma^{\sigma} e_t^{-\alpha\sigma} N_t^{(1-\beta)\sigma} \text{ or } e_t = \Gamma^{\frac{1}{\alpha}} y_t^{-\frac{1}{\alpha\sigma}} N_t^{\frac{1-\beta}{\alpha}} \tag{29}$$

hence

$$n_t = \Gamma^{\frac{\sigma}{m}} e_t^{\frac{-\alpha(\sigma-1)}{m}} N_t^{\frac{(\sigma-\tilde{\sigma})(1-\beta)}{m}} \tag{30}$$

Equations (28) and (30) explain the forms of the $n_t = 1$ ($\Delta N_t = 0$) isoclines, and the signs of ΔN_t above and below these isoclines, in the (y, N) and (e, N) phase diagrams, respectively, as follows:

$$n_t \gtreqless 1 (\Leftrightarrow \Delta N_t \gtreqless 0) \Leftrightarrow N_t \lesseqgtr \Gamma y_t^{\frac{\sigma-1}{\sigma}} \tag{31}$$

$$n_t \gtreqless 1 (\Leftrightarrow \Delta N_t \gtreqless 0) \Leftrightarrow N_t \ \gtreqless\ \Gamma^{\frac{-\sigma}{(\sigma-\tilde{\sigma})(1-\beta)}} e_t^{\frac{\alpha(\sigma-1)}{(\sigma-\tilde{\sigma})(1-\beta)}} \tag{32}$$

Note that (28) and (30) $\Longrightarrow \frac{\partial n_t}{\partial N_t}\Big|_{y=\text{constant}} < 0$ but $\frac{\partial n_t}{\partial N_t}\Big|_{e=\text{constant}} > 0$, i.e. n_t rises as we move vertically downwards in (y, N)-space, but falls as we move downwards in (e, N)-space. Note also that the exponent of e_t in (32), $\frac{\alpha(\sigma-1)}{(\sigma-\tilde{\sigma})(1-\beta)} < 1$ if $\sigma > \sigma^{\dagger}$, giving the concave $n = 1$ isocline for the *high* substitutability case in Fig 7b, but $\frac{\alpha(\sigma-1)}{(\sigma-\tilde{\sigma})(1-\beta)} > 1$ if $\tilde{\sigma} < \sigma < \sigma^{\dagger}$, giving the convex isocline for the *medium* substitutability case in Fig 8b.[16]

From Appendix B2, the $\Delta y_t = 0$ isoclines in Figs 7a and 8a are determined by:

$$\frac{1 + \alpha(\sigma - 1) + \left(\frac{1-\beta}{1-\alpha-\beta}\right) \Gamma y_t^{\frac{\sigma-1}{\sigma}}}{1 + \Gamma y_t^{\frac{\sigma-1}{\sigma}}} (n_t - 1)\, \Delta\ln(y_t) \tag{33}$$

[16] We do not show the *low* substitutability, $\sigma < \tilde{\sigma}$ case; its results are the same as in the *medium* case, except that the $n = 1$ isocline in (*e*,*N*) phase-space is now downward-sloping because $\frac{\alpha(\sigma-1)}{(\sigma-\tilde{\sigma})(1-\beta)} < 0$.)

$$= \sigma(1-\beta)\left(n_t - \frac{1-\beta}{1-\alpha-\beta}\right)\Delta\ln(N_t) \; - \; (n_t - 1)\alpha\sigma \, \Delta\ln(L_t)$$

This equation immediately recovers $n_t = 1$, hence $N_t = \Gamma y_t^{\frac{\sigma-1}{\sigma}}$ (31), as the equation for the $\Delta N_t = 0$ isocline. With *constant population*, $\Delta\ln(L_t) = 0$, we have from (33):

$$\begin{gathered} \Delta y_t \gtreqless 0 \text{ where } n_t \gtreqless \frac{1-\beta}{1-\alpha-\beta} > 1; \\ \Rightarrow \text{from (28)}, N_t \lesseqgtr \left(\frac{1-\alpha-\beta}{1-\beta}\right)^m \Gamma y_t^{\frac{\sigma-1}{\sigma}} \end{gathered} \qquad (34)$$

with $\Delta y_t = 0$ being below the $\Delta N_t = 0$ isocline as shown in the Figures; and since $\frac{1-\beta}{1-\alpha-\beta} > 1$, $\Delta y_t > 0$ below the $\Delta y_t = 0$ isocline and < 0 above it, also as shown.

With *population growth*, the $\Delta y_t = 0$ isocline is given by $\sigma(1-\beta)\left(n_t - \frac{1-\beta}{1-\alpha-\beta}\right)\Delta\ln(N_t) - (n_t - 1)\alpha\sigma\Delta\ln(L_t) = 0$, so that:

$$n_t > \frac{1-\beta}{1-\alpha-\beta} \text{ and thus } N_t < \left(\frac{1-\alpha-\beta}{1-\beta}\right)^m \Gamma y_t^{\frac{\sigma-1}{\sigma}} \qquad (35)$$

From (27), the $\Delta N_t = 0$ isocline in (e, N)-space is as already described after (32), but with the added result from (36) below that $\Delta N_t > 0$ above the isocline and < 0 below it, as shown in Fig. 7b. From (26), (29) and (33) the following relationship holds (see Appendix B3):

$$\begin{aligned} &\left(\frac{1+\alpha(\sigma-1)}{1-\beta} + \frac{N_t n_t^m}{1-\alpha-\beta}\right)(n_t - 1)\,\Delta\ln(e_t) \\ &= [\{\sigma - \sigma^\dagger + (\sigma^\dagger - 1)(1 + N_t n_t^m)\}n_t - (\sigma - \sigma^\dagger)]\,\Delta\ln(N_t) \\ &\quad + \left(\frac{1 + N_t n_t^m}{1-\beta}\right)(n_t - 1)\,\Delta\ln(L_t) \end{aligned} \qquad (36)$$

However, unlike finding a $\Delta y_t = 0$ isocline from (33), finding a $\Delta e_t = 0$ isocline from (36) is not straightforward, and will be explored later.

4.5 Malthusian Sluggishness (MS) Region in (y, N)-Space under High Substitutability

We now prove the striking property of the *high* substitutability ($\sigma > \sigma^\dagger$) case shown in Figure 7a, stated as Proposition 1 below: that the (y,N) phase-space is separated into a lower region of MS, and an upper region of IR. An analytic proof exists only given the extra, counterfactual assumption of constant population, but we discuss below the extension by continuity to the historical case of population growth. Even given constant population, the proof is indirect, requiring two prior lemmas.

LEMMA 1. *Given high substitutability and constant population, a development path at any point on any curve in* (y, N)*-space satisfying* $\frac{1-\beta}{1-\alpha-\beta} \leq n(y,N) = \Gamma^{\frac{1}{m}} y^{\frac{\sigma-1}{m\sigma}} N^{\frac{-1}{m}} = \bar{n} \ \leq n_{\infty} \equiv \frac{(\sigma-1)(1-\beta)-1}{(\sigma-1)(1-\alpha-\beta)-1}$ *has a steeper slope than that curve at that point.*

Proof. See Appendix B4.

Lemma 1 shows that at any point in the region of (*y*,*N*)-space bounded by the rising, concave curve $n = \Gamma^{\frac{1}{m}} y^{\frac{\sigma-1}{m\sigma}} N^{\frac{-1}{m}} = \frac{1-\beta}{1-\alpha-\beta}$, shown in Figure 7a as the $\Delta y = 0$ isocline, and a second, rising, concave curve (not shown), $\Gamma^{\frac{1}{m}} y^{\frac{\sigma-1}{m\sigma}} N^{\frac{-1}{m}} = \frac{(\sigma-1)(1-\beta)-1}{(\sigma-1)(1-\alpha-\beta)-1}$, which lies beneath $\Delta y = 0$, the economy's development path has a steeper slope than the curve $\Gamma^{\frac{1}{m}} y^{\frac{\sigma-1}{m\sigma}} N^{\frac{-1}{m}} = \text{constant}$ passing through that point. Hence any path in this region must escape the region upwards across the $\Delta y = 0$ isocline as shown, meaning by (34) and (31) that it is an IR path.

LEMMA 2. *Given high substitutability and constant population:*

(i) the $\Delta n_t = 0$ *locus in* (y, N)*-space is*

$$\Gamma^{\frac{1}{m}} y^{\frac{\sigma-1}{m\sigma}} N^{\frac{-1}{m}} = \frac{\frac{1-\beta}{1-\alpha-\beta}\left(1+\Gamma y^{\frac{\sigma-1}{\sigma}}\right)(\sigma-1)(1-\beta) - \left[1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y^{\frac{\sigma-1}{\sigma}}\right]}{\left(1+\Gamma y^{\frac{\sigma-1}{\sigma}}\right)(\sigma-1)(1-\beta) - \left[1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y^{\frac{\sigma-1}{\sigma}}\right]} \tag{37}$$

(ii) this lies strictly below, and asymptotically (as $y_t \to \infty$*) approaches, the locus defined by*

$$\Gamma^{\frac{1}{m}} y^{\frac{\sigma-1}{m\sigma}} N^{\frac{-1}{m}} = \frac{(\sigma-1)(1-\beta)-1}{(\sigma-1)(1-\alpha-\beta)-1} \equiv n_{\infty} \tag{38}$$

(iii) at any point on the $\Delta n_t = 0$ *locus (37),* $\frac{\sigma-1}{\sigma}$*, the slope of the economy's path through that point, is shallower than the locus slope there.*

Proof. See Appendix B4.

Lemma 2 shows that the locus of all points on different development paths locally satisfying $\Delta n_t = 0$ forms a third, rising, concave, even lower curve (also not shown in Figure 7a), and further that the economy's development path at each point on this curve has a shallower slope than this third curve. Hence development can never cross this third curve upwards, so that any path beneath it is trapped there forever in a region of $\Delta y > 0$ and $\Delta N > 0$, that is, it is an MS path. Proposition 1 then follows from these two lemmas:

PROPOSITION 1. *Given high substitutability and constant population, there is a monotone increasing separatrix in* (y, N)*-space lying strictly below the* $\Delta y_t = 0$ *isocline,* $\Gamma^{\frac{1}{m}} y^{\frac{\sigma-1}{m\sigma}} N^{\frac{-1}{m}} = \frac{1-\beta}{1-\alpha-\beta}$, *with all paths below it being MS (*$\Delta N_t > 0$ *and* $\Delta y_t > 0$ *forever, staying below the separatrix), and all paths above this separatrix being IR (initially with* $\Delta N_t > 0$ *and* $\Delta y_t > 0$*, but then crossing the* $\Delta y_t = 0$ *isocline and thereafter the* $\Delta N_t = 0$ *isocline, thus with* $\Delta N_t < 0$ *and* $\Delta y_t < 0$ *forever after some time), as in Figure 7a.*

Proof. By Lemma 1, any path through a point with $\frac{1-\beta}{1-\alpha-\beta} < \Gamma^{\frac{1}{m}} y^{\frac{\sigma-1}{m\sigma}} N^{\frac{-1}{m}} = \text{constant} \leq n_\infty$ is an IR path, because it must eventually cross the $\Delta y_t = 0$ isocline upwards, and hence by the path directions in (31) and (34) also eventually cross the $\Delta N_t = 0$ isocline leftwards. By Lemma 2, any path in the region below the $\Delta n_t = 0$ locus (37), which lies strictly below the curve $\Gamma^{\frac{1}{m}} y^{\frac{\sigma-1}{m\sigma}} N^{\frac{-1}{m}} = n_\infty$, is trapped there; and by (31) and (34), $\Delta N_t > 0$ and $\Delta y_t > 0$ there, so the region is one of MS. By continuity, there must thus be an IR/MS separatrix between the $\Gamma^{\frac{1}{m}} y^{\frac{\sigma-1}{m\sigma}} N^{\frac{-1}{m}} = n_\infty$ and $\Delta n_t = 0$ loci, hence beneath the $\Delta y_t = 0$ isocline, as shown in Figure 7a. ■

With *population growth*, $\Delta \ln(L_t) > 0$ in (33) means that no simple analytic comparisons of $\Delta \ln(N_t)/\Delta \ln(y_t)$ values on the development path and the log-slopes of the $\Gamma^{\frac{1}{m}} y^{\frac{\sigma-1}{m\sigma}} N^{\frac{-1}{m}} = n_\infty$ or $\Delta n_t = 0$ loci are possible. But by continuity, Proposition 1 holds for some level of population growth, and our numerical simulations found that it does hold for historical British population growth, and for a wide range of variants on our baseline simulation.

Note from (37) and (38) that as $\alpha \to 0$, $n \to 1$ on both loci, so the MS region then occupies the entire $n_t > 1$ region. Conversely, the more important energy is (i.e. the higher α is), the smaller is the MS region, and hence the more likely that an economy lies in the IR region. In (e, N)-space under *high* substitutability, an IR/MS separatrix must also exist to separate IR paths with $N \to 0$ from MS paths with $N \to \infty$, as shown in Figure 7b. There is also another feature peculiar to (e, N)-space, as follows:

4.6 Modern Economic Growth Region in (e, N)-Space under High Substitutability

PROPOSITION 2. *Given high substitutability and constant population, an upward-sloping isocline* $\Delta e_t = 0$ *occurs in the region below the* $\Delta N_t = 0$ *isocline in* (e, N)*-space, with* $\Delta e_t > 0$ *above the former isocline and* $\Delta e_t < 0$ *(modern economic growth) below it, as in Figure 7b.*

Proof. From rearranging (36) with $\Delta\ln(L_t) = 0$, $\Delta e_t = 0$ when:

$$n_t = \frac{\sigma - \sigma^\dagger}{\sigma - \sigma^\dagger + (\sigma^\dagger - 1)(1 + N_t n_t^m)} < 1 \text{ when } \sigma > \sigma^\dagger \tag{39}$$

and with $\sigma > \sigma^\dagger$, this does have a solution with $0 < n_t < 1$ for any permitted parameter values. If we substitute $n_t = \Gamma^{\frac{\sigma}{m}} N_t^{\frac{(\sigma-\tilde{\sigma})(1-\beta)}{m}} e_t^{\frac{-\alpha(\sigma-1)}{m}}$ and differentiate implicitly (see Appendix B5 for details), we can show the isocline is upward sloping ($\frac{\Delta N_t}{\Delta e_t} > 0$). The result that $\Delta e_t > 0$ above the isocline, and $<\ 0$ below it then follows from the signs in (36). ■

With our baseline scenario's (in Section 5) realistic *growing population*, $\Delta\ln(L_t) > 0$, and for a wide range of variants on our baseline, we find empirically that Proposition 2 still holds. From (36), $\Delta\ln(L_t) > 0$ requires a lower N_t value to attain $\Delta e_t = 0$, i.e. population growth shifts the isocline down. This suggests, but does not prove, that population growth makes a significant later feature of industrial development – the peaking of the energy price ratio – happen sooner.

<u>4.7 Results for Medium/Low Substitutability</u>

Propositions 3 and 4 now explain the paths respectively shown in Figures 8a and 8b for *medium* substitutability, and the proofs here also apply to paths under *low* substitutability. Neither Proposition needs to assume constant population, so their proofs are not simple converses of the proofs of the corresponding Propositions 1 and 2 under *high* substitutability.

PROPOSITION 3. *Given medium or low substitutability, all development paths undergo an industrial revolution.*

Proof. Appendix B6 shows why, given $\sigma < \sigma^\dagger$, all paths under the $\Delta y_t = 0$ isocline in (*y*,*N*)-space eventually rise to cross that locus upwards; and from (34) and (31), all paths above that isocline eventually cross the $\Delta N_t = 0$ isocline leftwards into the region where $\Delta N_t < 0, \Delta y_t < 0$ forever, as in Fig 8a. ■

PROPOSITION 4. *Given medium or low substitutability,* $\Delta e_t > 0$ *everywhere, as in Figure 8b, i.e. no modern economic growth phase exists.*

Proof. Rearranging (36) gives

$$\left(\frac{1 + \alpha(\sigma - 1)}{1 - \beta} + \frac{N_t n_t^m}{1 - \alpha - \beta}\right) \Delta\ln(e_t) = \tag{40}$$

$$\frac{[\{\sigma - 1 + (\sigma^\dagger - 1)N_t n_t^m\}n_t + \sigma^\dagger - \sigma\,]\Delta\ln(N_t)}{n_t - 1} + \left(\frac{1 + N_t n_t^m}{1 - \beta}\right) \Delta\ln(L_t)$$

With $\sigma < \sigma^{\dagger}$, $\{\sigma - 1 + (\sigma^{\dagger} - 1)N_t n_t^m\}n_t + \sigma^{\dagger} - \sigma > 0$; from (27), $\frac{\Delta\ln(N_t)}{n_{t-1}} > 0$; and both $\left(\frac{1+\alpha(\sigma-1)}{1-\beta} + \frac{N_t n_t^m}{1-\alpha-\beta}\right) > 0$ and $\left(\frac{1+N_t n_t^m}{1-\beta}\right)\Delta\ln(L_t) > 0$ everywhere; so $\Delta e_t > 0$ everywhere. ■

4.8 Faster IR Growth than MS Growth under High Substitutability

We now investigate growth rates under the empirically relevant case of *high* substitutability. All results here assume population growth modest enough for Proposition 1 (the existence of separate IR and MS regions of the phase diagrams given *high* substitutability, whose formal proof assumes constant population) to still hold by continuity for a growing population (which is what we have found in all our empirical simulations). Routine algebra (see Appendix B7) transforms the Malthus and Solow versions of (20) for the growth rates of machine varieties respectively into:

$$n_{M,t} = \lambda N_{M,t-1}^{\frac{\mu+\nu-1}{h\nu}} \bar{E}_M^{\frac{\alpha}{h(1-\beta)}} \left(y_t^{-\frac{\sigma-1}{\sigma}} + \Gamma\right)^{-\left[\frac{(1-\alpha-\beta)(\sigma-\sigma^{\dagger})}{h(\sigma-1)(1-\beta)}\right]} L_t^{\frac{1-\alpha-\beta}{h(1-\beta)}} \tag{41}$$

$$n_{S,t} = \lambda N_{S,t-1}^{\frac{\mu+\nu-1}{h\nu}} N_{S,t}^{\frac{\alpha}{h(1-\alpha-\beta)}} \left(1 + \Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{-\left[\frac{\sigma-\sigma^{\dagger}}{h(\sigma-1)}\right]} L_t^{\frac{1}{h}} \tag{42}$$

Here and everywhere later λ is an arbitrary positive constant, which may differ from equation to equation. For example, its definition differs between (41) and (42), but to show this detail would just add clutter as we are interested only in the growth rates of $n_{M,t}$ and $n_{S,t}$.

For Proposition 5 only, we use the general formula (17) for innovation. The $N_{i,t-1}^{\frac{\mu+\nu-1}{h\nu}}$ terms in (41) and (42) then remain present, showing the effect on growth rates in different sectors of different assumptions about $\mu + \nu$. To prove Proposition 6, though, we need to revert to our standard assumption of CRS innovation ($\mu + \nu = 1$, so the $N_{i,t-1}^{\frac{\mu+\nu-1}{h\nu}}$ terms disappear and *h* becomes *m*).

PROPOSITION 5. *Given high substitutability:*
(i) the growth rate of the number of Malthus varieties rises on an MS path if $\nu \geq 1 - \mu$*; (ii) the growth rate of the number of Solow varieties rises along an IR path once* $\Delta y_t < 0$ *forever if* $\nu \geq \frac{1-\alpha-\beta}{1-\beta}(1-\mu)$*; (iii) if either of these conditions on μ and ν do not hold, then in either case the varieties growth rate could either rise or fall over time.*
Proof. See Appendix B7.

PROPOSITION 6. *Given high substitutability and* $\nu = 1 - \mu$*, economic growth (i.e. the growth of* Y_t/L_t*, final output per capita) under an IR eventually accelerates, and eventually is faster than under MS.*

Proof. See Appendix B7.

Note the asymmetries above between the Malthus and Solow sectors: the constant $\bar{E}_M^{\frac{\alpha}{h(1-\beta)}}$ in (41) compared to the rising $N_{S,t}^{\frac{\alpha}{h(1-\alpha-\beta)}}$ in (42), hence both the tighter MS-path restriction in Proposition 5 needed to cause a rising growth rate of varieties, and the eventually higher IR economic growth rate in Proposition 6. These asymmetries all stem from the expandable coal supply, $E_{S,t}$, present in the Solow-sector version of (20), compared to the non-expandable wood supply, $\bar{E}_M$, in the Malthus-sector version.

4.9 Comparative Static Effects of Parameters on Goods Ratio y_t and Price Ratio e_t

Equations (21)-(23) are simultaneous, and computing the general equilibrium comparative statics via the multivariate implicit function theorem is intractable. Nevertheless, we can find local comparative static effects of several parameters on the Malthus/Solow goods ratio y_t and price ratio e_t, taking $N_{M,t}$ and $N_{S,t}$ as momentarily fixed and hence denoted as $\bar{N}_{M,t}$ and $\bar{N}_{S,t}$. None of the *high* substitutability, constant population, or CRS innovation assumptions is needed for the following results to hold.

PROPOSITION 7. *Any of a lower number of wood-using varieties,* $\bar{N}_{M,t}$*, a lower wood quantity,* $\bar{E}_M$*, a lower coal price* $\bar{e}_S$*, higher numbers of coal-using varieties,* $\bar{N}_{S,t}$ *or higher population,* L_t*, move the economy towards locally lower* y_t *(i.e. higher industrialization):*

$$\partial y_t/\partial \bar{N}_{M,t}\,, \partial y_t/\partial \bar{E}_M\,, \partial y_t/\partial \bar{e}_S > 0;\ \partial y_t/\partial \bar{N}_{S,t}\,, \partial y_t/\partial L_t < 0 \tag{43}$$

Additionally, an equi-proportional increase in $\bar{N}_{M,t}$ *and* $\bar{N}_{S,t}$*. i.e.* $\Delta ln(\bar{N}_{M,t}) = \Delta ln(\bar{N}_{S,t}) > 0$*, hence* $\Delta ln(\bar{N}_t) = 0$*, results in lower* y_t*.*

Proof. See Appendix B8, which starts by expressing $y_t = Y_{M,t}(y_t, N_{M,t})/Y_{S,t}(y_t, N_{S,t})$ from (21) as $f(y_t, \bar{N}_{M,t}, \bar{N}_{S,t}, \mathbf{\Omega}) = 0$, where $\mathbf{\Omega} \equiv [\bar{E}_M, \bar{e}_S, \alpha, \beta, \gamma, \sigma, \mu, \eta, L_t]$ are selected exogenous parameters.

