CENTRE FOR APPLIED MACROECONOMIC ANALYSIS

The Australian National University

CAMA Working Paper Series **July, 2010**

ON A UNIQUE NONDEGENERATE DISTRIBUTION OF AGENTS IN THE HUGGETT MODEL

Timothy Kam
Research School of Economics, The Australian National University

Junsang Lee
Research School of Economics, The Australian National University

CAMA Working Paper 19/2010
http://cama.anu.edu.au

ON A UNIQUE NONDEGENERATE DISTRIBUTION OF AGENTS IN A HETEROGENEOUS AGENT MODEL

TIMOTHY KAM* & JUNSANG LEE

Research School of Economics
H. W. Arndt Building 25a
The Australian National University
A.C.T. 0200, Australia.
Email: timothy.kam@anu.edu.au*
Tel: +61-2-612-51072
Fax: +61-2-612-55124

ABSTRACT

The seminal work of Huggett ["The risk-free rate in heterogeneous-agent incomplete-insurance economies", *Journal of Economic Dynamics and Control*, 1993, 17(5-6), 953-969] showed that there exists a unique stationary distribution of agent types, given by their individual states of asset and income endowment pairs. However, the question remains open if the equilibrium individual state space might turn out to be trivial, in the sense that every agent's common borrowing constraint binds forever. If so, the invariant probability measure of agent types will place all mass on this minimal credit level. By invoking a simple comparative-static argument, we provide closure to this open question. We thus reinforce Huggett's result of a unique stationary equilibrium distribution of agents by showing that it must also be one that is nontrivial or nondegenerate.

KEYWORDS: Incomplete markets; Compactness; Individual state space; Stationary distribution
JEL CODES: C62; D31; D52

1. Introduction

The seminal work of Huggett [1993] showed that there exists a unique stationary distribution of agent types, given by their individual states of asset and income endowment pairs. In the setting of Huggett [1993], the key insight on the risk-free rate anomaly arising from representative agent models, was obtained by an appeal to incomplete asset markets (via agents' borrowing constraints) and precautionary saving motives. This framework is one of the key foundations for further quantitative research using heterogenous agent macroeconomics. In this class of models, important questions such as asset pricing puzzles [see e.g. Huggett, 1993; Aiyagari, 1994], and fiscal policy and taxation [see e.g. Heathcote, 2005], can now be seriously addressed.

Proving the existence of a unique stationary distribution of agent types in the model of Huggett [1993] is vital since the *stationary equilibrium* risk-free rate depends on this object. To establish this result, *i.e.* Theorem 2 in Huggett [1993], certain sufficiency conditions in Theorem 2 of Hopenhayn and Prescott [1992] are required to be satisfied by the model. One of the requirements of the model is compactness of agents' equilibrium individual state space, denoted by S. Huggett [1993] showed the existence of a compact individual state space in any equilibrium where agents optimize. Intuitively, one needs to show that each agent indexed by an asset-endowment pair, $