PROPOSITION 8. *Either of a lower wood quantity or higher current population raises the wood/coal price ratio (which generally increases industrialization, except in the modern economic growth phase shown in Figure 7b):*

$$\partial e_t/\partial \bar{E}_M < 0\,;\ \partial e_t/\partial L_t > 0 \tag{44}$$

Additionally, an equi-proportional increase in $\overline{N}_{M,t}$ *and* $\overline{N}_{S,t}$ *results in higher* e_t.

Proof. See Appendix B8, which starts by expressing (21) as $f\left(e_t, \overline{N}_{M,t}, \overline{N}_{S,t}, \mathbf{\Omega}\right) = 0$.

All these comparative-static effects accord with intuition; but quantifying their total effect on development over relevant time periods requires numerical simulations, to which we now turn.

5 Simulations

In this section we first show how a Baseline simulation of our model, fitted to the stylized facts of the British Industrial Revolution using reasonable parameter values, demonstrates various analytical results from Section 4. We then show counterfactual simulations which illustrate our analytic comparative static Proposition 7, namely that the Industrial Revolution would have been delayed by either a higher elasticity of substitution, more abundant wood, a higher coal price, less initial Solow knowledge, or less population growth. We use a Matlab program to find numerical solutions for (21)-(23), period by period, for $p_t, N_{M,t}$, and $N_{S,t}$.

5.1 Population Calibration

For our historical baseline and counterfactual simulation scenarios we provide the exogenous population input parametrically. We refitted Marchetti *et al.*'s (1996) bilogistic function model using Broadberry *et al.*'s (2015) data for the population of the United Kingdom at 20-year intervals, resulting in the following fit:

$$S_\tau = \frac{9.7}{1 + exp\left(-\frac{\ln(81)}{267}(\tau - 1530)\right)} + \frac{47.4}{1 + exp\left(-\frac{\ln(81)}{171}(\tau - 1870)\right)} \tag{45}$$

where τ is the calendar year and population $S_\square$ is measured in millions. Then we assume that the total (normalized) labor force is given by $L_t = S_t/S_1$, where time t counts 20-year periods from $t = 1$ in 1560, the first year of Warde's (2007) energy data, to $t = 18$ in 1900, so that $t = (\tau - 1540)/20$. Figure 6 shows the original data and the fitted curve.

5.2 Baseline Simulation

Following Kander and Stern (2014), our Baseline scenario uses an elasticity of substitution of $\sigma = 4.4$ in the production of the final output.[17] We take the cost share of

[17] Kander and Stern (2014) estimate that the elasticity of substitution between traditional (mainly wood) and modern (mainly coal) energy carriers was 4.4 in Sweden from 1850 to 1950, but with a wide confidence interval. We adopt their estimate for the elasticity of substitution between the intermediate goods even though this elasticity should be larger than that between the energy carriers.

energy in 1800 in Britain to be around 25% not including human and animal power (Gentvilaite *et al.*, 2015), so we set the energy output elasticity to $\alpha = 0.25$. We normalize the quantity of wood, $\bar{E}_M$, to 1. We set the output elasticity of machines to $\beta = 0.225$ based on Table 13 in Clark (2010), so that our Baseline elasticity is *high* ($\sigma = 4.4 > \sigma^{\dagger} = 2.90$ from (26)). We set the innovation exponent to $\mu = 0.5$ arbitrarily as we have no evidence on this. We normalize the stock of machine varieties in the Solow sector in 1540 (t = 0) to $N_{S,0} = 1$. The remaining parameters are $N_{M,0}$, η, $\bar{e}_S$ and γ. We optimize these by minimizing the sum of squared proportional deviations from six Stylized Facts, three based on the initial state in Britain in 1560 and three based on the change in the variables over its industrial revolution. Using calendar year time subscripts, the chosen stylized facts and proportional deviations are respectively:

1. In 1560 90% of the workforce was in the Malthus sector: $\ln\left(\frac{L_{M,1560}}{L_{1560}}\right) - ln(0.9)$.[18]
2. The price of wood is double the price of coal in 1560 (Allen, 2009): $\ln\left(\frac{e_{M,1560}}{e_{S,1560}}\right) - \ln(2)$.
3. In 1560, coal use is 30% of wood use (Warde, 2007): $\ln\left(\frac{E_{S,1560}}{\bar{E}_M}\right) - \ln(0.3)$.
4. Output per capita rises 5.4-fold from 1560 to 1900 (Broadberry *et al.*, 2015): $\ln\left(\frac{(Y/L)_{1900}}{(Y/L)_{1560}}\right) - \ln(5.4)$.
5. The price of wood doubles from 1560 to its peak (Allen, 2009): $\ln\left(\frac{max(e_{M,t})}{e_{M,1560}}\right) - \ln(2)$.
6. Energy intensity doubles from its minimum to 1900 (Warde, 2007; Broadberry *et al.*, 2015): $\ln\left(\frac{(E/Y)_{1900}}{(E/Y)_{1560}}\right) - \ln(2)$.[19]

Our full set of Baseline parameters, whether selected from the literature or optimized as just described, is shown in Table 1.

Figure 10a graphs our Baseline simulation results over time for two ratios, the Malthus sector's share of labor input ($L_{M,t}/L_t$) and the wood/coal relative price (e_t), and two absolute quantities, coal use ($E_{S,t}$) and output per capita (Y_t/L_t). Coal use and output per capita are

[18] This is a somewhat arbitrary assumption, as available data on the agricultural share of the workforce or the share of urban population are not relevant to our "wood-using" definition of the Malthus sector.

[19] This reflects the increase in total energy intensity in Figure 5. We tried using a ratio of 4 instead, to reflect the increase in firewood and coal energy intensity, but this gave a much poorer fit to the other stylized facts.

normalized to 1 in 1560, and coal use is also converted to logarithms because its overall growth is so huge. Results are broadly comparable to the historical data shown in Figures 1 to 5. The peak in the simulated wood/coal price – an illustration of Proposition 2 extended to a growing population – comes somewhat later than it does historically in Figure 3. Growth of output per capita accelerates, fulfilling the potential allowed by Proposition 6, but more slowly after 1650 in the simulation than GDP per capita does in Figure 1. Hence our simulation's 19th-Century growth rate is higher than it was historically – reaching 3.3% p.a. in 1880-1900 – in order to reduce the deviation from Stylized Fact 4 above. The share of labor in the Malthus sector falls from 85% in 1560 to 50% just after 1800 and 4% in 1900. Simulated coal use increases more than 250-fold by 1900 and its growth rate accelerates, though this is slower than in reality, consistent with the lower than historical increase in energy intensity in the simulation.

Another noteworthy result not shown in Figure 10a is that in our Baseline simulation, $n_{1560} \equiv \frac{\Delta N_{M,1560}/N_{M,1560}}{\Delta N_{S,1560}/N_{S,1560}} = 0.54$. So the economy was already industrializing ($n < 1$) in 1560, even though $\frac{\Delta N_{M,1560}}{\Delta N_{S,1560}} = 5.9$, meaning there was a larger absolute increase of Malthus-sector machine varieties then, which remained true up till 1820. Thus according to our model, Britain in 1560 was already on an inevitable path to an industrial revolution, though how long it might take to get there would depend greatly on long-term population growth, as our counterfactual simulations will show.

5.3 Counterfactual Simulations

In Figures 10b-f we simulate the following five counterfactual scenarios to highlight the potential effects on economic growth of changing energy resource abundance and scarcity, or other key parameters:

b. Abundant Wood: Wood quantity is 10 times higher than in the Baseline scenario ($\bar{E}_M = 10$ instead of 1).
c. Expensive Coal: The coal price is 4 times higher ($\bar{e}_S = 3$ instead of 0.75).
d. More Substitutability: The elasticity of inter-sectoral substitution σ is 10 instead of 4.4.
e. Low Solow Knowledge: The initial stock of Solow sector varieties is halved ($N_{S,0} = 0.5$ instead of 1).
f. Constant Population ($L_t = 1$ always).

With the exception of Scenario f, we assume that population followed its historical path. In all five cases, as predicted by our comparative static Proposition 7, an industrial revolution (falling y_t) is much delayed: GDP per capita grows slowly or declines, less labor shifts to the Solow sector, and coal use and growth is lower. However, none of the cases shown, result in actual Malthusian sluggishness (forever rising y_t, which Proposition 1 shows is possible since $\sigma > \sigma^{\dagger}$), though combinations of them (e.g. Abundant Wood and Constant Population together) do in fact result in MS.

In the Abundant Wood scenario (Figure 10b) output per capita is twice the Baseline level in 1560 but the relative price of wood and use of coal are both lower. Output per capita fluctuates and sustained growth only starts from 1880, while the price of wood rises throughout the period. The price of wood rises fivefold from 1560 to 1900 but still does not reach the peak level seen in the Baseline scenario. Energy intensity (not shown) declines till 1880 and the share of labor in the Malthus sector starts higher (96%) and falls much less, being still 68% in 1900. This scenario clearly illustrates the paradox where an abundance of wood stalls development despite much higher initial output per capita.

The Expensive Coal scenario (Figure 10c) looks similar to the Abundant Wood scenario, except that the initial income level is a little below the Baseline scenario and coal use is even lower. Not shown is that the price of wood relative to output here is about the same as in the Baseline scenario, so that here both fuels are relatively expensive, whereas in the Abundant Wood scenario, wood is much cheaper relative to output than in the Baseline scenario.

An alternative way of modeling abundant fuels is to let wood and coal be more fungible with each other, which we simulate in the More Substitutability scenario (Figure 10d) by assuming a much higher elasticity of intermediate goods substitution than our already *high* Baseline value. Here the Industrial Revolution is postponed by centuries, as output per capita declines throughout the period and less than 2% of the workforce transfers to the Solow sector by 1900. Interestingly, the price of wood starts higher than in the Baseline scenario and rises more, but because of the increased substitutability this rise is much slower to shift innovation to the Solow Sector. Assuming that the economy had low Solow sector knowledge in the Solow sector in 1560 produces similar results (Figure 10e) but with higher coal use.

Our final counterfactual simulation, with Constant Population (Figure 10f) is very different to the other five scenarios. Here there is sustained but very slow growth in GDP per capita: 0.05% p.a. in 1560-80, rising to only 0.08% p.a. in 1900. The price of wood rises only

9% and the use of coal slightly more than doubles in 340 years, and the Malthus sector's labor share falls only slightly, from 85% in 1560 to 74% in 1900. Running the simulation into the future, Solow sector output exceeds Malthus sector output around 2400, so there is eventually a transition but it is extremely delayed. These results are consistent with the comparative-static effects of population growth in Propositions 7 and 8. An in-between scenario (not shown in Figure 10) with half the Baseline population growth, so that population at time τ is $L_{1560} + 0.5(L_\tau - L_{1560})$, results in far more than half the Baseline scenario's overall change in the goods output ratio and wood/coal price. This suggests how important population growth was for raising the relative wood/coal price as a key mechanism that drove the British Industrial Revolution.

6 Alternative Histories and a Transition Back to Renewables

In the previous section we examined some counterfactual scenarios. A broader question, often raised by our seminar audiences, is whether there would have been an industrial revolution under other historical circumstances. For example, imagine if political and institutional conditions worsened in Britain after the American colonies were founded but before the Industrial Revolution, so that the British environment was no longer supportive of innovation. Would there have been an industrial revolution in America instead? Given our results, we think that unlikely because of America's relative abundance of wood versus coal. While lumber's relative price to other goods in the U.S. rose throughout the 19th Century and the first half of the 20th Century (Cleveland and Stern, 1993), this was presumably at least in part due to technology imported from Britain that first used wood as fuel before turning to coal. Without industrialization would there have been as much emigration to America either? By contrast with the US and its abundant sources of traditional and modern energy, some countries with few modern energy resources, such as Denmark or Japan, industrialized by importing coal, following the breakthroughs made in Britain. Based on the Expensive Coal scenario, we suggest that the Industrial Revolution would have been unlikely to start in such countries, with no lead from Britain or a similarly endowed country.

What do our results imply for the current and potential energy transition from fossil fuels to modern renewable energy? If a cap is placed on annual carbon emissions, then in the absence of sequestration, fossil fuels will be available in a fixed quantity, similar to wood in our model. By contrast, the supply of modern renewables could be effectively infinitely elastic, since total annual energy use today is similar in magnitude to the solar radiation falling on the Earth in one hour. Presumably there is also a high elasticity of substitution between products produced with fossil and renewable energy. So our model could be adapted for future scenarios by switching the quantity and price constraints on our two energy sources, and a cap on carbon emissions could drive innovation to the non-fossil energy-using sector, as in Acemoglu *et al*. (2012).

7 Conclusions

We have shown here the potential importance of the differential abundance of energy resources − wood and coal − in driving a transition from pre-industrial to modern economic growth, using a model that both yields theoretical insights and reproduces key empirical features of the British Industrial Revolution. We extended and calibrated an increasing machine varieties, directed technical change model which, unlike previous related research (Hansen and Prescott, 2002; Fröling 2011; Eren and Garcia-Macia, 2013), does not assume productivity or productivity growth to be inherently higher in the modern, industrial, coal-using, "Solow" sector than in the traditional, “wood”-using "Malthus" sector. Rather, we assume resource supply conditions differ inherently, so that wood is inelastically and coal elastically supplied, which is a stylized representation of the British historical record.

Analytically, our model shows that an industrial revolution, where goods output becomes ever more concentrated in the coal-using sector, is possible, but not inevitable if substitutability between the intermediate (wood- and coal-using) goods is high enough. Given high enough substitutability, we showed that Malthusian sluggishness — slow growth with goods output ever more concentrated in the Malthus, wood-using sector — is possible, depending on the economy's starting point. We also showed that when there is an industrial revolution economic growth eventually accelerates, and is eventually at a higher rate than under Malthusian sluggishness, though this result depends on our assumption of constant returns to scale in innovation. Lastly, comparative static analysis showed the effect of key parameters on the economy’s state of development: notably, at any time, any of a lower coal price, lower wood quantity, or higher population will further industrialize the economy.

Given some parameter values from the literature, fitting our model to some basic stylized historical facts results in a baseline simulation with sensible values for the free parameters, and a development path that reproduces the key features of the British Industrial Revolution. From the start, the growth rate of coal-using machine varieties exceeds that of wood-using varieties, though its absolute growth is less until 1820. The only exogenous driver in our model is the historical rate of population growth. This should be endogenized in future research, but leaving it exogenous here better highlights the role of natural resource scarcity in driving growth.

Compared to the previous literature (see Ashraf and Galor, 2011), our model introduces a new reason for why an economy may remain forever in Malthusian sluggishness, or fail to make a timely industrial transition, since either may be caused by abundant wood, high elasticities of substitution and/or slow population growth. Our model's counterfactual simulations show that a much higher fixed quantity of wood input or fixed price of coal, and or slower population growth would have greatly delayed growth of GDP per capita and the rate of innovation. In our model, it is the growing relative scarcity of wood caused by population growth that results in innovation to develop coal-using machines. Necessity is thus indeed the mother of invention: on its own, the unlimited supply of coal does not trigger a transition if wood is not relatively scarce.

Our model thus partly supports views by Allen (2009) and Wrigley (2010) that the Industrial Revolution first happened in Britain mainly because of its cheap, abundant coal. Counter to Clark and Jacks (2007), Madsen *et al*. (2010), and Harley and Crafts (2000), our model tells a plausible story of how coal could have played a central role in the Industrial Revolution.

However, we stress that our support is partial, because our model does not imply that cheap coal alone would have been sufficient for the Industrial Revolution to happen in Britain in the 18th Century. Good institutions, human capital, and endogenous population growth have all been suggested as key factors (Clark, 2014), and our results should not be seen as disagreeing with this view. Good institutions − for example, a patenting system to protect innovators' property rights, which Madsen *et al.* (2010) stress was developed much earlier in Britain than anywhere else, and scientific progress, likewise stressed by Mokyr (2009b) − are invisibly assumed in the mathematical structure of most economic growth models, including ours, so we implicitly treat them as also being necessary for growth. If economic analysis can be developed to take the major conventional factors *and* renewable energy scarcity *and* fossil fuel availability all into account, then the Industrial Revolution may not "remain[s] one of history's mysteries" (Clark, 2014, 260) for much longer.

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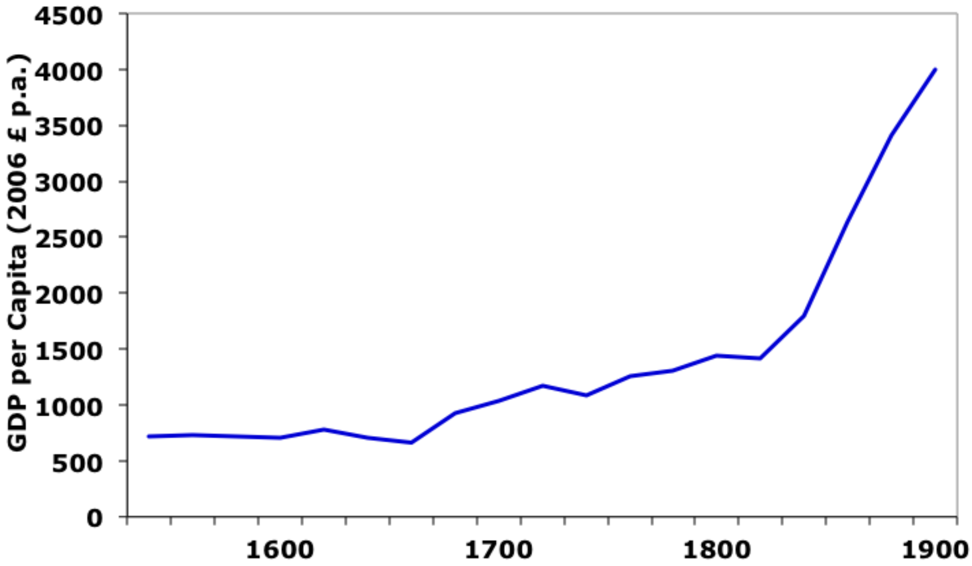
Table 1: Baseline Parameters:

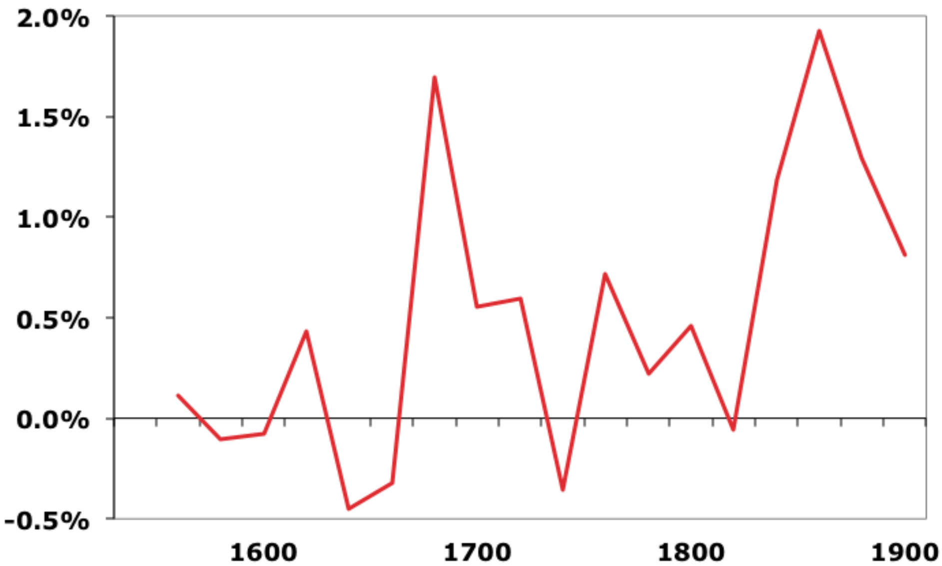
Parameter	Symbol	Value	Sources
CES elasticity in final production	σ	4.4	Kander and Stern (2014)
Distribution parameter in CES final production	γ	0.29	Optimized
Energy output elasticity	α	0.25	Energy cost share in 1800 in the UK was about 25% not counting animal and human power (Gentvilaite et al., 2015).
Capital (machine) output elasticity	β	0.225	This is based on a share of capital that fluctuates between about 0.2 and 0.25 in Clark (2010).
Productivity innovation in M sector	η_M	0.44	Optimized
Productivity innovation in S sector	η_S	0.44	Optimized
Parameter in innovation production	μ	0.5	Arbitrary
Initial idea stock in M sector	$N_{M,0}$	11	Optimized
Initial idea stock in S sector	$N_{S,0}$	1	Normalized
Constant price of coal	$\bar{e}_S$	0.75	Optimized
Constant consumption of wood	$\bar{E}_M$	1	Normalized

Figure 1. GDP per Capita



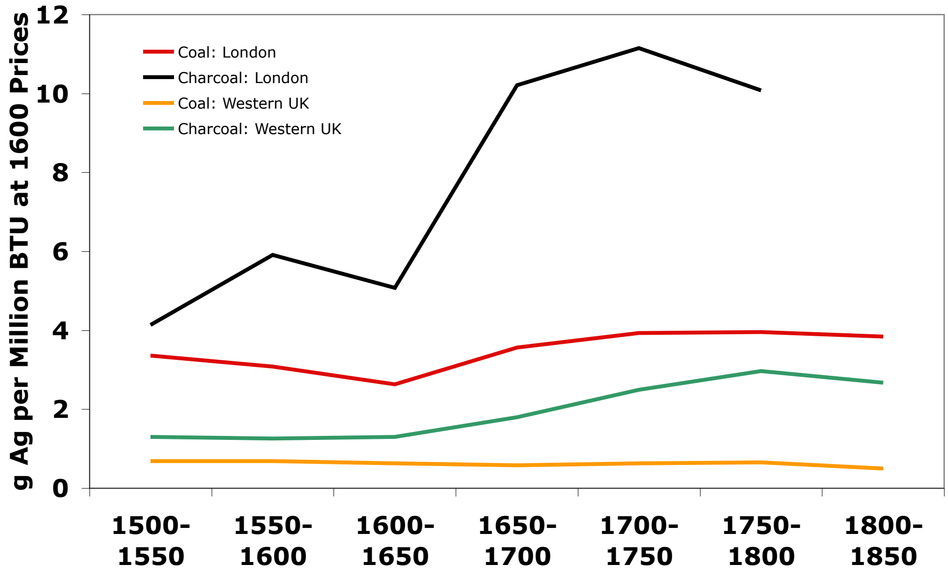
Source: Broadberry *et al*. (2015), Hills *et al*. (2010).

Figure 2. Real GDP per Capita Annual Growth Rate



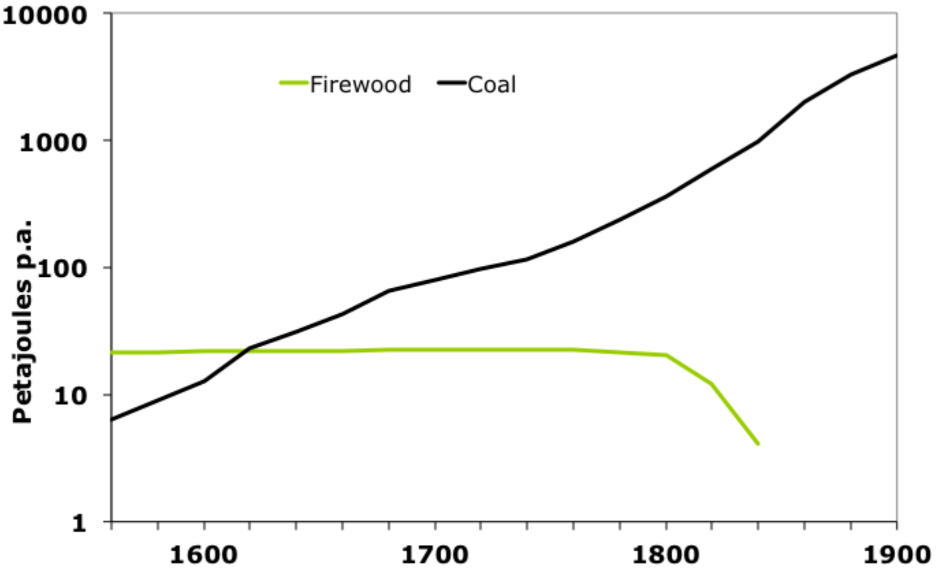
Source: Broadberry *et al*. (2015), Hills *et al*. (2010).

Figure 3. Real Prices of Coal and Charcoal in London and the Western UK



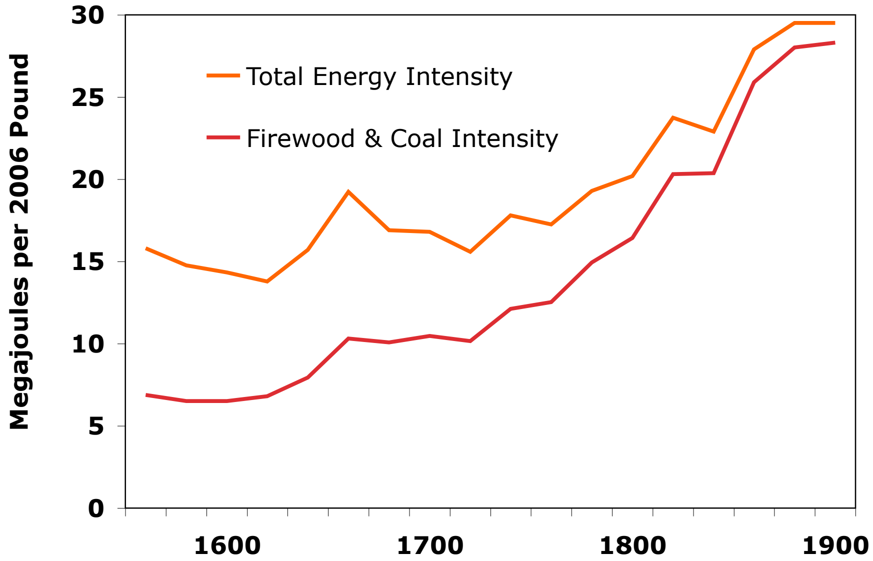
Source: Allen (2009), Table 4.3. Units are grams of silver per million BTU at constant prices of 1550.

Figure 4. Quantities of Firewood and Coal



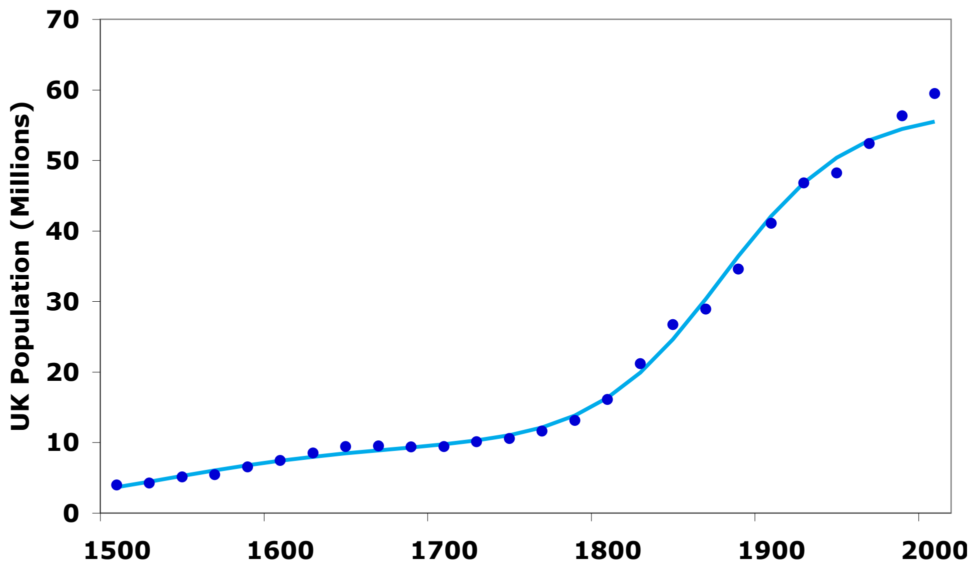
Source: Warde (2007).

Figure 5. Energy Intensity



Sources: Authors' calculations from data in Warde (2007), Broadberry *et al*. (2015), and Hills *et al*. (2010).

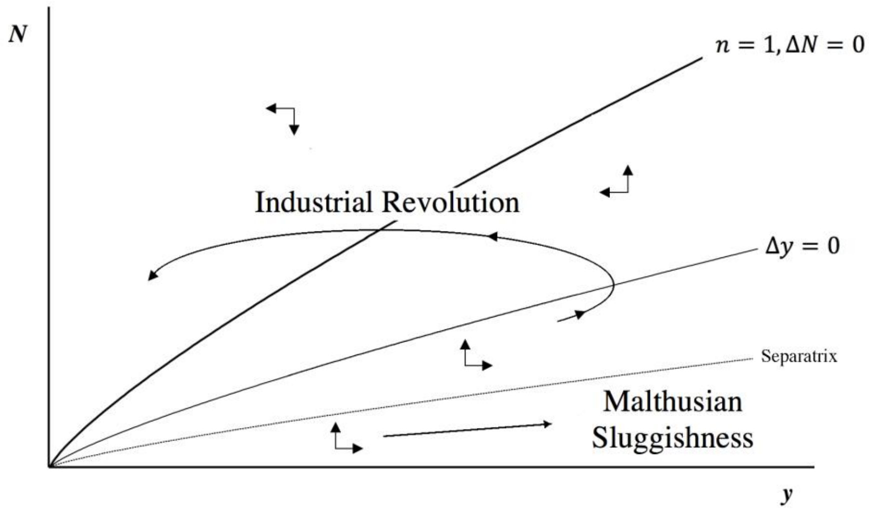
Figure 6. United Kingdom Population



Sources: Broadberry *et al*. (2015), authors' estimates.

Figure 7. Phase Diagrams for High Elasticity of Substitution

a. Malthus/Solow machine varieties ratio, *N*, and goods ratio, *y*.



b. Malthus/Solow machine varieties ratio, *N*, and energy price ratio, *e*.

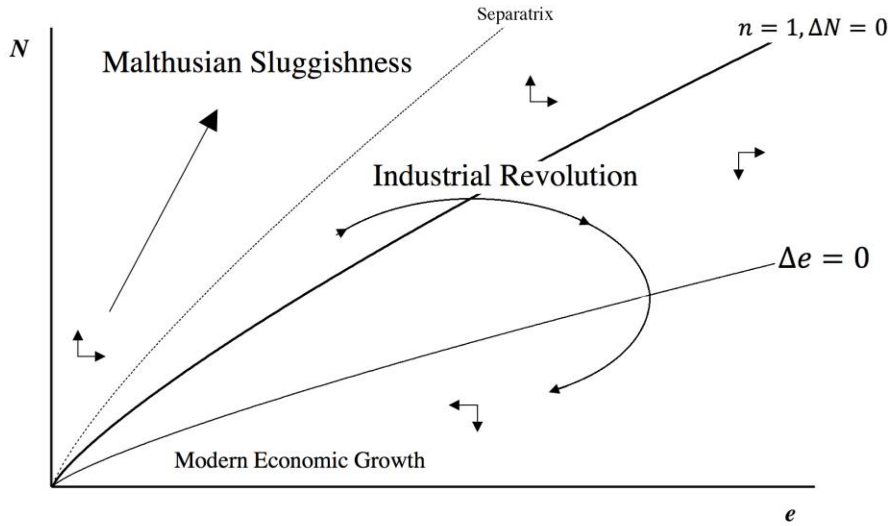
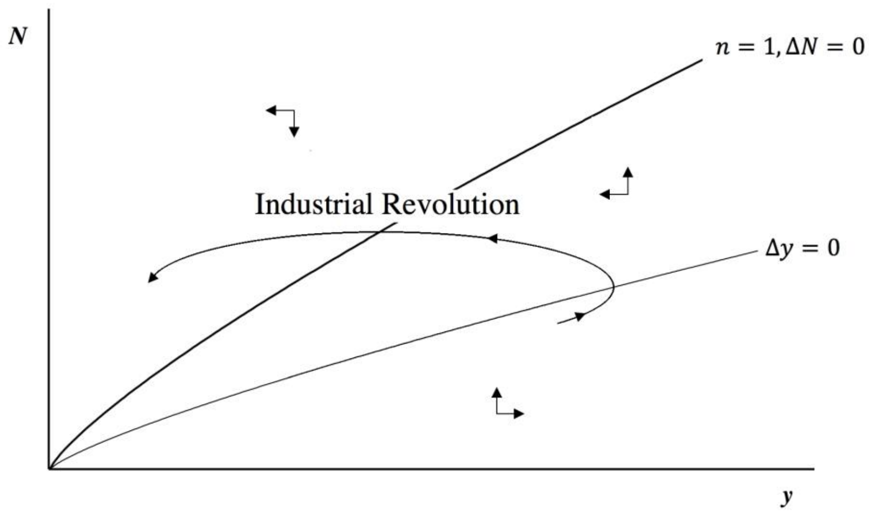


Figure 8. Phase Diagrams for Medium Elasticity of Substitution

a. Malthus/Solow machine varieties ratio, *N*, and goods ratio, *y*.



b. Malthus/Solow machine varieties ratio, *N*, and energy price ratio, *e*.

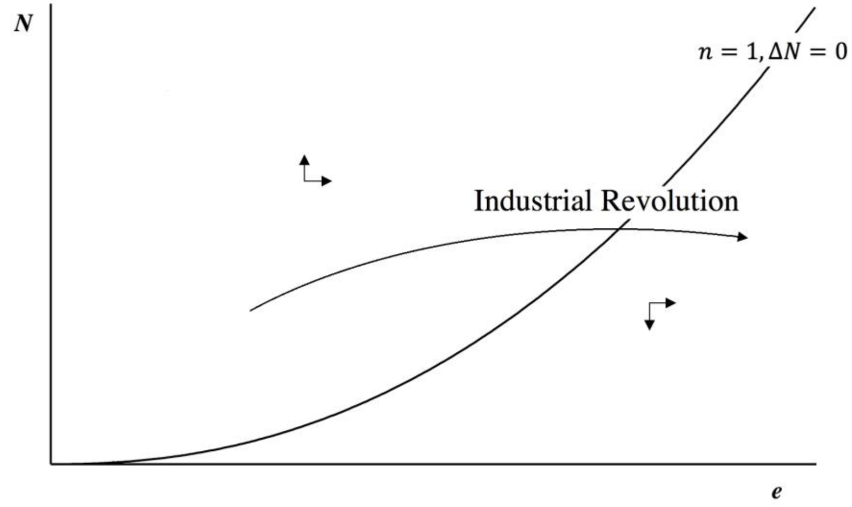
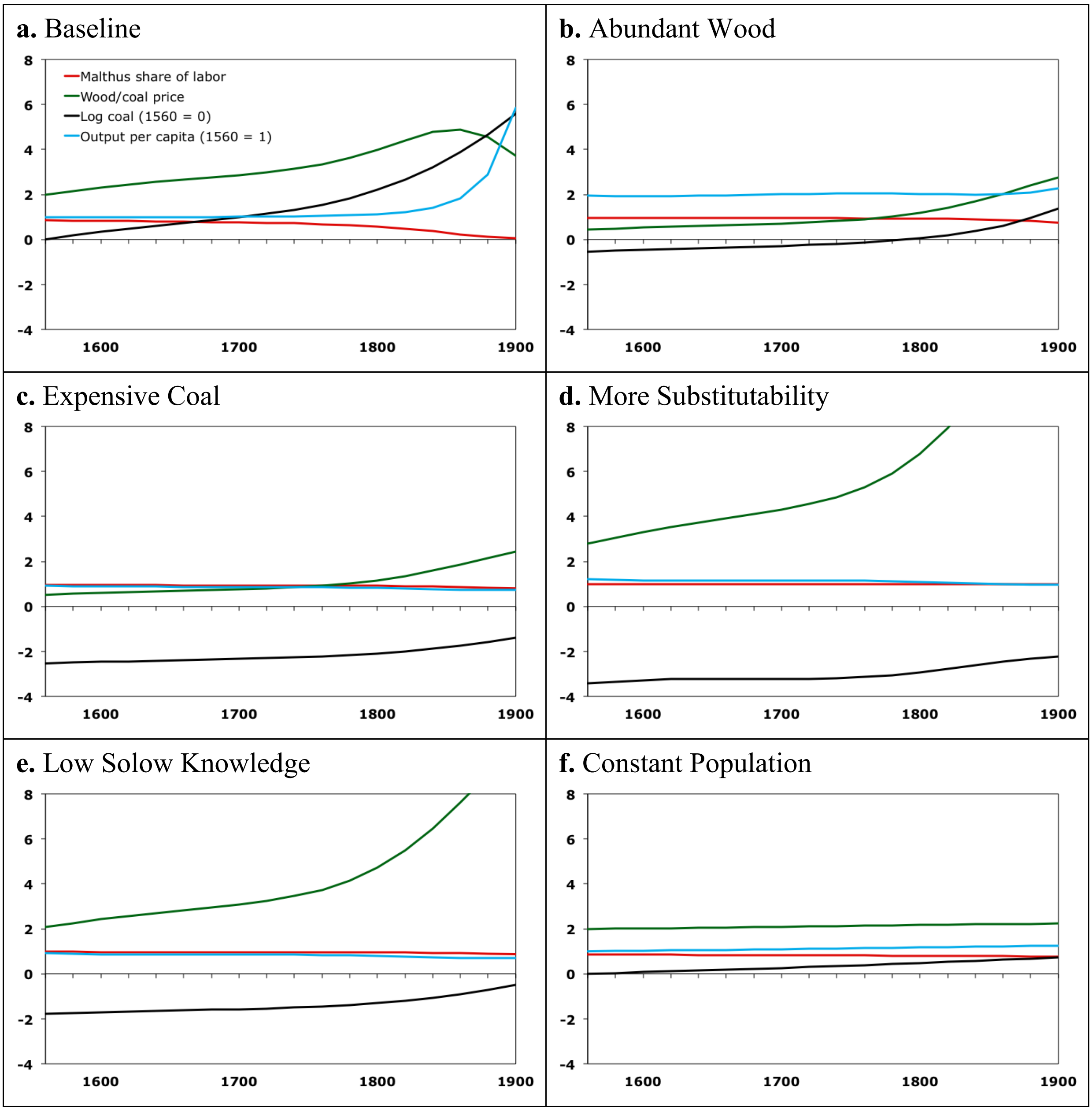


Figure 9. Baseline and Counterfactual Simulations



Notes: The share of labor in the Malthus sector and the wood/coal price are expressed as ratios. The log of coal use is normalized to zero and the level of output per capita to unity in 1560 in the Baseline simulation. Therefore, in Figures 10b-f the levels of coal and output are relative to those in the Baseline simulation.

APPENDIX (FOR ONLINE PUBLICATION)

Appendix A: Derivation of Equilibrium Equations in Section 3

Intermediate goods prices and the labor allocation are jointly determined economy-wide because of the labor adding-up condition (4) and the numeraire equation (5). Then, given goods prices and the labor allocation, all other quantities can be determined for each sector. First, we substitute $p_{M,t} = p_t p_{S,t}$ into the LHS of the numeraire equation (5):

$$\gamma^\sigma \left(p_t p_{S,t}\right)^{1-\sigma} + (1-\gamma)^\sigma p_{S,t}^{1-\sigma} = 1$$

Dividing both sides by $p_{S,t}^{1-\sigma}$ and raising them to the power of $\frac{1}{\sigma-1}$ gives three forms of $p_{S,t}$ for use in (23) and elsewhere (the second and third using $\Gamma \equiv \frac{\gamma}{1-\gamma}$ and $\Gamma^\sigma p_t^{1-\sigma} = \Gamma y_t^{\frac{\sigma-1}{\sigma}}$ from (6)):

$$\begin{aligned} p_{S,t} &= [\gamma^\sigma p_t^{1-\sigma} + (1-\gamma)^\sigma]^{\frac{1}{\sigma-1}} = (1-\gamma)^{\frac{\sigma}{\sigma-1}}(1+\Gamma^\sigma p_t^{1-\sigma})^{\frac{1}{\sigma-1}} \\ &= (1-\gamma)^{\frac{\sigma}{\sigma-1}}\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} \end{aligned} \tag{A1}$$

The price in the Malthus sector for use in (22) is then:

$$\begin{aligned} p_{M,t} = p_t p_{S,t} &= (1-\gamma)^{\frac{\sigma}{\sigma-1}} p_t (1+\Gamma^\sigma p_t^{1-\sigma})^{\frac{1}{\sigma-1}} = (1-\gamma)^{\frac{\sigma}{\sigma-1}}(p_t^{\sigma-1}+\Gamma^\sigma)^{\frac{1}{\sigma-1}} \\ &= (1-\gamma)^{\frac{\sigma}{\sigma-1}}\Gamma\left(y_t^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)^{\frac{1}{\sigma-1}} \end{aligned} \tag{A2}$$

Next we find the optimal levels of labor. Using (9):

$$\begin{aligned} w_t = (1-\alpha-\beta)p_{M,t}\frac{Y_{M,t}}{L_{M,t}} = (1-\alpha-\beta)p_{S,t}\frac{Y_{S,t}}{L_{S,t}} &\Longrightarrow \frac{Y_{M,t}}{Y_{S,t}} = y_t = (\Gamma/p_t)^\sigma \\ &= \frac{1}{p_t}\frac{L_{M,t}}{L_{S,t}} \Longrightarrow \frac{L_{M,t}}{L_{S,t}} = \Gamma^\sigma {p_t}^{1-\sigma} \end{aligned} \tag{A3}$$

Given $L_t = L_{M,t} + L_{S,t}$, $L_{S,t}$ and $L_{M,t}$, for use in (23) and (22) and elsewhere, are given by:

$$L_t = (\Gamma^\sigma {p_t}^{1-\sigma} + 1)L_{S,t}(\,p_t) \Rightarrow L_{S,t} = \frac{L_t}{1+\Gamma^\sigma p_t^{1-\sigma}} = \frac{L_t}{1+\Gamma {y_t}^{\frac{\sigma-1}{\sigma}}} \tag{A4}$$

and

$$L_{M,t}(\,p_t) = L_t - L_{S,t}(\,p_t) = \frac{L_t\Gamma^\sigma p_t^{1-\sigma}}{\Gamma^\sigma p_t^{1-\sigma}+1} = \frac{L_t\Gamma}{\Gamma + {y_t}^{-\frac{\sigma-1}{\sigma}}} \tag{A5}$$

Then we substitute the optimal amount of machines sold, $x_{i,t}^*(j)$ from (12), into the goods production functions (2) and (3). Noting that $x_{i,t}^*(j)$ does not vary with j, this yields:

$$Y_{i,t} = \frac{1}{\beta}\left(\int_0^{N_{i,t}} \left(\frac{p_{i,t}E_{i,t}^{\alpha}L_{i,t}^{1-\alpha-\beta}}{x_{i,t}(j)^{1-\beta}}\right)^{\beta} dj\right)E_{i,t}^{\alpha}L_{i,t}^{1-\alpha-\beta} = \frac{1}{\beta}\left(N_{i,t}(p_{i,t}E_{i,t}^{\alpha}L_{i,t}^{1-\alpha-\beta})^{\frac{\beta}{1-\beta}}\right)E_{i,t}^{\alpha}L_{i,t}^{1-\alpha-\beta}$$

hence

$$Y_{M,t}(p_t, N_{M,t}) = \frac{1}{\beta}N_{M,t}p_{M,t}^{\frac{\beta}{1-\beta}}(p_t)\bar{E}_M^{\frac{\alpha}{1-\beta}}L_{M,t}^{\frac{1-\alpha-\beta}{1-\beta}}(p_t) \tag{14}$$

and

$$Y_{S,t}(p_t, N_{S,t}) = \frac{1}{\beta}N_{S,t}p_{S,t}^{\frac{\beta}{1-\beta}}(p_t)E_{S,t}^{\frac{\alpha}{1-\beta}}(p_t, N_{S,t})L_{S,t}^{\frac{1-\alpha-\beta}{1-\beta}}(p_t) \tag{15}$$

We also need to find $E_{S,t}(p_t, N_{S,t})$, the optimal amount of coal, in terms of the endogenous variables p_t and $N_{S,t}$. Substituting (15) into (8) and rearranging yields:

$$\bar{e}_S = \alpha p_{S,t}\frac{Y_{S,t}}{E_{S,t}} = \frac{\alpha}{\beta}p_{S,t}{}^{\frac{1}{1-\beta}}N_{S,t}E_{S,t}^{\frac{-(1-\alpha-\beta)}{1-\beta}}L_{S,t}^{\frac{1-\alpha-\beta}{1-\beta}}$$

Then solving for the coal quantity we have:

$$E_{S,t}(p_t, N_{S,t}) = \left(\frac{\alpha N_{S,t}}{\beta\bar{e}_S}\right)^{\frac{1-\beta}{1-\alpha-\beta}} p_{S,t}{}^{\frac{1}{1-\alpha-\beta}}(p_t)L_{S,t}(p_t) \tag{16}$$

Inserting (16) back into (15) gives:

$$\begin{aligned} &Y_{S,t}(p_t, N_{S,t}) \\ &\quad = \frac{1}{\beta}N_{S,t}\, p_{S,t}^{\frac{\beta}{1-\beta}}(p_t)\left[\left(\frac{\alpha N_{S,t}}{\beta\bar{e}_S}\right)^{\frac{1-\beta}{1-\alpha-\beta}} p_{S,t}{}^{\frac{1}{1-\alpha-\beta}}(p_t)L_{S,t}(p_t)\right]^{\frac{\alpha}{1-\beta}} L_{S,t}^{\frac{1-\alpha-\beta}{1-\beta}}(p_t) \end{aligned}$$

i.e.

$$Y_{S,t}(p_t, N_{S,t}) = \left(\frac{N_{S,t}}{\beta}\right)^{\frac{1-\beta}{1-\alpha-\beta}}\left(\frac{\alpha}{\bar{e}_S}\right)^{\frac{\alpha}{1-\alpha-\beta}} p_{S,t}{}^{\frac{\alpha+\beta}{1-\alpha-\beta}}(p_t)L_{S,t}(p_t) \tag{A6}$$

and inserting (A6) and (14) into (6) gives the equilibrium output price ratio in the form:

$$p_t = \Gamma\left(\frac{Y_{M,t}(p_t, N_{M,t})}{Y_{S,t}(p_t, N_{S,t})}\right)^{-\frac{1}{\sigma}} \tag{21}$$

Lastly, (A1)-(A2) and (A4)-(A5) are the functional forms used in (22)-(23).

Appendix B1: Derivation of $n_t(y_t, N_t)$ and $n_t(e_t, N_t)$

Dividing (22) by (23) and using ratio definitions from (24)-(26) gives:

$$n_t \equiv \frac{\Delta N_{M,t}/N_{M,t-1}}{\Delta N_{S,t}/N_{S,t-1}}$$

$$= \left(\frac{p_{M,t}(p_t)}{p_{S,t}(p_t)}\right)^{\frac{1}{m(1-\beta)}} \left(\frac{\bar{E}_{M,t}}{E_{S,t}(p_t, N_{S,t})}\right)^{\frac{\alpha}{m(1-\beta)}} \left(\frac{L_{M,t}(p_t)}{L_{S,t}(p_t)}\right)^{\frac{1-\alpha-\beta}{m(1-\beta)}}$$

$$= p_t^{\frac{1}{m(1-\beta)}} E_t^{\frac{\alpha}{m(1-\beta)}} l_t^{\frac{1-\alpha-\beta}{m(1-\beta)}} \tag{B1}$$

Equation (B1) is not in the most intuitively useful form, so we substitute $p_t = \Gamma y_t^{-\frac{1}{\sigma}}$ (6) and use other equations, again with ratios defined as in (24)-(26), as follows:

(7)/(8), (6): $$e_t = p_t y_t / E_t = \Gamma y_t^{\frac{\sigma-1}{\sigma}} / E_t \tag{B2}$$

(14)/(15), (B1): $$y_t = N_t p_t^{\frac{\beta}{1-\beta}} E_t^{\frac{\alpha}{1-\beta}} l_t^{\frac{1-\alpha-\beta}{1-\beta}} = N_t n_t^m / p_t \tag{B3}$$

(9): $$y_t = l_t / p_t \tag{B4}$$

Substitute (6) into (B3) and solve for y_t:

$$y_t = N_t n_t^m / \Gamma y_t^{-\frac{1}{\sigma}} \Rightarrow n_t = \Gamma^{\frac{1}{m}} y_t^{\frac{\sigma-1}{m\sigma}} N_t^{\frac{-1}{m}} \tag{28}$$

Substituting in (B3) for p_t from (6), for E_t from (B2), and for l_t from (B4) and (6) gives:

$$y_t = N_t \left(\Gamma y_t^{-\frac{1}{\sigma}}\right)^{\frac{\beta}{1-\beta}} \left(\frac{\Gamma y_t^{\frac{\sigma-1}{\sigma}}}{e_t}\right)^{\frac{\alpha}{1-\beta}} \left(\Gamma y_t^{-\frac{1}{\sigma}} y_t\right)^{\frac{1-\alpha-\beta}{1-\beta}} \tag{B5}$$

Rearranging gives this conversion between (y_t, N_t) and (e_t, N_t) spaces:

$$y_t = \Gamma^\sigma e_t^{-\alpha\sigma} N_t^{(1-\beta)\sigma}; \text{ hence } e_t = \Gamma^{\frac{1}{\alpha}} y_t^{-\frac{1}{\alpha\sigma}} N_t^{\frac{1-\beta}{\alpha}} \tag{29}$$

Inserting (26) for y_t into (25), and using $\tilde{\sigma} \equiv 1 + \frac{1}{1-\beta}$ from (26), finally gives:

$$n_t = \Gamma^{\frac{\sigma}{m}} e_t^{\frac{-\alpha(\sigma-1)}{m}} N_t^{\frac{(\sigma-1)(1-\beta)-1}{m}} = \Gamma^{\frac{\sigma}{m}} e_t^{\frac{-\alpha(\sigma-1)}{m}} N_t^{\frac{(\sigma-\tilde{\sigma})(1-\beta)}{m}} \tag{30}$$

Appendix B2: Derivation of (33)

Substituting (B2) into (29) gives:

$$y_t = \Gamma^\sigma \left(N_t^{(1-\beta)\sigma} \Gamma^{-\alpha\sigma} y_t^{-\alpha\sigma\left(\frac{\sigma-1}{\sigma}\right)} / E_t^{-\alpha\sigma}\right) N_t^{(1-\beta)\sigma} = \Gamma^{(1-\alpha)\sigma} N_t^{(1-\beta)\sigma} y_t^{-\alpha(\sigma-1)} E_t^{\alpha\sigma}$$

$$\Rightarrow y_t^{1+\alpha(\sigma-1)} = \Gamma^{(1-\alpha)\sigma} N_t^{(1-\beta)\sigma} E_t^{\alpha\sigma} \tag{B6}$$

Inserting (A4) and (A1) into (16):

$$\Rightarrow E_{S,t} = \left(\frac{\alpha N_{S,t}}{\beta \bar{e}_S}\right)^{\frac{1-\beta}{1-\alpha-\beta}} \left[\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)(1-\gamma)^{\sigma}\right]^{\frac{1}{\sigma-1}\left(\frac{1}{1-\alpha-\beta}\right)} \frac{L_t}{1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}}$$

$$\Rightarrow E_t = \frac{\bar{E}_M}{E_{S,t}} = \left(\frac{\beta \bar{e}_S}{\alpha N_{S,t}}\right)^{\frac{1-\beta}{1-\alpha-\beta}} \frac{\bar{E}_M}{L_t} \frac{\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{1-\frac{1}{\sigma-1}\left(\frac{1}{1-\alpha-\beta}\right)}}{(1-\gamma)^{\frac{\sigma}{\sigma-1}\left(\frac{1}{1-\alpha-\beta}\right)}}$$

Substituting this into (B6) we have:

$$y_t^{1+\alpha(\sigma-1)} = \Gamma^{(1-\alpha)\sigma} N_t^{(1-\beta)\sigma} \left(\frac{\beta \bar{e}_S}{\alpha N_{S,t}}\right)^{\frac{\alpha\sigma(1-\beta)}{1-\alpha-\beta}} \left(\frac{\bar{E}_M}{L_t}\right)^{\alpha\sigma} \frac{\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{\alpha\sigma\left(1-\frac{1}{\sigma-1}\left(\frac{1}{1-\alpha-\beta}\right)\right)}}{(1-\gamma)^{\frac{\alpha\sigma^2}{\sigma-1}\left(\frac{1}{1-\alpha-\beta}\right)}} \tag{B7}$$

Taking logs and then differences,[1] and substituting $\Delta y_t / y_t = \Delta\ln(y_t)$ gives (see the Annex):

$$\frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}}{1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}} \Delta\ln(y_t) = \sigma(1-\beta)\,\Delta\ln\left(N_{M,t}\right) - \sigma\,\frac{(1-\beta)^2}{1-\alpha-\beta}\Delta\ln\left(N_{S,t}\right) - \alpha\sigma\,\Delta\ln(L_t) \tag{B8}$$

Using $\Delta\ln\left(N_{M,t}\right) = n_t \Delta\ln\left(N_{S,t}\right) = n_t \Delta\ln(N_t)/(n_t - 1)$ then gives, after further algebra (again see the Annex):

$$\frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}}{1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}} (n_t-1)\,\Delta\ln(y_t) = \sigma(1-\beta)\left(n_t - \frac{1-\beta}{1-\alpha-\beta}\right)\Delta\ln(N_t) \;-\; (n_t-1)\alpha\sigma\,\Delta\ln(L_t) \tag{33}$$

Appendix B3: Derivation of (36)

Taking differences of the log of (29) gives $\Delta\ln(y_t) = \sigma[(1-\beta)\Delta\ln(N_t) - \alpha\,\Delta\ln(e_t)]$, and substituting this, and $\Gamma y_t^{\frac{\sigma-1}{\sigma}} = N_t n_t^m$ from (6) and (B3), into (33) gives:

[1] As noted early in Section 4, first differences in time throughout this paper are treated as if they were differentials, so that all time variables are treated as continuous functions of time.

$$\sigma\left[\frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)N_t n_t^m}{1+N_t n_t^m}\right](n_t-1)[(1-\beta)\Delta\ln(N_t)-\alpha\Delta\ln(e_t)]$$
$$\approx\sigma\left[(1-\beta)\left(n_t-\frac{1-\beta}{1-\alpha-\beta}\right)\Delta\ln(N_t)-(n_t-1)\alpha\Delta\ln(L_t)\right] \tag{B9}$$

After much further algebra in the Annex, including using $\frac{1-\beta}{1-\alpha-\beta}-1=\alpha(\sigma^\dagger-1)$ from (26), this yields:

$$\left(\frac{1+\alpha(\sigma-1)}{1-\beta}+\frac{N_t n_t^m}{1-\alpha-\beta}\right)(n_t-1)\,\Delta\ln(e_t)$$
$$=[\{\sigma-\sigma^\dagger+(\sigma^\dagger-1)(1+N_t n_t^m)\}n_t-(\sigma-\sigma^\dagger)]\,\Delta\ln(N_t) \tag{36}$$
$$+\left(\frac{1+Nn_t^m}{1-\beta}\right)(n_t-1)\,\Delta\ln(L_t)$$

Appendix B4: Malthusian Sluggishness under High Substitutability

Proof of Lemma 1

As a preliminary, note that

$$-(1-\beta)<-(1-\alpha-\beta)$$
$$\Rightarrow[(\sigma-1)(1-\alpha-\beta)-1]\,(1-\beta)<[(\sigma-1)(1-\beta)-1]\,(1-\alpha-\beta)$$
$$\Rightarrow\frac{1-\beta}{1-\alpha-\beta}<n_\infty\equiv\frac{(\sigma-1)(1-\beta)-1}{(\sigma-1)(1-\alpha-\beta)-1}$$

Then rearrange (33) to show that at any path-point, given constant population ($\Delta\ln(L_t)=0$),

$$\frac{\Delta\ln(N_t)}{\Delta\ln(y_t)}=\frac{\left[1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right](n_t-1)}{\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)\sigma(1-\beta)\left(n_t-\frac{1-\beta}{1-\alpha-\beta}\right)}$$

We can then prove (see the Annex for details) that at any point in (y,N)-space on or above the $n=\frac{(\sigma-1)(1-\beta)-1}{(\sigma-1)(1-\alpha-\beta)-1}\equiv n_\infty$ locus (38), but not above the $\Delta y_t=0$ locus, i.e. with $\frac{1-\beta}{1-\alpha-\beta}\le n(y,N)=\Gamma^{\frac{1}{m}}y^{\frac{\sigma-1}{m\sigma}}N^{\frac{-1}{m}}\le n_\infty$, the path through that point must have log-slope $\frac{\Delta\ln(N_t)}{\Delta\ln(y_t)}>\frac{\sigma-1}{\sigma}$, so that its slope is steeper than the curve $\Gamma^{\frac{1}{m}}y^{\frac{\sigma-1}{m\sigma}}N^{\frac{-1}{m}}=$ constant through that point. ■

Proof of Lemma 2

(i) We find the $\Delta(n_t)=0$ locus in (y,N)-space by taking logs then the differences of (28) $n_t=\Gamma^{\frac{1}{m}}N_t^{\frac{-1}{m}}y_t^{\frac{\sigma-1}{m\sigma}}$, and then setting $\Delta\ln(n_t)=0$

$$0 = -\frac{1}{m}\Delta\ln(N_t) + \frac{\sigma-1}{m\sigma}\Delta\ln(y_t) \Rightarrow \frac{\Delta\ln(N_t)}{\Delta\ln(y_t)} = \frac{\sigma-1}{\sigma} \tag{B10}$$

Substitute this in (33) with $\Delta\ln(L_t) = 0$, which relates the growth rate of *y* and *N* given constant population, multiplied by $\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)$:

$$\left(1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)(n_t-1)\,\Delta\ln(y_t)$$
$$= \left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)\sigma(1-\beta)\left(n_t-\frac{1-\beta}{1-\alpha-\beta}\right)\Delta\ln(N_t)$$
$$= \left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)\sigma(1-\beta)\left(n_t-\frac{1-\beta}{1-\alpha-\beta}\right)\frac{\sigma-1}{\sigma}\Delta\ln(y_t)$$

Divide by $\Delta\ln(y_t)$ and rearrange:

$$\Rightarrow \left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)(1-\beta)\left(n_t-\frac{1-\beta}{1-\alpha-\beta}\right)(\sigma-1) = \left[1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right](n_t-1) \tag{B11}$$

Rearranging this, the $\Delta(n_t) = 0$ locus is thus:

$$n(y,N) = \Gamma^{\frac{1}{m}} y^{\frac{\sigma-1}{m\sigma}} N^{\frac{-1}{m}} = \frac{\frac{1-\beta}{1-\alpha-\beta}\left(1+\Gamma y^{\frac{\sigma-1}{\sigma}}\right)(\sigma-1)(1-\beta) - \left[1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y^{\frac{\sigma-1}{\sigma}}\right]}{\left(1+\Gamma y^{\frac{\sigma-1}{\sigma}}\right)(\sigma-1)(1-\beta) - \left[1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y^{\frac{\sigma-1}{\sigma}}\right]} \tag{37}$$

$$= \frac{\frac{1-\beta}{1-\alpha-\beta}\left(y^{-\left(\frac{\sigma-1}{\sigma}\right)}+\Gamma\right)(\sigma-1)(1-\beta) - \left[\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma + \{1+\alpha(\sigma-1)\}y^{-\left(\frac{\sigma-1}{\sigma}\right)}\right]}{\left(y^{-\left(\frac{\sigma-1}{\sigma}\right)}+\Gamma\right)(\sigma-1)(1-\beta) - \left[\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma + \{1+\alpha(\sigma-1)\}y^{-\left(\frac{\sigma-1}{\sigma}\right)}\right]} \tag{B12}$$

$$\rightarrow \frac{\frac{1-\beta}{1-\alpha-\beta}(\sigma-1)(1-\beta) - \frac{1-\beta}{1-\alpha-\beta}}{(\sigma-1)(1-\beta) - \frac{1-\beta}{1-\alpha-\beta}} = n_\infty \ \text{ as } \ y \rightarrow \infty \tag{38}$$

(ii) We can then show (see the Annex) that for any $0 < y < \infty$, (B12) > (38), i.e. has higher $N^{\frac{-1}{m}}$, i.e. has lower *N*; hence the $\Delta\ln(n_t) = 0$ locus lies below the $n = n_\infty$ locus (38).

(iii) To show that the $\Delta\ln(n_t) = 0$ locus in (N, y)-space is locally steeper than the development path through any point on the locus, we first insert (28) into (B11) and rearrange:

$$\frac{\left[1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y^{\frac{\sigma-1}{\sigma}}\right]\left(\Gamma^{\frac{1}{m}} y^{\frac{\sigma-1}{m\sigma}} N^{\frac{-1}{m}} - 1\right)}{\left(1+\Gamma y^{\frac{\sigma-1}{\sigma}}\right)(\sigma-1)(1-\beta)\left(\Gamma^{\frac{1}{m}} y^{\frac{\sigma-1}{m\sigma}} N^{\frac{-1}{m}} - \frac{1-\beta}{1-\alpha-\beta}\right)} = 1 \tag{B13}$$

The log-slope of this locus in (*y*,*N*) space (again see the Annex) is:

$$\frac{\Delta\ln(N)}{\Delta\ln(y)} = \frac{\left[(1-\alpha-\beta)+(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right]\left(\sigma-\sigma^{\dagger}\right)\frac{n}{m}(\sigma-1)}{\left[1-\alpha-\beta+(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right](\sigma-\sigma^{\dagger})\frac{\sigma n}{m}}$$
$$+\frac{\left[\left(\sigma-\sigma^{\dagger}\right)(n-1)-(\sigma-1)\left(\frac{\alpha}{1-\alpha-\beta}\right)\right](\sigma-1)(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}}{\left[1-\alpha-\beta+(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right](\sigma-\sigma^{\dagger})\frac{\sigma n}{m}}$$

We can then show (by straightforward but tedious algebra, again in the Annex) that this slope is steeper than the (log-)slope of the path through that point, which since by construction $\Delta\ln(n_t) = 0$ at that point, is $\frac{\Delta\ln(N_t)}{\Delta 1(y_t)} = \frac{\sigma-1}{\sigma}$ from (B10). ■

Appendix B5: Proof of upward-sloping $\Delta e_t = 0$ isocline (part-proof of Proposition 2)

$$n_t = \frac{\sigma-\sigma^{\dagger}}{\sigma-\sigma^{\dagger}+(\sigma^{\dagger}-1)(1+N_t n_t^m)} \tag{39}$$

$$n_t = \Gamma^{\frac{\sigma}{m}} N_t^{\frac{(\sigma-\tilde{\sigma})(1-\beta)}{m}} e_t^{\frac{-\alpha(\sigma-1)}{m}} \tag{30}$$

Substitute (30) into (39):

$$\Rightarrow n_t = \frac{\sigma-\sigma^{\dagger}}{\sigma-1+(\sigma^{\dagger}-1)\Gamma^{\sigma} N_t^{(\sigma-\tilde{\sigma})(1-\beta)+1} e_t^{-\alpha(\sigma-1)}} = \Gamma^{\frac{\sigma}{m}} N_t^{\frac{(\sigma-\tilde{\sigma})(1-\beta)}{m}} e_t^{\frac{-\alpha(\sigma-1)}{m}}$$

$$\Rightarrow \frac{\sigma-\sigma^{\dagger}}{\Gamma^{\frac{\sigma}{m}}} = N_t^{\frac{(\sigma-\tilde{\sigma})(1-\beta)}{m}} e_t^{\frac{-\alpha(\sigma-1)}{m}}\left[\sigma-1+(\sigma^{\dagger}-1)\Gamma^{\sigma} N_t^{(\sigma-\tilde{\sigma})(1-\beta)+1} e_t^{-\alpha(\sigma-1)}\right]$$

Now take total differences:

$$0 = \frac{(\sigma-\tilde{\sigma})(1-\beta)}{m} N_t^{\frac{(\sigma-\tilde{\sigma})(1-\beta)}{m}-1} e_t^{\frac{-\alpha(\sigma-1)}{m}}\left[\sigma-1+(\sigma^{\dagger}-1)\Gamma^{\sigma} N_t^{(\sigma-\tilde{\sigma})(1-\beta)+1} e_t^{-\alpha(\sigma-1)}\right]\Delta N_t$$
$$-\frac{\alpha(\sigma-1)}{m} N_t^{\frac{(\sigma-\tilde{\sigma})(1-\beta)}{m}} e_t^{\frac{-\alpha(\sigma-1)}{m}-1}\left[\sigma-1+(\sigma^{\dagger}-1)\Gamma^{\sigma} N_t^{(\sigma-\tilde{\sigma})(1-\beta)+1} e_t^{-\alpha(\sigma-1)}\right]\Delta e_t$$
$$+N_t^{\frac{(\sigma-\tilde{\sigma})(1-\beta)}{m}} e_t^{\frac{-\alpha(\sigma-1)}{m}}(\sigma^{\dagger}-1)\Gamma^{\sigma}[(\sigma-\tilde{\sigma})(1-\beta)+1] N_t^{(\sigma-\tilde{\sigma})(1-\beta)} e_t^{-\alpha(\sigma-1)}\Delta N_t$$
$$-N_t^{\frac{(\sigma-\tilde{\sigma})(1-\beta)}{m}} e_t^{\frac{-\alpha(\sigma-1)}{m}}(\sigma^{\dagger}-1)\Gamma^{\sigma}[\alpha(\sigma-1)] N_t^{(\sigma-\tilde{\sigma})(1-\beta)+1} e_t^{-\alpha(\sigma-1)-1}\Delta e_t$$

$$\Rightarrow \left\{\frac{(\sigma-\tilde{\sigma})(1-\beta)}{m} N_t^{\frac{(\sigma-\tilde{\sigma})(1-\beta)}{m}-1} e_t^{\frac{-\alpha(\sigma-1)}{m}}\left[\sigma-1+(\sigma^{\dagger}-1)\Gamma^{\sigma} N_t^{(\sigma-\tilde{\sigma})(1-\beta)+1} e_t^{-\alpha(\sigma-1)}\right] + N_t^{\frac{(\sigma-\tilde{\sigma})(1-\beta)}{m}} e_t^{\frac{-\alpha(\sigma-1)}{m}}(\sigma^{\dagger}-1)\Gamma^{\sigma}[(\sigma-\tilde{\sigma})(1-\beta)+1] N_t^{(\sigma-\tilde{\sigma})(1-\beta)} e_t^{-\alpha(\sigma-1)}\right\}\Delta N_t$$

$$= \left\{\tfrac{\alpha(\sigma-1)}{m} N_t^{\frac{(\sigma-\tilde{\sigma})(1-\beta)}{m}} e_t^{\frac{-\alpha(\sigma-1)}{m}-1}\left[\sigma - 1 + (\sigma^{\dagger}-1)\Gamma^{\sigma} N_t^{(\sigma-\tilde{\sigma})(1-\beta)+1} e_t^{-\alpha(\sigma-1)}\right] + N_t^{\frac{(\sigma-\tilde{\sigma})(1-\beta)}{m}} e_t^{\frac{-\alpha(\sigma-1)}{m}} (\sigma^{\dagger}-1)\Gamma^{\sigma}\alpha(\sigma-1) N_t^{(\sigma-\tilde{\sigma})(1-\beta)+1} e_t^{-\alpha(\sigma-1)-1}\right\} \Delta e_t$$

The bracketed expressions multiplying ΔN_t and Δe_t are both unambiguously positive, so $\Delta N_t/\Delta e_t > 0$. ■

Appendix B6 (part-proof of Proposition 3)

To prove that all paths under the $\Delta y_t = 0$ locus in (*y*,*N*)-space eventually rise to cross that locus upwards, we have to show that at any point under this locus, the slope of the path through that point is steeper than the curve $n(y, N) \equiv \Gamma^{\frac{1}{m}} y^{\frac{\sigma-1}{m\sigma}} N^{\frac{-1}{m}} = \bar{n}$, where $\bar{n}$ is a constant,, through that point. That is, from (35) and (33), we need to show that:

$$\left\{\bar{n} > \frac{1-\beta}{1-\alpha-\beta} (> 1) \text{ and } \sigma - 1 < \frac{1}{1-\alpha-\beta}\right\}$$

$$\Rightarrow \frac{\Delta \ln(N_t)}{\Delta \ln(y_t)} = \frac{\left[1 + \alpha(\sigma-1) + \left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right](\bar{n}-1)}{\left(1 + \Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)\sigma(1-\beta)\left(\bar{n} - \frac{1-\beta}{1-\alpha-\beta}\right)} + \frac{(\bar{n}-1)\alpha\sigma\, \Delta \ln(L_t)}{\Delta \ln(y_t)} > \frac{\sigma-1}{\sigma}$$

Since $\bar{n} > 1$, $\Delta \ln(L_t) > 0$ always and $\Delta \ln(y_t) > 0$ below the $\Delta \ln(y_t) = 0$ locus, the second term on the LHS is > 0, so it will be enough just to prove that

$$\left[1 + \alpha(\sigma-1) + \left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right](\bar{n}-1)\sigma > \left(1 + \Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)\sigma(1-\beta)\left(\bar{n} - \frac{1-\beta}{1-\alpha-\beta}\right)(\sigma-1)$$

$$\text{i.e.} \frac{\left[1 + \alpha(\sigma-1) + \left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right](\bar{n}-1)}{\left(1 + \Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)\sigma(1-\beta)\left(\bar{n} - \frac{1-\beta}{1-\alpha-\beta}\right)} > \frac{\sigma-1}{\sigma}$$

The proof of this by straightforward but tedious algebra is given in the Annex. ■

Appendix B7: Growth Rates

Algebra for Proposition 5

We start with these minor rearrangements of (20) for the growth rates of Malthus-sector and Solow-sector machine varieties:

$$n_{M,t} \equiv \frac{\Delta N_{M,t}}{N_{M,t-1}} = \lambda N_{M,t-1}^{\frac{\mu+\nu-1}{h\nu}} p_{M,t}^{\frac{1}{h(1-\beta)}}(p_t) \bar{E}_M^{\frac{\alpha}{h(1-\beta)}} L_{M,t}^{\frac{1-\alpha-\beta}{h(1-\beta)}}(p_t) \quad \text{(B15)}$$

$$n_{S,t} \equiv \frac{\Delta N_{S,t}}{N_{S,t-1}} = \lambda N_{S,t-1}^{\frac{\mu+\nu-1}{h\nu}} p_{S,t}^{\frac{1}{h(1-\beta)}}(p_t) E_{S,t}^{\frac{\alpha}{h(1-\beta)}}(p_t, N_{S,t}) L_{S,t}^{\frac{1-\alpha-\beta}{h(1-\beta)}}(p_t) \quad \text{(B16)}$$

where λ is an arbitrary positive constant which may differ from equation to equation. Substituting for coal use $E_{S,t}(p_t, N_{S,t})$ from (16) into (B16) gives after routine algebra (see the Annex) this equation for the growth rate of Solow-sector machine varieties, $n_{S,t}$:

$$n_{S,t} = \frac{\Delta N_{S,t}}{N_{S,t-1}} = \lambda N_{S,t-1}^{\frac{\mu+\nu-1}{h\nu}} p_{S,t}^{\frac{1}{h(1-\alpha-\beta)}} N_{S,t}^{\frac{\alpha}{h(1-\alpha-\beta)}} L_{S,t}^{\frac{1}{h}} \quad \text{(B17)}$$

Substituting for $p_{M,t}^{\frac{1}{m(1-\beta)}} L_{M,t}^{\frac{1-\alpha-\beta}{m(1-\beta)}}$ from (A2) and (A5) into (B15) and for $p_{S,t}^{\frac{1}{m(1-\alpha-\beta)}} L_{S,t}^{\frac{1}{m}}$ from (A1) and (A4) into (B17) then yields, after further routine algebra (again see the Annex), these growth rates for each sector:

$$n_{M,t} = \lambda N_{M,t-1}^{\frac{\nu+\mu-1}{h\nu}} \bar{E}_M^{\frac{\alpha}{h(1-\beta)}} \left(y_t^{-\frac{\sigma-1}{\sigma}} + \Gamma\right)^{-\left[\frac{(\sigma-\sigma^\dagger)(1-\alpha-\beta)}{h(\sigma-1)(1-\beta)}\right]} L_t^{\frac{1-\alpha-\beta}{h(1-\beta)}} \quad (41)$$

and

$$n_{S,t} = \lambda N_{S,t-1}^{\frac{\nu+\mu-1}{h\nu}} N_{S,t}^{\frac{\alpha}{h(1-\alpha-\beta)}} \left(1 + \Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{-\frac{\sigma-\sigma^\dagger}{h(\sigma-1)}} L_t^{\frac{1}{m}} \quad (42)$$

Proof of Proposition 5

(i) In (41), all terms are rising or constant: $\Delta N_{M,t-1} > 0$ & $\nu \geq 1-\mu \Rightarrow \Delta N_{M,t-1}^{\frac{\mu+\nu-1}{h\nu}} > 0$; $\Delta \bar{E}_M^{\frac{\alpha}{h(1-\beta)}} = 0$; $\Delta y_t > 0$ & $\sigma > \sigma^\dagger$ on an MS path $\Rightarrow \Delta\left(y_t^{-\frac{\sigma-1}{\sigma}} + \Gamma\right)^{-\left[\frac{(1-\alpha-\beta)(\sigma-\sigma^\dagger)}{h(\sigma-1)(1-\beta)}\right]} > 0$; and $\Delta L_t^{\frac{1-\alpha-\beta}{h(1-\beta)}} > 0$; so $n_{M,t}$, the growth rate of Malthus varieties rises forever.

(ii) In (42), once $\Delta y_t < 0$ forever on an IR path, for $\nu \geq 1-\mu$, all terms are rising or constant for the same reasons as in (i); while for $\frac{1-\alpha-\beta}{1-\beta}(1-\mu) \leq \nu < 1-\mu$, let $\frac{N_{S,t}}{N_{S,t-1}} = k_t (> 1)$ with $\dot{k}_t > 0$; then $\Delta N_{S,t-1}^{\frac{\mu+\nu-1}{h\nu}} N_{S,t}^{\frac{\alpha}{h(1-\alpha-\beta)}} = \Delta k_t^{\frac{1-\mu-\nu}{h\nu}} N_{S,t}^{\frac{(1-\beta)\nu-(1-\mu)(1-\alpha-\beta)}{h\nu(1-\alpha-\beta)}} > 0$: so in either case, $n_{S,t}$, the growth rate of Solow-sector varieties, rises forever. ■

Proof of Proposition 6

In the following, we denote growth rates and asymptotic growth rates for variable X_t thus:

$$\frac{\Delta X_t/\Delta t}{X_t} \equiv g(X_t) \text{ and } \lim_{t\to\infty}\frac{\Delta X_t/\Delta t}{X_t} \equiv g_\infty(X_t),$$

with *MS* and *IR* subscripts added as needed. However, note that $g_{\infty MS}\left(N_{M,t}\right)$ and $g_{\infty IR}\left(N_{S,t}\right)$, also written as $n_{M,t\infty MS}$ and $n_{S,t\infty IR}$, are not conventional limits, since we will show that while these growth rates may be signed asymptotically, they generally do not approach fixed limits.

By definition $y_t \to \infty$ under MS; and by (A2) and (A5):

$$\lim_{y_t\to\infty} p_{M,t} = \lim_{y_t\to\infty}(1-\gamma)^{\frac{\sigma}{\sigma-1}}\Gamma\left(y_t^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)^{\frac{1}{\sigma-1}} = (1-\gamma)^{\frac{\sigma}{\sigma-1}}\Gamma^{\frac{\sigma}{\sigma-1}} \Longrightarrow g_{\infty MS}\left(p_{M,t}\right) = 0$$

$$\bar{E}_M = \text{constant} \Longrightarrow g_\infty\left(\bar{E}_M^{\frac{\alpha}{h(1-\beta)}}\right) = 0;\ \lim_{y_t\to\infty} L_{M,t} = \lim_{y_t\to\infty}\frac{L_t\Gamma}{\Gamma + y_t^{-\frac{\sigma-1}{\sigma}}} = L_t$$

and inserting these limits into (41) (with $\nu = 1-\mu$ so the $N_{M,t-1}^{\frac{\mu+\nu-1}{h\nu}}$ term disappears and *h* becomes *m*) gives:

$$g_{\infty MS}\left(n_{M,t}\right) = 0 + 0 + g_{\infty MS}\left(L_{M,t}^{\frac{1-\alpha-\beta}{m(1-\beta)}}\right) = \frac{1-\alpha-\beta}{m(1-\beta)}g_\infty(L_t) \tag{B18}$$

By definition $y_t \to 0$ under IR, and by (A1) and (A4),

$$\lim_{y_t\to 0} p_{S,t} = \lim_{y_t\to 0}(1-\gamma)^{\frac{\sigma}{\sigma-1}}\Gamma\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} = (1-\gamma)^{\frac{\sigma}{\sigma-1}} \Longrightarrow g_{\infty IR}\left(p_{S,t}\right) = 0$$

$$\lim_{y_t\to 0} L_{S,t} = \lim_{y_t\to 0}\frac{L_t}{\Gamma y_t^{\frac{\sigma-1}{\sigma}}+1} = L_t$$

and inserting these limits into (42) gives:

$$g_{\infty IR}\left(n_{S,t}\right) = \frac{\alpha}{m(1-\alpha-\beta)}n_{S,t\infty IR} + \frac{1}{m}g_\infty(L_t) \tag{B19}$$

Next we find the growth rates of labor productivity (output per capita) for the Malthus and Solow sectors. For the Malthus sector, substituting (A2) for $p_{M,t}(y_t)$ and (A5) for $L_{M,t}(y_t)$ into (14) for $Y_{M,t}$ and then rearranging, gives (see the Annex):

$$\frac{Y_{M,t}}{L_{M,t}} = \lambda N_{M,t}\left(y_t^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)^{\frac{(\sigma-1)\alpha+\beta}{(\sigma-1)(1-\beta)}}\bar{E}_M^{\frac{\alpha}{1-\beta}}L_t^{\frac{-\alpha}{1-\beta}} \tag{B20}$$

$$\Rightarrow g_{\infty MS}\left(\frac{Y_{M,t}}{L_{M,t}}\right) = g_{\infty MS}(N_{M,t}) - \frac{\alpha}{1-\beta} g_\infty(L_t)$$
$$= n_{M,t\,\infty MS} - \frac{\alpha}{1-\beta} g_\infty(L_t) \qquad \text{(B21)}$$

For the Solow sector, substituting (A1) for $p_{S,t}(y_t)$ and (16) for $E_{S,t}$ into (15) for $Y_{S,t}$ and rearranging yields (see the Annex):

$$\frac{Y_{S,t}}{L_{S,t}} = \lambda N_{S,t}^{\frac{1-\beta}{1-\alpha-\beta}} \left(1 + \Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\alpha+\beta}{(\sigma-1)(1-\alpha-\beta)}} \qquad \text{(B22)}$$

$$\Rightarrow g_{\infty IR}\left(\frac{Y_{S,t}}{L_{S,t}}\right) = \frac{1-\beta}{1-\alpha-\beta} n_{S,t\infty IR} \qquad \text{(B23)}$$

Now $y_t \to 0$ on an IR path, $L_{S,t} \to L_t$ and (by (1)) $Y_t \to (1-\gamma)^{\frac{\sigma-1}{\sigma}} Y_{S,t}$, hence $g_{\infty IR}\left(\frac{Y_{S,t}}{L_{S,t}}\right) = g_{\infty IR}\left(\frac{Y_t}{L_t}\right)$, the economy's "growth rate" (i.e. of final output per capita). So (B23) shows the IR growth rate is eventually a multiple of $n_{S,t\,\infty IR}$, which by (B19) is rising then, i.e. growth eventually accelerates on an IR path. Similar algebra shows that since $y_t \to \infty$ on an MS path, $g_{\infty MS}\left(\frac{Y_{M,t}}{L_{M,t}}\right) = g_{\infty MS}\left(\frac{Y_t}{L_t}\right)$. So (B21) shows that under MS, economic growth is eventually slower than $n_{M,t}$, the growth rate of machine varieties on that path, because of the drag of population growth at rate $g_\infty(L_t)$. From the asymptotic growth rates in (B18) and (B19), for any pair of MS and IR paths starting with the same parameters except for different initial varieties $(N_{M,0}, N_{S,0})$ (a difference needed to make one path MS and the other IR), after some finite time eventually $n_{S,t}$ on the IR path must exceed $n_{M,t}$ on the MS path. Hence $g_{\infty IR}\left(\frac{Y_t}{L_t}\right) > g_{\infty MS}\left(\frac{Y_t}{L_t}\right)$: economic growth is eventually at on an IR path than on an MS path.

■

Appendix B8: Proof of Propositions 7 and 8

If we fix $N_{M,t}$ and $N_{S,t}$ – i.e. treat them as constants – the equation system consists of $p_t = \Gamma\left(\frac{Y_{M,t}(p_t,N_{M,t})}{Y_{S,t}(p_t,N_{S,t})}\right)^{-\frac{1}{\sigma}}$ (21) alone, or even simpler from (6), $y_t = \frac{Y_{M,t}(y_t,N_{M,t})}{Y_{S,t}(y_t,N_{S,t})}$.

Comparative static effects on y_t

Substitute (14) for $Y_{M,t}(p_t, N_{M,t})$ and (A6) for $Y_{S,t}(p_t, N_{S,t})$ into the latter equation:

$$y_t = \frac{Y_{M,t}(y_t, N_{M,t})}{Y_{S,t}(y_t, N_{S,t})} = \frac{N_{M,t}p_{M,t}^{\frac{\beta}{1-\beta}}(y_t)\bar{E}_M^{\frac{\alpha}{1-\beta}}L_{M,t}^{\frac{1-\alpha-\beta}{1-\beta}}(y_t)}{N_{S,t}^{\frac{1-\beta}{1-\alpha-\beta}}p_{S,t}^{\frac{\alpha+\beta}{1-\alpha-\beta}}(y_t)\left(\frac{\alpha}{\beta\bar{e}_S}\right)^{\frac{\alpha}{1-\alpha-\beta}}L_{S,t}(y_t)} \tag{B24}$$

Take logs, let $\mathbf{\Omega}$ be the vector of all model parameters, and momentarily fix $N_{M,t}$ and $N_{S,t}$, which we then denote as $\bar{N}_{M,t}$ and $\bar{N}_{S,t}$. This gives:

$$\begin{aligned} f(y_t, \bar{N}_{M,t}, \bar{N}_{S,t}, \mathbf{\Omega}) \equiv \ln(\bar{N}_{M,t}) - \left(\frac{1-\beta}{1-\alpha-\beta}\right)\ln(\bar{N}_{S,t}) \\ + \frac{\beta}{1-\beta}\ln\left(p_{M,t}(y_t)\right) - \left(\frac{\alpha+\beta}{1-\alpha-\beta}\right)\ln\left(p_{S,t}(y_t)\right) \\ + \left(\frac{\alpha}{1-\beta}\right)\ln(\bar{E}_M) - \left(\frac{\alpha}{1-\alpha-\beta}\right)\ln\left(\frac{\alpha}{\beta\bar{e}_S}\right) \\ + \left(\frac{1-\alpha-\beta}{1-\beta}\right)\ln\left(L_{M,t}(y_t)\right) - \ln\left(L_{S,t}(y_t)\right) - \ln(y_t) = 0 \end{aligned} \tag{B25}$$

We next calculate $\partial f/\partial y_t$:

$$\begin{aligned} \frac{\partial f}{\partial y_t} = \left(\frac{\beta}{1-\beta}\right)\frac{\partial \ln\left(p_{M,t}(y_t)\right)}{\partial y_t} - \left(\frac{\alpha+\beta}{1-\alpha-\beta}\right)\frac{\partial \ln\left(p_{S,t}(y_t)\right)}{\partial y_t} \\ + \left(\frac{1-\alpha-\beta}{1-\beta}\right)\frac{\partial \ln\left(L_{M,t}(y_t)\right)}{\partial y_t} - \frac{\partial \ln\left(L_{S,t}(y_t)\right)}{\partial y_t} - \frac{1}{y_t} \end{aligned} \tag{B26}$$

We find the four partial derivatives in (B26) as follows. From (A1), taking logs and then the derivative (see the Annex for the full algebra):

$$\frac{\partial \ln\left(p_{S,t}(y_t)\right)}{\partial y_t} = \frac{\Gamma}{\sigma y_t^{\frac{1}{\sigma}}\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}} + 1\right)} > 0$$

From (A2) and the previous result:

$$\frac{\partial \ln\left(p_{M,t}(y_t)\right)}{\partial y_t} = \frac{-1}{\sigma y_t\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}}+1\right)} < 0$$

From (A4):

$$\frac{\partial \ln\left(L_{S,t}(y_t)\right)}{\partial y_t} = \frac{-(\sigma-1)\Gamma}{\sigma y_t^{\frac{1}{\sigma}}\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}}+1\right)} < 0$$

From (A5) and the previous result: $\frac{\partial \ln\left(L_{M,t}(y_t)\right)}{\partial y_t} = \frac{\sigma-1}{\sigma y_t\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}}+1\right)} > 0$

Hence (B26) becomes:

$$\frac{\partial f}{\partial y_t} = \left(\frac{\beta}{1-\beta}\right)\frac{-1}{\sigma y_t\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}}+1\right)} - \left(\frac{\alpha+\beta}{1-\alpha-\beta}\right)\frac{\Gamma}{\sigma y_t^{\frac{1}{\sigma}}\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}}+1\right)}$$
$$+\left(\frac{1-\alpha-\beta}{1-\beta}\right)\frac{\sigma-1}{\sigma y_t\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}}+1\right)} + \frac{(\sigma-1)\Gamma}{\sigma y_t^{\frac{1}{\sigma}}\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)} - \frac{1}{y_t}$$

which after several lines of algebra (see the Annex) simplifies to:

$$\frac{\partial f}{\partial y_t} = -\frac{(1-\alpha-\beta)(1-\alpha+\sigma\alpha)+(1-\beta)\Gamma y_t^{\frac{\sigma-1}{\sigma}}}{\sigma(1-\beta)(1-\alpha-\beta)y_t\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}}+1\right)} < 0 \qquad \text{(B27)}$$

Lastly, we calculate $\partial f\left(y_t, \bar{N}_{M,t}, \bar{N}_{S,t}, \mathbf{\Omega}\right)/\partial \Omega_j$ from (B25) for selected parameters Ω_j, and insert the results and $\partial f/\partial y_t < 0$ from (B27) into $\frac{\partial y_t}{\partial \mathbf{\Omega}} \approx -\frac{\partial f\left(y_t,\bar{N}_{M,t},\bar{N}_{S,t},\mathbf{\Omega}\right)/\partial \mathbf{\Omega}}{\partial f/\partial y_t}$ and $\frac{\partial y_t}{\partial \bar{N}_{i,t}} = -\frac{\partial f\left(y_t,\bar{N}_{M,t},\bar{N}_{S,t},\mathbf{\Omega}\right)/\partial \bar{N}_{i,t}}{\partial f/\partial y_t}$ from the implicit function theorem, to yield the results shown in Proposition 7 :

$$\partial y_t/\partial \bar{N}_{M,t} = -\frac{1}{\bar{N}_{M,t}(\partial f/\partial y_t)} > 0$$

$$\partial y_t/\partial \bar{N}_{S,t} = \frac{\left(\frac{1-\beta}{1-\alpha-\beta}\right)}{\bar{N}_{S,t}(\partial f/\partial y_t)} < 0$$

$$\partial y_t/\partial \bar{E}_M = -\frac{\left(\frac{\alpha}{1-\beta}\right)}{\bar{E}_M(\partial f/\partial y_t)} > 0$$

$$\partial y_t/\partial \bar{e}_S = -\frac{\left(\frac{\alpha}{1-\alpha-\beta}\right)}{\bar{e}_S(\partial f/\partial y_t)} > 0$$

$$\partial y_t/\partial L_t = -\frac{\left(\frac{1-\alpha-\beta}{1-\beta}\right)-1}{L_t(\partial f/\partial y_t)} < 0$$

And since (B25) $\Rightarrow \frac{\partial y_t}{\partial \ln(\bar{N}_{S,t})} = \left(\frac{1-\beta}{1-\alpha-\beta}\right)\frac{\partial y_t}{\partial \ln(\bar{N}_{M,t})} < 0$, we also have that equal increases in $\ln(\bar{N}_{M,t})$ and $\ln(\bar{N}_{S,t})$, i.e. a larger total number of varieties $\bar{N}_{M,t} + \bar{N}_{S,t}$ which leaves the ratio $\bar{N}_t$ unchanged, will increase industrialisation (lower y_t). ■

Comparative static effects on the energy price ratio, e_t

Finding the effects of parameters on e_t starts by transforming (29) with (25) into $\ln(y_t) = \sigma\ln(\Gamma) + (1-\beta)\sigma\left(\ln(N_{M,t}) - \ln(N_{S,t})\right) - \alpha\sigma\ln(e_t)$ and using this to substitute for $\ln(y_t)$ in equation (B25) with dependencies of e_t. After much algebra (see the Annex) this yields:

$$\begin{aligned}
&f\left(e_t, \bar{N}_{M,t}, \bar{N}_{S,t}, \mathbf{\Omega}\right) \\
&\quad = -\sigma\ln(\Gamma) + [1-(1-\beta)\sigma]\ln(\bar{N}_{M,t}) - \left(\frac{1-\beta}{1-\alpha-\beta} - (1-\beta)\sigma\right)\ln(\bar{N}_{S,t}) \\
&\qquad - \left(\frac{\beta}{1-\beta}\right)[\ln(\Gamma) + (1-\beta)\ln(\bar{N}_t) - \alpha\ln(e_t)] \\
&\qquad + \left(\frac{\beta}{1-\beta}\right)\left\{\frac{1}{\sigma-1}\ln\left(\Gamma^\sigma \bar{N}_t^{(1-\beta)(\sigma-1)} e_t^{-\alpha(\sigma-1)} + 1\right) + \frac{\sigma}{\sigma-1}\ln(1-\gamma)\right\} \\
&\qquad - \left(\frac{\alpha+\beta}{1-\alpha-\beta}\right)\left\{\frac{1}{\sigma-1}\ln\left(\Gamma^\sigma \bar{N}_t^{(1-\beta)(\sigma-1)} e_t^{-\alpha(\sigma-1)} + 1\right) + \frac{\sigma}{\sigma-1}\ln(1-\gamma)\right\} \qquad \text{(B28)} \\
&\qquad + \left(\frac{\alpha}{1-\beta}\right)\ln(\bar{E}_M) - \left(\frac{\alpha}{1-\alpha-\beta}\right)\ln\left(\frac{\alpha}{\beta\bar{e}_S}\right) \\
&\qquad + \left(\frac{1-\alpha-\beta}{1-\beta}\right)\{\ln(L_t) + \ln(\Gamma) + (\sigma-1)[\ln(\Gamma) + (1-\beta)\ln(\bar{N}_t) - \alpha\ln(e_t)]\} \\
&\qquad - \left(\frac{1-\alpha-\beta}{1-\beta}\right)\ln\left(\Gamma^\sigma \bar{N}_t^{(1-\beta)(\sigma-1)} e_t^{-\alpha(\sigma-1)} + 1\right) - \ln(L_t) \\
&\qquad + \ln\left(\Gamma^\sigma \bar{N}_t^{(1-\beta)(\sigma-1)} e_t^{-\alpha(\sigma-1)} + 1\right) + \alpha\sigma\ln(e_t)
\end{aligned}$$

Equation (29), in the form $\ln(y_t) = \sigma\ln(\Gamma) + (1-\beta)\sigma\ln(\bar{N}_t) - \alpha\sigma\ln(e_t)$, means that $\frac{\partial f}{\partial e_t} = -\alpha\sigma\frac{\partial f}{\partial y_t} > 0$. This, combined with the only two tractable partial derivatives from (B28), gives the comparative static results shown in Proposition 8:

$$\partial e_t/\partial \bar{E}_M = -\frac{\left(\frac{\alpha}{1-\beta}\right)}{\bar{E}_M(\partial f/\partial e_t)} < 0$$

$$\partial e_t/\partial L_t = -\frac{\left(\frac{1-\alpha-\beta}{1-\beta}\right)-1}{L_1(\partial f/\partial e_t)} > 0$$

Lastly, since (B28) $\Rightarrow \left.\frac{\partial e_t}{\partial \ln(\bar{N}_{M,t})}\right|_{\bar{N}_t \text{ constant}} = \frac{\frac{1-\beta}{1-\alpha-\beta}-(1-\beta)\sigma}{1-(1-\beta)\sigma} \left.\frac{\partial e_t}{\partial \ln(\bar{N}_{S,t})}\right|_{\bar{N}_t \text{ constant}} > 0$, we also have that equal rises in $\ln(\bar{N}_{M,t})$ and $\ln(\bar{N}_{S,t})$, i.e. a larger total number of varieties $\bar{N}_{M,t} + \bar{N}_{S,t}$ which keeps the ratio $\bar{N}_t$ constant, will raise the energy price ratio e_t. ■

ANNEX (FOR ONLINE PUBLICATION)

Derivations in Appendix B2

Steps from (B7) to (B8)

Take logs then differences of (B7):

$$[1+\alpha(\sigma-1)]\Delta\ln(y_t) = \sigma(1-\beta)\,\Delta\left(\ln\left(N_{M,t}\right)-\ln\left(N_{S,t}\right)\right)-\frac{\alpha\sigma(1-\beta)}{1-\alpha-\beta}\Delta\ln\left(N_{S,t}\right)$$

$$-\alpha\sigma\,\Delta\ln(L_t)+\alpha\left(1-\frac{1}{\sigma-1}\left(\frac{1}{1-\alpha-\beta}\right)\right)\frac{\Gamma(\sigma-1)y_t^{\frac{-1}{\sigma}}\Delta y_t}{\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)}$$

$$[1+\alpha(\sigma-1)]\Delta\ln(y_t) = \sigma(1-\beta)\,\Delta\left(\ln N_{M,t}-\ln N_{S,t}\right)-\frac{\alpha\sigma(1-\beta)}{1-\alpha-\beta}\Delta\ln\left(N_{S,t}\right)$$

$$-\alpha\sigma\,\Delta\ln(L_t)+\alpha\left[\sigma-1-\left(\frac{1}{1-\alpha-\beta}\right)\right]\frac{\Gamma y_t^{\frac{\sigma-1}{\sigma}}\Delta y_t/y_t}{\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)}$$

Substituting $\Delta y_t/y_t = \Delta\ln(y_t)$ and rearranging:

$$[1+\alpha(\sigma-1)]\frac{\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)}{1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}}\Delta\ln(y_t)$$

$$= \sigma(1-\beta)\,\Delta\ln\left(N_{M,t}\right)-\left[\sigma(1-\beta)+\frac{\alpha\sigma(1-\beta)}{1-\alpha-\beta}\right]\Delta\ln\left(N_{S,t}\right)$$

$$-\alpha\sigma\,\Delta\ln(L_t)+\alpha\left[\sigma-1-\left(\frac{1}{1-\alpha-\beta}\right)\right]\frac{\Gamma y_t^{\frac{\sigma-1}{\sigma}}\Delta\ln(y_t)}{\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)}$$

$$\Rightarrow \frac{1+\alpha(\sigma-1)+\left[1+\alpha(\sigma-1)-\alpha(\sigma-1)+\frac{\alpha}{1-\alpha-\beta}\right]\Gamma y_t^{\frac{\sigma-1}{\sigma}}}{1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}}\Delta\ln(y_t)$$

$$= \sigma(1-\beta)\,\Delta\ln\left(N_{M,t}\right)-\sigma(1-\beta)\,\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Delta\ln\left(N_{S,t}\right)-\alpha\sigma\,\Delta\ln(L_t)$$

$$\Rightarrow \frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}}{1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}}\Delta\ln(y_t) = \sigma(1-\beta)\,\Delta\ln\left(N_{M,t}\right) - \sigma\,\frac{(1-\beta)^2}{1-\alpha-\beta}\Delta\ln\left(N_{S,t}\right) - \alpha\sigma\,\Delta\ln(L_t) \quad \text{(B8)}$$

Steps from (B8) to (27)

To progress from (B8), we need to replace $\Delta\ln\left(N_{S,t}\right)$, using this:

$$n_t = \frac{\Delta N_{M,t}/N_{M,t}}{\Delta N_{S,t}/N_{S,t}} \Rightarrow n_t\,\Delta\ln\left(N_{S,t}\right) = \Delta\ln\left(N_{M,t}\right) = \Delta\ln\left(N_t N_{S,t}\right) = \Delta\ln(N_t) + \Delta\ln\left(N_{S,t}\right)$$

$$\Rightarrow \Delta\ln\left(N_{S,t}\right) = \frac{\Delta\ln(N_t)}{n_t-1} \Rightarrow \Delta\ln\left(N_{M,t}\right) = n_t\Delta\ln\left(N_{S,t}\right) = \frac{n_t\Delta\ln(N_t)}{n_t-1}$$

So (B8) becomes:

$$\frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}}{1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}}\Delta\ln(y_t) = \sigma(1-\beta)\left(\frac{n_t\Delta\ln(N_t)}{n_t-1}\right) - \sigma\,\frac{(1-\beta)^2}{1-\alpha-\beta}\left(\frac{\Delta\ln(N_t)}{n_t-1}\right) - \alpha\sigma\Delta\ln(L_t)$$

$$= \frac{\sigma(1-\beta)}{n_t-1}\left(n_t - \frac{1-\beta}{1-\alpha-\beta}\right)\Delta\ln(N_t) - \alpha\sigma\Delta\ln(L_t)$$

Multiplying both sides by $n_t - 1$:

$$\Rightarrow \frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}}{\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)}(n_t-1)\Delta\ln(y_t) = \sigma(1-\beta)\left(n_t - \frac{1-\beta}{1-\alpha-\beta}\right)\Delta\ln(N_t) - (n_t-1)\alpha\sigma\Delta\ln(L_t) \quad (27)$$

Derivations in Appendix B3

$$\sigma\left[\frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)N_t n_t^m}{1+N_t n_t^m}\right](n_t-1)[(1-\beta)\Delta\ln(N_t) - \alpha\Delta\ln(e_t)] = \sigma\left[(1-\beta)\left(n_t - \frac{1-\beta}{1-\alpha-\beta}\right)\Delta\ln(N_t) - (n_t-1)\alpha\Delta\ln(L_t)\right] \quad \text{(B9)}$$

$$\Rightarrow (1-\beta)\left[\frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)N_t n_t^m}{1+N_t n_t^m}(n_t-1)-\left(n_t-\frac{1-\beta}{1-\alpha-\beta}\right)\right]\Delta\ln(N_t)$$

$$+(n_t-1)\alpha\Delta\ln(L_t)$$

$$=\frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)N_t n_t^m}{1+N_t n_t^m}(n_t-1)\alpha\,\Delta\ln(e_t)$$

$$\Rightarrow\left(\frac{\frac{1+\alpha(\sigma-1)}{1-\beta}+\frac{N_t n_t^m}{1-\alpha-\beta}}{1+N_t n_t^m}\right)(n_t-1)\alpha\,\Delta\ln(e_t)$$

$$=\left[\left(\frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)N_t n_t^m}{1+N_t n_t^m}-1\right)n_t+\frac{1-\beta}{1-\alpha-\beta}\right.$$

$$\left.-\frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)N_t n_t^m}{1+N_t n_t^m}\right]\Delta\ln(N_t)+\frac{(n_t-1)\alpha}{(1-\beta)}\Delta\ln(L_t)$$

$$=\left[\left(\frac{1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)N_t n_t^m-1-N_t n_t^m}{1+N_t n_t^m}\right)n_t\right.$$

$$\left.+\frac{\left(\frac{1-\beta}{1-\alpha-\beta}\right)(1+N_t n_t^m-N_t n_t^m)-1-\alpha(\sigma-1)}{1+N_t n_t^m}\right]\Delta\ln(N_t)$$

$$+\frac{(n_t-1)\alpha}{(1-\beta)}\Delta\ln(L_t)$$

$$=\left[\left(\frac{\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)N_t n_t^m-N_t n_t^m}{1+N_t n_t^m}\right)n_t+\frac{\left(\frac{1-\beta}{1-\alpha-\beta}\right)-1-\alpha(\sigma-1)}{1+N_t n_t^m}\right]\Delta\ln(N_t)$$

$$+\frac{(n_t-1)\alpha}{(1-\beta)}\Delta\ln(L_t)$$

Now substitute $\frac{1-\beta}{1-\alpha-\beta}-1=\frac{\alpha}{1-\alpha-\beta}=\alpha(\sigma^\dagger-1)$, which makes this expression:

$$=\left[\left(\frac{\alpha(\sigma-1)+\alpha(\sigma^\dagger-1)N_t n_t^m}{1+N_t n_t^m}\right)n_t+\frac{\alpha(\sigma^\dagger-1)-\alpha(\sigma-1)}{1+N_t n_t^m}\right]\Delta\ln(N_t)$$

$$+\frac{(n_t-1)\alpha}{(1-\beta)}\Delta\ln(L_t)$$

$$\Rightarrow \left(\frac{\frac{1+\alpha(\sigma-1)}{1-\beta} + \frac{N_t n_t^m}{1-\alpha-\beta}}{1+N_t n_t^m} \right) (n_t - 1)\alpha\, \Delta\ln(e_t)$$

$$= [\{\sigma - 1 + (\sigma^{\dagger} - 1) N_t n_t^m\} n_t - (\sigma - \sigma^{\dagger})] \frac{\alpha\, \Delta\ln(N_t)}{1 + N_t n_t^m} + \frac{(n_t - 1)\alpha}{(1-\beta)} \Delta\ln(L_t)$$

$$\Rightarrow \left(\frac{1+\alpha(\sigma-1)}{1-\beta} + \frac{N_t n_t^m}{1-\alpha-\beta} \right) (n_t - 1)\, \Delta\ln(e_t)$$
$$= [\{\sigma - \sigma^{\dagger} + (\sigma^{\dagger} - 1)(1 + N_t n_t^m)\} n_t - (\sigma - \sigma^{\dagger})]\, \Delta\ln(N_t) \qquad (29)$$
$$+ \left(\frac{1 + N_t n_t^m}{1-\beta} \right) (n_t - 1)\, \Delta\ln(L_t)$$

Derivations in Appendix B4

Algebra for Lemma 1

We need to show that:

$$\frac{1-\beta}{1-\alpha-\beta} \leq \bar{n} \leq \frac{(\sigma-1)(1-\beta)-1}{(\sigma-1)(1-\alpha-\beta)-1} \text{ and } \sigma > \sigma^{\dagger}, \text{i.e.}\ \sigma - 1 > \frac{1}{1-\alpha-\beta},$$

$$\Rightarrow \frac{\Delta\ln(N_t)}{\Delta\ln(y_t)} = \frac{\left[1 + \alpha(\sigma-1) + \left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right](\bar{n} - 1)}{\left(1 + \Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)\sigma(1-\beta)\left(\bar{n} - \frac{1-\beta}{1-\alpha-\beta}\right)} > \frac{\sigma-1}{\sigma}.$$

$$\text{Well, this inequality } \Leftrightarrow \left[1 + \alpha(\sigma-1) + \left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right](\bar{n} - 1)$$
$$> \left(1 + \Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)(\sigma-1)(1-\beta)\left(\bar{n} - \frac{1-\beta}{1-\alpha-\beta}\right)$$

$$\Leftrightarrow [1 + \alpha(\sigma-1)](\bar{n} - 1) + \left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}(\bar{n} - 1)$$
$$> (\sigma-1)(1-\beta)\left(\bar{n} - \frac{1-\beta}{1-\alpha-\beta}\right)$$
$$+ \Gamma y_t^{\frac{\sigma-1}{\sigma}}(\sigma-1)(1-\beta)\left(\bar{n} - \frac{1-\beta}{1-\alpha-\beta}\right)$$

$$\Leftrightarrow \left(\frac{1-\beta}{1-\alpha-\beta}\right)(\bar{n} - 1)\Gamma y_t^{\frac{\sigma-1}{\sigma}} - (\sigma-1)(1-\beta)\left(\bar{n} - \frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}$$
$$> (\sigma-1)(1-\beta)\left(\bar{n} - \frac{1-\beta}{1-\alpha-\beta}\right) - [1 + \alpha(\sigma-1)](\bar{n} - 1)$$

$$\Leftrightarrow \left[\left(\frac{1-\beta}{1-\alpha-\beta}\right)(\bar{n}-1)-(\sigma-1)(1-\beta)\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)\right]\Gamma y_t^{\frac{\sigma-1}{\sigma}}$$
$$> (\sigma-1)(1-\beta)\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)-[1+\alpha(\sigma-1)](\bar{n}-1)$$

So we can prove the $\frac{\Delta\ln(N_t)}{\Delta\ln(y_t)}$ inequality true by showing [LHS] >0 and RHS < 0 as follows:

$$[\text{LHS}] = \left(\frac{1-\beta}{1-\alpha-\beta}\right)(\bar{n}-1)-(\sigma-1)(1-\beta)\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)$$

which (because $\sigma-1>\frac{1}{1-\alpha-\beta}$)

$$> \left(\frac{1-\beta}{1-\alpha-\beta}\right)(\bar{n}-1)-\left(\frac{1-\beta}{1-\alpha-\beta}\right)\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)$$

$$= \left(\frac{1-\beta}{1-\alpha-\beta}\right)\left(\frac{1-\beta}{1-\alpha-\beta}-1\right)>0$$

$$\text{RHS} = (\sigma-1)(1-\beta)\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)-[1+\alpha(\sigma-1)](\bar{n}-1)$$

$$= [(\sigma-1)(1-\alpha-\beta)-1]\bar{n}-(\sigma-1)(1-\beta)\frac{1-\beta}{1-\alpha-\beta}+1+\alpha(\sigma-1)$$

which (because $\bar{n}\leq\frac{(\sigma-1)(1-\beta)-1}{(\sigma-1)(1-\alpha-\beta)-1}$)

$$\leq [(\sigma-1)(1-\alpha-\beta)-1]\frac{(\sigma-1)(1-\beta)-1}{(\sigma-1)(1-\alpha-\beta)-1}+(\sigma-1)\frac{\alpha(1-\alpha-\beta)-(1-\beta)^2}{1-\alpha-\beta}+1$$

$$= (\sigma-1)(1-\beta)-1-(\sigma-1)\frac{\alpha^2+(1-\alpha-\beta)(1-\beta)}{1-\alpha-\beta}+1$$

$$= (\sigma-1)(1-\beta)-1-\frac{(\sigma-1)\alpha^2}{1-\alpha-\beta}-[(\sigma-1)(1-\beta)-1] = -\frac{(\sigma-1)\alpha^2}{1-\alpha-\beta}<0.$$

Detailed proofs for Lemma 2

(ii) Proof that (B12) > (38), i.e. that

$$\frac{\frac{1-\beta}{1-\alpha-\beta}\left(y^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)(\sigma-1)(1-\beta)-\left[\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma+\{1+\alpha(\sigma-1)\}y^{-\frac{\sigma-1}{\sigma}}\right]}{\left(y^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)(\sigma-1)(1-\beta)-\left[\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma+\{1+\alpha(\sigma-1)\}y^{-\frac{\sigma-1}{\sigma}}\right]}$$
$$> \frac{(\sigma-1)(1-\beta)-1}{(\sigma-1)(1-\alpha-\beta)-1}$$

Now let $\left(y^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)(\sigma-1)(1-\beta)\equiv W,\ \left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma+\{1+\alpha(\sigma-1)\}y^{-\frac{\sigma-1}{\sigma}}\equiv X$

then we must show that: $\dfrac{\left(\frac{1-\beta}{1-\alpha-\beta}\right)W-X}{W-X} > \dfrac{(\sigma-1)(1-\beta)-1}{(\sigma-1)(1-\alpha-\beta)-1}$

$$\Leftrightarrow \left[\left(\frac{1-\beta}{1-\alpha-\beta}\right)W-X\right][(\sigma-1)(1-\alpha-\beta)-1] > (W-X)[(\sigma-1)(1-\beta)-1]$$

$$\Leftrightarrow (\sigma-1)(1-\beta)W-(\sigma-1)(1-\alpha-\beta)X-\left(\frac{1-\beta}{1-\alpha-\beta}\right)W+X$$
$$> (\sigma-1)(1-\beta)W-W-(\sigma-1)(1-\beta)X+X$$

$$\Leftrightarrow -(\sigma-1)(1-\alpha-\beta)X-\left(\frac{1-\beta}{1-\alpha-\beta}\right)W > -W-(\sigma-1)(1-\beta)X$$

$$\Leftrightarrow (\sigma-1)\alpha X > \left(\frac{1-\beta}{1-\alpha-\beta}-1\right)W = \left(\frac{\alpha}{1-\alpha-\beta}\right)W$$

$$\Leftrightarrow (\sigma-1)X > \frac{W}{1-\alpha-\beta}$$

$$\Leftrightarrow (\sigma-1)\left[\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma+\{1+\alpha(\sigma-1)\}y^{-\frac{\sigma-1}{\sigma}}\right] > \frac{\left(y^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)(\sigma-1)(1-\beta)}{1-\alpha-\beta}$$

$$\Leftrightarrow \left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma+\{1+\alpha(\sigma-1)\}y^{-\frac{\sigma-1}{\sigma}} > y^{-\frac{\sigma-1}{\sigma}}\left(\frac{1-\beta}{1-\alpha-\beta}\right)+\Gamma\left(\frac{1-\beta}{1-\alpha-\beta}\right)$$

$0<y<\infty$ means we can divide by $y^{-\frac{\sigma-1}{\sigma}}$ without changing the inequality. So we must show

$$1+\alpha(\sigma-1) > \frac{1-\beta}{1-\alpha-\beta} \Leftrightarrow \alpha(\sigma-1) > \frac{\alpha}{1-\alpha-\beta} \Leftrightarrow \sigma > 1+\frac{1}{1-\alpha-\beta} = \sigma^{\dagger}$$

The last statement is true, given the High Substitutability condition assumed in this case, so reversing the chain of implications means we have proved (B12) > (38).

(iii) Proof that the $\Delta\ln(n_t)=0$ locus is locally steeper than the development path:

The $\Delta\ln(n_t)=0$ locus in (N,y)-space is the curve:

$$\frac{\left[1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y^{\frac{\sigma-1}{\sigma}}\right]\left(\Gamma^{\frac{1}{m}}y^{\frac{\sigma-1}{m\sigma}}N^{\frac{-1}{m}}-1\right)}{\left(1+\Gamma y^{\frac{\sigma-1}{\sigma}}\right)\sigma(1-\beta)\left(\Gamma^{\frac{1}{m}}y^{\frac{\sigma-1}{m\sigma}}N^{\frac{-1}{m}}-\frac{1-\beta}{1-\alpha-\beta}\right)} = \frac{\sigma-1}{\sigma} \tag{B13}$$

To compute the log-slope, $\frac{\Delta\ln(N)}{\Delta\ln(y)}$, of this locus, first set the difference of cross-products = 0:

$$\sigma\left(1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y^{\frac{\sigma-1}{\sigma}}\right)\left(\Gamma^{\frac{1}{m}}y^{\frac{\sigma-1}{m\sigma}}N^{\frac{-1}{m}}-1\right)-$$
$$(\sigma-1)\left(1+\Gamma y^{\frac{\sigma-1}{\sigma}}\right)\sigma(1-\beta)\left(\Gamma^{\frac{1}{m}}y^{\frac{\sigma-1}{m\sigma}}N^{\frac{-1}{m}}-\frac{1-\beta}{1-\alpha-\beta}\right)=0 \tag{B14}$$

To expand Δ(B14) = 0, we will use these two first differences:

$$\Delta\left[\Gamma y^{\frac{\sigma-1}{\sigma}}\right] = \Gamma\frac{\sigma-1}{\sigma}y^{\frac{-1}{\sigma}}\Delta y = \Gamma\frac{\sigma-1}{\sigma}y^{\frac{\sigma-1}{\sigma}}\Delta\ln(y), \qquad \text{and}$$

$$\Delta\left(\Gamma^{\frac{1}{m}}y^{\frac{\sigma-1}{m\sigma}}N^{\frac{-1}{m}}\right) = \Gamma^{\frac{1}{m}}\left(\frac{\sigma-1}{m\sigma}y^{\frac{\sigma-1}{m\sigma}-1}N^{\frac{-1}{m}}\Delta y - \frac{1}{m}y^{\frac{\sigma-1}{m\sigma}}N^{\frac{-1}{m}-1}\Delta N\right)$$

$$= \frac{\Gamma^{\frac{1}{m}}y^{\frac{\sigma-1}{m\sigma}}N^{\frac{-1}{m}}}{m}\left(\frac{\sigma-1}{\sigma}\Delta\ln(y) - \Delta\ln(N)\right) = \frac{n}{m}\left(\frac{\sigma-1}{\sigma}\Delta\ln(y) - \Delta\ln(N)\right).$$

$$\Longrightarrow 0 = \Delta(\text{M7}) = \sigma\left(1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y^{\frac{\sigma-1}{\sigma}}\right)\frac{n}{m}\left(\frac{\sigma-1}{\sigma}\Delta\ln(y) - \Delta\ln(N)\right)$$

$$+\sigma\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma\frac{\sigma-1}{\sigma}y^{\frac{\sigma-1}{\sigma}}(n-1)\Delta\ln(y)$$

$$-(\sigma-1)\Gamma\frac{\sigma-1}{\sigma}y^{\frac{\sigma-1}{\sigma}}\Delta\ln(y)\sigma(1-\beta)\left(n-\frac{1-\beta}{1-\alpha-\beta}\right)$$

$$-(\sigma-1)\left(1+\Gamma y^{\frac{\sigma-1}{\sigma}}\right)\sigma(1-\beta)\frac{n}{m}\left(\frac{\sigma-1}{\sigma}\Delta\ln(y) - \Delta\ln(N)\right)$$

$$= \sigma\left(1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y^{\frac{\sigma-1}{\sigma}}\right)\frac{n}{m}\left(\frac{\sigma-1}{\sigma}\Delta\ln(y) - \Delta\ln(N)\right)$$

$$+\left(\frac{1-\beta}{1-\alpha-\beta}\right)(\sigma-1)\Gamma y^{\frac{\sigma-1}{\sigma}}(n-1)\Delta\ln(y)$$

$$-(\sigma-1)^2(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\left(n-\frac{1-\beta}{1-\alpha-\beta}\right)\Delta\ln(y)$$

$$-(\sigma-1)\left(1+\Gamma y^{\frac{\sigma-1}{\sigma}}\right)\sigma(1-\beta)\frac{n}{m}\left(\frac{\sigma-1}{\sigma}\Delta\ln(y) - \Delta\ln(N)\right)$$

$$= \left[1+\alpha(\sigma-1)+\frac{(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}}{1-\alpha-\beta} - (\sigma-1)\left(1+\Gamma y^{\frac{\sigma-1}{\sigma}}\right)(1-\beta)\right]\frac{\sigma n}{m}\left(\frac{\sigma-1}{\sigma}\Delta\ln(y) - \Delta\ln(N)\right)$$

$$+\left(\frac{1}{1-\alpha-\beta}\right)(n-1)(\sigma-1)(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\Delta\ln(y)$$

$$-(\sigma-1)\left(n-\frac{1-\beta}{1-\alpha-\beta}\right)(\sigma-1)(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\Delta\ln(y)$$

$$= \Bigg[1 + \alpha(\sigma - 1) - (\sigma - 1)(1 - \beta) + \frac{(1 - \beta)\Gamma y^{\frac{\sigma-1}{\sigma}}}{1 - \alpha - \beta} - (\sigma - 1)(1 - \beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\Bigg]\frac{\sigma n}{m}\left(\frac{\sigma - 1}{\sigma}\Delta\ln(y) - \Delta\ln(N)\right)$$

$$+ \left[\left(\frac{1}{1 - \alpha - \beta}\right)(n - 1) - (\sigma - 1)\left(n - 1 - \frac{\alpha}{1 - \alpha - \beta}\right)\right](\sigma - 1)(1 - \beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\Delta\ln(y)$$

$$= \left[1 - (\sigma - 1)(1 - \alpha - \beta) - \left(\sigma - 1 - \frac{1}{1 - \alpha - \beta}\right)(1 - \beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right]\frac{\sigma n}{m}\left(\frac{\sigma - 1}{\sigma}\Delta\ln(y) - \Delta\ln(N)\right)$$

$$= \left[-(1 - \alpha - \beta)(\sigma - \sigma^{\dagger}) - (\sigma - \sigma^{\dagger})(1 - \beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right]\frac{\sigma n}{m}\left(\frac{\sigma - 1}{\sigma}\Delta\ln(y) - \Delta\ln(N)\right)$$

$$+ \left[-(\sigma - \sigma^{\dagger})(n - 1) + (\sigma - 1)\left(\frac{\alpha}{1 - \alpha - \beta}\right)\right](\sigma - 1)(1 - \beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\Delta\ln(y)$$

$$= -\left[(1 - \alpha - \beta) + (1 - \beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right](\sigma - \sigma^{\dagger})\frac{n}{m}(\sigma - 1)\Delta\ln(y)$$

$$-\left[(\sigma - \sigma^{\dagger})(n - 1) - (\sigma - 1)\left(\frac{\alpha}{1 - \alpha - \beta}\right)\right](\sigma - 1)(1 - \beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\Delta\ln(y)$$

$$+ \left[1 - \alpha - \beta + (1 - \beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right](\sigma - \sigma^{\dagger})\frac{\sigma n}{m}\Delta\ln(N)$$

$$\Rightarrow \frac{\Delta\ln(N)}{\Delta\ln(y)} = \frac{\left[(1-\alpha-\beta)+(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right](\sigma-\sigma^{\dagger})\frac{n}{m}(\sigma-1) + \left[(\sigma-\sigma^{\dagger})(n-1)-(\sigma-1)\left(\frac{\alpha}{1-\alpha-\beta}\right)\right](\sigma-1)(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}}{\left[1-\alpha-\beta+(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right](\sigma-\sigma^{\dagger})\frac{\sigma n}{m}}$$

is the log-slope of the $\Delta\ln(n_t) = 0$ locus in (*y*,*N*) space.

We next show that the log-slope of the locus exceeds the path's slope, i.e from (B10):

$$\frac{\left[1-\alpha-\beta+(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right](\sigma-\sigma^{\dagger})\frac{n}{m}(\sigma-1) + \left[(\sigma-\sigma^{\dagger})(n-1)-(\sigma-1)\left(\frac{\alpha}{1-\alpha-\beta}\right)\right](\sigma-1)(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}}{\left[1-\alpha-\beta+(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right](\sigma-\sigma^{\dagger})\frac{\sigma n}{m}} - \frac{\sigma-1}{\sigma} > 0$$

So we need to show the following:

$$\left\{\left[1 - \alpha - \beta + (1 - \beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right](\sigma - \sigma^{\dagger})\frac{n}{m}(\sigma - 1) + \left[(\sigma - \sigma^{\dagger})(n - 1) - (\sigma - 1)\left(\frac{\alpha}{1-\alpha-\beta}\right)\right](\sigma - 1)(1 - \beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right\}\sigma > \left[1 - \alpha - \beta + (1 - \beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right](\sigma - \sigma^{\dagger})(\sigma - 1)\frac{\sigma n}{m}$$

$$\Leftrightarrow \left[1-\alpha-\beta+(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right](\sigma-\sigma^{\dagger})\frac{n}{m}(\sigma-1) + \left[(\sigma-\sigma^{\dagger})(n-1)-(\sigma-1)\left(\frac{\alpha}{1-\alpha-\beta}\right)\right](\sigma-1)(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}} > \left[1-\alpha-\beta+(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}}\right](\sigma-\sigma^{\dagger})(\sigma-1)\frac{n}{m}$$

$$\Leftrightarrow \left[(\sigma-\sigma^{\dagger})(n-1)-(\sigma-1)\left(\frac{\alpha}{1-\alpha-\beta}\right)\right](\sigma-1)(1-\beta)\Gamma y^{\frac{\sigma-1}{\sigma}} > 0$$

$$\Leftrightarrow (\sigma-\sigma^{\dagger})(n-1) = \left(\sigma-1-\frac{1}{1-\alpha-\beta}\right)(n-1) > (\sigma-1)\left(\frac{\alpha}{1-\alpha-\beta}\right)$$

$$\Leftrightarrow (\sigma-1)\left(n-1-\frac{\alpha}{1-\alpha-\beta}\right) > \frac{n-1}{1-\alpha-\beta}$$

$$\Leftrightarrow \sigma-1 > \frac{\frac{n-1}{1-\alpha-\beta}}{n-1-\frac{\alpha}{1-\alpha-\beta}} = \frac{n-1}{(n-1)(1-\alpha-\beta)-\alpha} \quad \Leftrightarrow \sigma > 1+\frac{n-1}{(n-1)(1-\alpha-\beta)-\alpha} = 1+\frac{1}{1-\alpha-\beta-\frac{\alpha}{(n-1)}}$$

We now use the result that:

$$1+\frac{1}{1-\alpha-\beta-\frac{\alpha}{(n_\infty-1)}} = 1+\frac{1}{1-\alpha-\beta-\frac{\alpha}{\left[\frac{\left(\frac{1-\beta}{1-\alpha-\beta}\right)\left(\sigma-1-\frac{1}{1-\beta}\right)}{\sigma-1-\frac{1}{1-\alpha-\beta}}-1\right]}}$$

$$= 1+\frac{1}{1-\alpha-\beta-\frac{\alpha}{\left[\frac{(\sigma-1)(1-\beta)-1}{(\sigma-1)(1-\alpha-\beta)-1}-1\right]}}$$

$$= 1+\frac{1}{1-\alpha-\beta-\frac{\alpha}{\left[\frac{(\sigma-1)(1-\beta)-(\sigma-1)(1-\alpha-\beta)}{(\sigma-1)(1-\alpha-\beta)-1}\right]}}$$

$$= 1+\frac{1}{1-\alpha-\beta-\frac{\alpha}{\left[\frac{(\sigma-1)\alpha}{(\sigma-1)(1-\alpha-\beta)-1}\right]}} = 1+\frac{1}{1-\alpha-\beta-\frac{(\sigma-1)(1-\alpha-\beta)-1}{(\sigma-1)}}$$

$$= 1+\frac{\sigma-1}{(\sigma-1)(1-\alpha-\beta)-(\sigma-1)(1-\alpha-\beta)+1} = \sigma$$

Hence we need to show $1+\frac{1}{1-\alpha-\beta-\frac{\alpha}{(n-1)}} < 1+\frac{1}{1-\alpha-\beta-\frac{\alpha}{(n_\infty-1)}} = \sigma$, and this last statement is true because $n_\infty := \frac{\left(\frac{1-\beta}{1-\alpha-\beta}\right)\left(\sigma-1-\frac{1}{1-\beta}\right)}{\sigma-1-\frac{1}{1-\alpha-\beta}}$ was shown earlier to be the *minimum* value of *n* on the $\Delta\ln(n_t) = 0$ locus, namely its asymptotic value, which is exceeded at any finite point on the locus.

Derivation in Appendix B6

Algebra for Proposition 3

As shown in the Proof of Lemma 1, showing

$$\frac{\left[1+\alpha(\sigma-1)+\left(\frac{1-\beta}{1-\alpha-\beta}\right)\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right](\bar{n}-1)}{\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)\sigma(1-\beta)\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)} > \frac{\sigma-1}{\sigma}$$

is the same as showing:

$$\left[\left(\frac{1-\beta}{1-\alpha-\beta}\right)(\bar{n}-1)-(1-\beta)\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)(\sigma-1)\right]\Gamma y_t^{\frac{\sigma-1}{\sigma}}$$
$$> (1-\beta)\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)(\sigma-1)-[1+\alpha(\sigma-1)](\bar{n}-1)$$

We prove this inequality is true by showing the [LHS] > 0 and the RHS < 0 as follows:

$$[\text{LHS}] = \left(\frac{1-\beta}{1-\alpha-\beta}\right)(\bar{n}-1)-(\sigma-1)(1-\beta)\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)$$

$$= \left(\frac{1-\beta}{1-\alpha-\beta}-(\sigma-1)(1-\beta)\right)\bar{n}-\left(\frac{1-\beta}{1-\alpha-\beta}\right)+(\sigma-1)(1-\beta)\left(\frac{1-\beta}{1-\alpha-\beta}\right)$$

$$= \left[\left(\frac{1}{1-\alpha-\beta}-(\sigma-1)\right)\bar{n}-\left(\frac{1}{1-\alpha-\beta}\right)+(\sigma-1)\left(\frac{1-\beta}{1-\alpha-\beta}\right)\right](1-\beta)$$

which (because $\sigma-1<\frac{1}{1-\alpha-\beta}$ and $\bar{n}>\frac{1-\beta}{1-\alpha-\beta}$)

$$> \left[\left(\frac{1}{1-\alpha-\beta}-(\sigma-1)\right)\frac{1-\beta}{1-\alpha-\beta}+\left(\sigma-1-\frac{1}{1-\beta}\right)\left(\frac{1-\beta}{1-\alpha-\beta}\right)\right](1-\beta)$$

$$= \left(\frac{1}{1-\alpha-\beta}-\frac{1}{1-\beta}\right)\frac{(1-\beta)^2}{1-\alpha-\beta} > 0.$$

$$\text{RHS} = (1-\beta)\left(\bar{n}-\frac{1-\beta}{1-\alpha-\beta}\right)(\sigma-1)-[1+\alpha(\sigma-1)](\bar{n}-1)$$

$$= [(1-\alpha-\beta)(\sigma-1)-1]\bar{n}-(1-\beta)\frac{1-\beta}{1-\alpha-\beta}(\sigma-1)+1$$

which (again because $\sigma-1<\frac{1}{1-\alpha-\beta}$ and $\bar{n}>\frac{1-\beta}{1-\alpha-\beta}$)

$$< [(1-\alpha-\beta)(\sigma-1)-1]\frac{1-\beta}{1-\alpha-\beta}-(1-\beta)\frac{1-\beta}{1-\alpha-\beta}(\sigma-1)+1$$

$$= \left[(1-\alpha-\beta)-(1-\beta)\right](\sigma-1)\frac{1-\beta}{1-\alpha-\beta}-\frac{1-\beta}{1-\alpha-\beta}+1$$

$$= -\alpha(\sigma-1)\frac{1-\beta}{1-\alpha-\beta}-\frac{\alpha}{1-\alpha-\beta}<0.$$

Derivations in Appendix B7

Derivation of (B17)

$$E_{S,t}(p_t,N_{S,t})=\left(\frac{\alpha N_{S,t}}{\beta\bar{e}_S}\right)^{\frac{1-\beta}{1-\alpha-\beta}}L_{S,t}(p_t)\,p_{S,t}{}^{\frac{1}{1-\alpha-\beta}}(p_t) \tag{16}$$

$$\Rightarrow E_{S,t}^{\frac{\alpha}{1-\beta}}=\left(\frac{\alpha N_{S,t}}{\beta\bar{e}_S}\right)^{\frac{\alpha}{1-\alpha-\beta}}L_{S,t}^{\frac{\alpha}{1-\beta}}\,p_{S,t}{}^{\frac{\alpha}{(1-\beta)(1-\alpha-\beta)}}$$

Inserting this into (B16) gives

$$n_{S,t}=\lambda N_{S,t-1}^{\frac{\mu+\nu-1}{h\nu}}p_{S,t}^{\frac{1}{h(1-\beta)}}N_{S,t}^{\frac{\alpha}{h(1-\alpha-\beta)}}L_{S,t}^{\frac{\alpha}{h(1-\beta)}}p_{S,t}^{\frac{\alpha}{h(1-\beta)(1-\alpha-\beta)}}L_{S,t}^{\frac{1-\alpha-\beta}{h(1-\beta)}}$$

$$=\lambda N_{S,t-1}^{\frac{\mu+\nu-1}{h\nu}}p_{S,t}^{\frac{1-\alpha-\beta+\alpha}{h(1-\beta)(1-\alpha-\beta)}}N_{S,t}^{\frac{\alpha}{h(1-\alpha-\beta)}}L_{S,t}^{\frac{1-\beta}{h(1-\beta)}}$$

$$\Rightarrow n_{S,t}=\lambda N_{S,t-1}^{\frac{\mu+\nu-1}{h\nu}}p_{S,t}{}^{\frac{1}{h(1-\alpha-\beta)}}N_{S,t}^{\frac{\alpha}{h(1-\alpha-\beta)}}L_{S,t}^{\frac{1}{h}} \tag{B17}$$

Derivations of (41) and (42):

$$\text{(A2)\& (A5)} \Rightarrow p_{M,t}^{\frac{1}{h(1-\beta)}}L_{M,t}^{\frac{1-\alpha-\beta}{h(1-\beta)}}$$

$$=\left[(1-\gamma)^{\frac{\sigma}{\sigma-1}}\Gamma y_t^{-\frac{1}{\sigma}}\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}}\right]^{\frac{1}{h(1-\beta)}}\left(\frac{L_t\Gamma y_t{}^{\frac{\sigma-1}{\sigma}}}{1+\Gamma y_t{}^{\frac{\sigma-1}{\sigma}}}\right)^{\frac{1-\alpha-\beta}{h(1-\beta)}}$$

$$=\lambda\left[y_t^{-\frac{1}{\sigma}}y_t^{\frac{\sigma-1}{\sigma}\frac{1}{\sigma-1}}\left(y_t{}^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)^{\frac{1}{\sigma-1}}\right]^{\frac{1}{h(1-\beta)}}\left(\frac{L_t}{y_t{}^{-\frac{\sigma-1}{\sigma}}+\Gamma}\right)^{\frac{1-\alpha-\beta}{h(1-\beta)}}$$

$$=\lambda\left(y_t{}^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)^{\left(\frac{1}{\sigma-1}\frac{1}{[h(1-\beta)]}-\frac{1-\alpha-\beta}{h(1-\beta)}\right)}L_t^{\frac{1-\alpha-\beta}{h(1-\beta)}}$$

$$=\lambda\left(y_t{}^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)^{\left(\frac{1}{1-\alpha-\beta}-(\sigma-1)\right)\frac{1-\alpha-\beta}{h(\sigma-1)(1-\beta)}}L_t^{\frac{1-\alpha-\beta}{h(1-\beta)}}$$

$$=\lambda\left(y_t{}^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)^{-\left[\frac{(\sigma-\sigma^{\dagger})(1-\alpha-\beta)}{h(\sigma-1)(1-\beta)}\right]}L_t^{\frac{1-\alpha-\beta}{h(1-\beta)}}$$

$$(A1)\&\ (A4) \Rightarrow p_{S,t}^{\frac{1}{h(1-\alpha-\beta)}} L_{S,t}^{\frac{1}{h}}$$

$$= \left[(1-\gamma)^{\frac{\sigma}{\sigma-1}}\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}}\right]^{\frac{1}{h(1-\alpha-\beta)}} \left[L_t\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{-1}\right]^{\frac{1}{h}}$$

$$= \lambda\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{\left[\frac{1}{(\sigma-1)(1-\alpha-\beta)}-1\right]\frac{1}{h}} L_t^{\frac{1}{h}}$$

$$= \lambda\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{\left(\frac{1}{1-\alpha-\beta}-\sigma+1\right)\frac{1}{h(\sigma-1)}} L_t^{\frac{1}{h}}$$

$$= \lambda\left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{-\frac{(\sigma-\sigma^{\dagger})}{h(\sigma-1)}} L_t^{\frac{1}{h}}$$

Then substitute the above into (B16) and (B17) respectively:

$$\begin{aligned} n_{M,t} &= \lambda N_{M,t-1}^{\frac{\nu+\mu-1}{h\nu}} p_{M,t}^{\frac{1}{h(1-\beta)}} \bar{E}_M^{\frac{\alpha}{h(1-\beta)}} L_{M,t}^{\frac{1-\alpha-\beta}{h(1-\beta)}} \\ &= \lambda N_{M,t-1}^{\frac{\nu+\mu-1}{h\nu}} \bar{E}_M^{\frac{\alpha}{h(1-\beta)}} \left(y_t^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)^{-\left[\frac{(\sigma-\sigma^{\dagger})(1-\alpha-\beta)(\sigma-\sigma^{\dagger})}{h(\sigma-1)(1-\beta)}\right]} L_t^{\frac{1-\alpha-\beta}{h(1-\beta)}} \end{aligned} \quad (41)$$

$$n_{S,t} = \lambda N_{S,t-1}^{\frac{\nu+\mu-1}{h\nu}} N_{S,t}^{\frac{\alpha}{h(1-\alpha-\beta)}} \left(1+\Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)^{-\frac{\sigma-\sigma^{\dagger}}{h(\sigma-1)}} L_t^{\frac{1}{h}} \quad (42)$$

Derivation of (B20)

From (14),

$$Y_{M,t} = \frac{1}{\beta} N_{M,t} p_{M,t}^{\frac{\beta}{1-\beta}} \bar{E}_M^{\frac{\alpha}{1-\beta}} L_{M,t}^{\frac{1-\alpha-\beta}{1-\beta}} \Longrightarrow \frac{Y_{M,t}}{L_{M,t}} = \frac{1}{\beta} N_{M,t} p_{M,t}^{\frac{\beta}{1-\beta}} \bar{E}_M^{\frac{\alpha}{1-\beta}} L_{M,t}^{\frac{-\alpha}{1-\beta}}$$

which, substituting (A2) for $p_{M,t}$ and (A5) for $L_{M,t}$,

$$= \frac{1}{\beta} N_{M,t}\left[(1-\gamma)^{\frac{\sigma}{\sigma-1}}\Gamma\left(y_t^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)^{\frac{1}{\sigma-1}}\right]^{\frac{\beta}{1-\beta}} \bar{E}_M^{\frac{\alpha}{1-\beta}} \left(\frac{L_t\Gamma}{y_t^{-\frac{\sigma-1}{\sigma}}+\Gamma}\right)^{\frac{-\alpha}{1-\beta}}$$

$$= \lambda N_{M,t}\left(y_t^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)^{\frac{1}{\sigma-1}\left(\frac{\beta}{1-\beta}\right)+\frac{\alpha}{1-\beta}} \bar{E}_M^{\frac{\alpha}{1-\beta}} L_t^{\frac{-\alpha}{1-\beta}}$$

$$= \lambda N_{M,t}\left(y_t^{-\frac{\sigma-1}{\sigma}}+\Gamma\right)^{\frac{(\sigma-1)\alpha+\beta}{(\sigma-1)(1-\beta)}} \bar{E}_M^{\frac{\alpha}{1-\beta}} L_t^{\frac{-\alpha}{1-\beta}} \quad (B20)$$

Derivation of (B22)

Substituting (16) for $E_{S,t}$ into (15) and rearranging:

$$E_{S,t} = \left(\frac{\alpha N_{S,t}}{\beta \bar{e}_S}\right)^{\frac{1-\beta}{1-\alpha-\beta}} L_{S,t}\, p_{S,t}{}^{\frac{1}{1-\alpha-\beta}} \tag{16}$$

$$Y_{S,t} = \frac{1}{\beta} N_{S,t} p_{S,t}^{\frac{\beta}{1-\beta}} E_{S,t}^{\frac{\alpha}{1-\beta}} L_{S,t}^{\frac{1-\alpha-\beta}{1-\beta}} \tag{15}$$

$$\Longrightarrow Y_{S,t} = \lambda N_{S,t} p_{S,t}{}^{\frac{\beta}{1-\beta}} \left[\left(\frac{N_{S,t}}{\bar{e}_S}\right)^{\frac{1-\beta}{1-\alpha-\beta}} L_{S,t}\, p_{S,t}{}^{\frac{1}{1-\alpha-\beta}}\right]^{\frac{\alpha}{1-\beta}} L_{S,t}^{\frac{1-\alpha-\beta}{1-\beta}}$$

(Powers on: $N_{S,t}$: $\frac{1-\alpha-\beta}{1-\alpha-\beta} + \frac{1-\beta}{1-\alpha-\beta}\frac{\alpha}{1-\beta} = \frac{1-\beta}{1-\alpha-\beta}$; $p_{S,t}$: $\frac{\beta(1-\alpha-\beta)+\alpha}{(1-\beta)(1-\alpha-\beta)} = \frac{\beta(1-\beta)+\alpha(1-\beta)}{(1-\beta)(1-\alpha-\beta)} = \frac{\alpha+\beta}{1-\alpha-\beta}$)

$$= \lambda N_{S,t}{}^{\frac{1-\beta}{1-\alpha-\beta}} p_{S,t}{}^{\frac{\alpha+\beta}{1-\alpha-\beta}} L_{S,t}$$

and then using (A1) for $p_{S,t}$

$$\Longrightarrow \frac{Y_{S,t}}{L_{S,t}} = \lambda N_{S,t}{}^{\frac{1-\beta}{1-\alpha-\beta}} \left[(1-\gamma)^{\frac{\sigma}{\sigma-1}} \left(1 + \Gamma y_t{}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}}\right]^{\frac{\alpha+\beta}{1-\alpha-\beta}}$$

$$= \lambda N_{S,t}{}^{\frac{1-\beta}{1-\alpha-\beta}} \left(1 + \Gamma y_t{}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\alpha+\beta}{(\sigma-1)(1-\alpha-\beta)}} \tag{B22}$$

Derivations in Appendix B8

Full algebra for comparative statics results stated after (B26).

Calculation of $\frac{\partial ln p_{S,t}(y_t)}{\partial y_t}$

$$\text{(A1)} \Rightarrow \ln\left(p_{S,t}(y_t)\right) = \frac{\sigma}{\sigma-1}\ln(1-\gamma) + \frac{1}{\sigma-1}\ln\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}} + 1\right)$$

$$\Rightarrow \frac{\partial \ln\left(p_{S,t}(y_t)\right)}{\partial y_t} = \frac{\frac{\gamma}{1-\gamma}\frac{\sigma-1}{\sigma} y_t^{\frac{-1}{\sigma}}}{(\sigma-1)\left(\frac{\gamma}{1-\gamma} y_t^{\frac{\sigma-1}{\sigma}} + 1\right)} = \frac{\Gamma}{\sigma y_t^{\frac{1}{\sigma}}\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}} + 1\right)}$$

Calculation of $\frac{\partial \ln p_{M,t}(y_t)}{\partial y_t}$

$$\text{(A2)} \Rightarrow \ln\left(p_{M,t}(y_t)\right) = \frac{\sigma}{\sigma-1}\ln(1-\gamma) + \frac{1}{\sigma-1}\ln\left(y_t^{-\frac{\sigma-1}{\sigma}} + \Gamma\right)$$

$$\Rightarrow \frac{\partial \ln\left(p_{M,t}(y_t)\right)}{\partial y_t} = \frac{-\frac{\sigma-1}{\sigma} y_t^{-\frac{\sigma-1}{\sigma}} y_t^{-1}}{(\sigma-1)\left(\Gamma + y_t^{-\frac{\sigma-1}{\sigma}}\right)} = \frac{-1}{\sigma y_t \left(\Gamma y_t^{\frac{\sigma-1}{\sigma}} + 1\right)}$$

Calculation of $\frac{\partial \ln\left(L_{S,t}(y_t)\right)}{\partial y_t}$

$$\text{(A4)} \Rightarrow \ \ln\left(L_{S,t}(\, y_t)\right) = \ln(L_t) - \ln\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}} + 1\right)$$

$$\Rightarrow \frac{\partial \ln\left(L_{S,t}(y_t)\right)}{\partial y_t} = -\frac{\Gamma \frac{\sigma-1}{\sigma} y_t^{\frac{-1}{\sigma}}}{1 + \Gamma y_t^{\frac{\sigma-1}{\sigma}}} = \frac{-(\sigma-1)\Gamma}{\sigma y_t^{\frac{1}{\sigma}} \left(\Gamma y_t^{\frac{\sigma-1}{\sigma}} + 1\right)}$$

Calculation of $\frac{\partial \ln\left(L_{M,t}(y_t)\right)}{\partial y_t}$

$$\text{(A5)} \Rightarrow \ \ln\left(L_{M,t}(\, y_t)\right) = \ln(L_t) + \ln(\Gamma) - \ln\left(\Gamma + y_t^{-\frac{\sigma-1}{\sigma}}\right) \Rightarrow \frac{\partial \ln\left(L_{M,t}(y_t)\right)}{\partial y_t}$$

$$= \frac{\sigma-1}{\sigma} \frac{y_t^{-\frac{\sigma-1}{\sigma}} y_t^{-1}}{\left(\Gamma + y_t^{-\frac{\sigma-1}{\sigma}}\right)} = \frac{\sigma-1}{\sigma y_t \left(\Gamma y_t^{\frac{\sigma-1}{\sigma}} + 1\right)}$$

Simplification of $\frac{\partial f}{\partial y_t}$

$$\frac{\partial f}{\partial y_t} = \left(\frac{\beta}{1-\beta}\right) \frac{-1}{\sigma y_t \left(\Gamma y_t^{\frac{\sigma-1}{\sigma}} + 1\right)} - \left(\frac{\alpha+\beta}{1-\alpha-\beta}\right) \frac{\Gamma}{\sigma y_t^{\frac{1}{\sigma}} \left(\Gamma y_t^{\frac{\sigma-1}{\sigma}} + 1\right)}$$

$$+ \left(\frac{1-\alpha-\beta}{1-\beta}\right) \frac{\sigma-1}{\sigma y_t \left(\Gamma y_t^{\frac{\sigma-1}{\sigma}} + 1\right)} + \frac{(\sigma-1)\Gamma}{\sigma y_t^{\frac{1}{\sigma}} \left(1 + \Gamma y_t^{\frac{\sigma-1}{\sigma}}\right)} - \frac{1}{y_t}$$

$$= \frac{-(1-\alpha-\beta)\beta + (1-\alpha-\beta)^2(\sigma-1) - (1-\beta)(1-\alpha-\beta)\sigma}{(1-\beta)(1-\alpha-\beta)\sigma y_t \left(\Gamma y_t^{\frac{\sigma-1}{\sigma}} + 1\right)}$$

$$+ \frac{[-(1-\beta)(\alpha+\beta) + (1-\beta)(1-\alpha-\beta)(\sigma-1) - (1-\beta)(1-\alpha-\beta)\sigma]\Gamma y_t^{\frac{\sigma-1}{\sigma}}}{(1-\beta)(1-\alpha-\beta)\sigma y_t \left(\Gamma y_t^{\frac{\sigma-1}{\sigma}} + 1\right)}$$

$$= \frac{-(1-\alpha-\beta)(\beta+1-\alpha-\beta)+(1-\alpha-\beta)\sigma[(1-\alpha-\beta)-(1-\beta)]-(1-\beta)\Gamma y_t^{\frac{\sigma-1}{\sigma}}}{(1-\beta)(1-\alpha-\beta)\sigma y_t\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}}+1\right)}$$

$$= -\frac{(1-\alpha-\beta)(1-\alpha+\sigma\alpha)+(1-\beta)\Gamma y_t^{\frac{\sigma-1}{\sigma}}}{\sigma(1-\beta)(1-\alpha-\beta)y_t\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}}+1\right)} \tag{B27}$$

Derivation of $f\left(e_t,\bar{N}_{M,t},\bar{N}_{S,t},\mathbf{\Omega}\right)$

$$\begin{aligned} f\left(y_t,\bar{N}_{M,t},\bar{N}_{S,t},\mathbf{\Omega}\right) & \\ \equiv \ln\left(\bar{N}_{M,t}\right) &- \left(\frac{1-\beta}{1-\alpha-\beta}\right)\ln\left(\bar{N}_{S,t}+\right)\frac{\beta}{1-\beta}\ln\left(p_{M,t}(y_t)\right) \\ &- \left(\frac{\alpha+\beta}{1-\alpha-\beta}\right)\ln\left(p_{S,t}(y_t)\right)+\left(\frac{\alpha}{1-\beta}\right)\ln(\bar{E}_M) \\ &- \left(\frac{\alpha}{1-\alpha-\beta}\right)\ln\left(\frac{\alpha}{\beta\bar{e}_S}\right)+\left(\frac{1-\alpha-\beta}{1-\beta}\right)\ln\left(L_{M,t}(y_t)\right) \\ &- \ln\left(L_{S,t}(y_t)\right)-\ln(y_t)=0 \end{aligned} \tag{B25}$$

Insert

$$\ln(y_t)=\sigma\ln(\Gamma)+(1-\beta)\sigma\left(\ln\left(N_{M,t}\right)-\ln\left(N_{S,t}\right)\right)-\alpha\sigma\ln(e_t)$$

$$\begin{aligned} \Rightarrow f\left(e_t,\bar{N}_{M,t},\bar{N}_{S,t},\mathbf{\Omega}\right) & \\ \equiv \ln\left(\bar{N}_{M,t}\right) &- \left(\frac{1-\beta}{1-\alpha-\beta}\right)\ln\left(\bar{N}_{S,t}\right)+\frac{\beta}{1-\beta}\ln\left(p_{M,t}(y_t)\right) \\ &- \left(\frac{\alpha+\beta}{1-\alpha-\beta}\right)\ln\left(p_{S,t}(y_t)\right)+\left(\frac{\alpha}{1-\beta}\right)\ln(\bar{E}_M)-\left(\frac{\alpha}{1-\alpha-\beta}\right)\ln\left(\frac{\alpha}{\beta\bar{e}_S}\right) \\ &+ \left(\frac{1-\alpha-\beta}{1-\beta}\right)\ln\left(L_{M,t}(y_t)\right)-\ln\left(L_{S,t}(y_t)\right)-\sigma\ln(\Gamma) \\ &- (1-\beta)\sigma\left(\ln\left(N_{M,t}\right)-\ln\left(N_{S,t}\right)\right)+\alpha\sigma\ln(e_t)=0 \end{aligned}$$

$$\begin{aligned} = -\sigma\ln(\Gamma)+[1-(1-\beta)\sigma]\ln\left(\bar{N}_{M,t}\right)-\left(\frac{1-\beta}{1-\alpha-\beta}-(1-\beta)\sigma\right)\ln\left(\bar{N}_{S,t}\right) & \\ + \frac{\beta}{1-\beta}\ln\left(p_{M,t}(y_t)\right)-\left(\frac{\alpha+\beta}{1-\alpha-\beta}\right)\ln\left(p_{S,t}(y_t)\right)+\left(\frac{\alpha}{1-\beta}\right)\ln(\bar{E}_M) & \\ - \left(\frac{\alpha}{1-\alpha-\beta}\right)\ln\left(\frac{\alpha}{\beta\bar{e}_S}\right)+\left(\frac{1-\alpha-\beta}{1-\beta}\right)\ln\left(L_{M,t}(y_t)\right)-\ln\left(L_{S,t}(y_t)\right) & \\ + \alpha\sigma\ln(e_t)=0 & \end{aligned}$$

Now using (29):

$$y_t=\Gamma^\sigma\bar{N}_t^{(1-\beta)\sigma}e_t^{-\alpha\sigma}\Rightarrow\Gamma y_t^{\frac{\sigma-1}{\sigma}}=\Gamma^\sigma\bar{N}_t^{(1-\beta)(\sigma-1)}e_t^{-\alpha(\sigma-1)}$$

$$\Rightarrow \ln\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}} + 1\right) = \ln\left(\Gamma^\sigma \bar{N}_t^{(1-\beta)(\sigma-1)} e_t^{-\alpha(\sigma-1)} + 1\right)$$

gives these versions of the logarithmic forms seen at intermediate stages of calculating $\partial\ln\left(p_{S,t}(y_t)\right)/\partial y_t$, $\partial\ln\left(p_{M,t}(y_t)\right)/\partial y_t$, $\partial\ln\left(L_{S,t}(y_t)\right)/\partial y_t$ and $\partial\ln\left(L_{M,t}(y_t)\right)/\partial y_t$ respectively:

$$\ln\left(p_{M,t}(e_t)\right) = -\frac{1}{\sigma}\ln(y_t) + \frac{1}{\sigma-1}\ln\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}} + 1\right) + \frac{\sigma}{\sigma-1}\ln(1-\gamma)$$

$$= -[\ln(\Gamma) + (1-\beta)\ln(\bar{N}_t) - \alpha\ln(e_t)] + \frac{1}{\sigma-1}\ln\left(\Gamma^\sigma \bar{N}_t^{(1-\beta)(\sigma-1)} e_t^{-\alpha(\sigma-1)} + 1\right) + \frac{\sigma}{\sigma-1}\ln(1-\gamma)$$

$$\ln\left(p_{S,t}(e_t)\right) = \frac{1}{\sigma-1}\ln\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}} + 1\right) + \frac{\sigma}{\sigma-1}\ln(1-\gamma)$$

$$= \frac{1}{\sigma-1}\ln\left(\Gamma^\sigma \bar{N}_t^{(1-\beta)(\sigma-1)} e_t^{-\alpha(\sigma-1)} + 1\right) + \frac{\sigma}{\sigma-1}\ln(1-\gamma)$$

$$\ln\left(L_{M,t}(e_t)\right) = \ln(L_t) + \ln(\Gamma) + \frac{\sigma-1}{\sigma}\ln(y_t) - \ln\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}} + 1\right)$$

$$= \ln(L_t) + \ln(\Gamma) + (\sigma-1)[\ln(\Gamma) + (1-\beta)\ln(\bar{N}_t) - \alpha\ln(e_t)] - \ln\left(\Gamma^\sigma \bar{N}_t^{(1-\beta)(\sigma-1)} e_t^{-\alpha(\sigma-1)} + 1\right)$$

$$\ln\left(L_{S,t}(e_t)\right) = \ln(L_t) - \ln\left(\Gamma y_t^{\frac{\sigma-1}{\sigma}} + 1\right)$$

$$= \ln(L_t) - \ln\left(\Gamma^\sigma \bar{N}_t^{(1-\beta)(\sigma-1)} e_t^{-\alpha(\sigma-1)} + 1\right)$$

$$
\begin{aligned}
\Rightarrow f\left(e_t, \bar{N}_{M,t}, \bar{N}_{S,t}, \mathbf{\Omega}\right)
&= -\sigma\ln(\Gamma) \; + [1-(1-\beta)\sigma]\ln\left(\bar{N}_{M,t}\right) - \left(\frac{1-\beta}{1-\alpha-\beta} - (1-\beta)\sigma\right)\ln\left(\bar{N}_{S,t}\right) \\
&\quad - \left(\frac{\beta}{1-\beta}\right)[\ln(\Gamma) + (1-\beta)\ln(\bar{N}_t) - \alpha\ln(e_t)] \\
&\quad + \left(\frac{\beta}{1-\beta}\right)\left\{\frac{1}{\sigma-1}\ln\left(\Gamma^{\sigma}\bar{N}_t^{(1-\beta)(\sigma-1)}e_t^{-\alpha(\sigma-1)}+1\right) + \frac{\sigma}{\sigma-1}\ln(1-\gamma)\right\} \\
&\quad - \left(\frac{\alpha+\beta}{1-\alpha-\beta}\right)\left\{\frac{1}{\sigma-1}\ln\left(\Gamma^{\sigma}\bar{N}_t^{(1-\beta)(\sigma-1)}e_t^{-\alpha(\sigma-1)}+1\right) + \frac{\sigma}{\sigma-1}\ln(1-\gamma)\right\} \\
&\quad + \left(\frac{\alpha}{1-\beta}\right)\ln(\bar{\mathrm{E}}_{\mathrm{M}}) - \left(\frac{\alpha}{1-\alpha-\beta}\right)\ln\left(\frac{\alpha}{\beta\bar{e}_S}\right) \\
&\quad + \left(\frac{1-\alpha-\beta}{1-\beta}\right)\{\ln(L_t) + \ln(\Gamma) \\
&\quad + (\sigma-1)[\ln(\Gamma) + (1-\beta)\ln(\bar{N}_t) - \alpha\ln(e_t)]\} \\
&\quad - \left(\frac{1-\alpha-\beta}{1-\beta}\right)\ln\left(\Gamma^{\sigma}\bar{N}_t^{(1-\beta)(\sigma-1)}e_t^{-\alpha(\sigma-1)}+1\right) - \ln(L_t) \\
&\quad + \ln\left(\Gamma^{\sigma}\bar{N}_t^{(1-\beta)(\sigma-1)}e_t^{-\alpha(\sigma-1)}+1\right) + \alpha\sigma\ln(e_t)
\end{aligned}
$$

$$
\begin{aligned}
&= -\sigma\ln(\Gamma) \; + [1-(1-\beta)\sigma]\ln\left(N_{M,t-1}\right) - \left(\frac{1-\beta}{1-\alpha-\beta} - (1-\beta)\sigma\right)\ln\left(N_{S,t-1}\right) \\
&\quad - \left(\frac{\beta}{1-\beta}\right)[\ln(\Gamma) + (1-\beta)\ln(\bar{N}_t) - \alpha\ln(e_t)] \\
&\quad + \left(\frac{\beta}{1-\beta}\right)\left\{\frac{1}{\sigma-1}\ln\left(\Gamma^{\sigma}\bar{N}_t^{(1-\beta)(\sigma-1)}e_t^{-\alpha(\sigma-1)}+1\right) + \frac{\sigma}{\sigma-1}\ln(1-\gamma)\right\} \\
&\quad - \left(\frac{\alpha+\beta}{1-\alpha-\beta}\right)\left\{\frac{1}{\sigma-1}\ln\left(\Gamma^{\sigma}\bar{N}_t^{(1-\beta)(\sigma-1)}e_t^{-\alpha(\sigma-1)}+1\right) + \frac{\sigma}{\sigma-1}\ln(1-\gamma)\right\} \\
&\quad + \left(\frac{\alpha}{1-\beta}\right)\ln(\bar{E}_M) - \left(\frac{\alpha}{1-\alpha-\beta}\right)\ln\left(\frac{\alpha}{\beta\bar{e}_S}\right) \\
&\quad + \left(\frac{1-\alpha-\beta}{1-\beta}\right)\{\ln(L_t) + \ln(\Gamma) \\
&\quad + (\sigma-1)[ln(\Gamma) + (1-\beta)\ln(\bar{N}_t) - \alpha\ln(e_t)]\} \\
&\quad - \left(\frac{1-\alpha-\beta}{1-\beta}\right)\ln\left(\Gamma^{\sigma}\bar{N}_t^{(1-\beta)(\sigma-1)}e_t^{-\alpha(\sigma-1)}+1\right) - \ln(L_t) \\
&\quad + \ln\left(\Gamma^{\sigma}\bar{N}_t^{(1-\beta)(\sigma-1)}e_t^{-\alpha(\sigma-1)}+1\right) + \alpha\sigma\ln(e_t)
\end{aligned}
$$

and then using $\frac{\beta}{1-\beta} - \frac{\alpha+\beta}{1-\alpha-\beta} = -\frac{\alpha}{(1-\beta)(1-\alpha-\beta)}$ and $-\left(\frac{1-\alpha-\beta}{1-\beta}\right) + 1 = \frac{\alpha}{1-\beta}$ gives (B28).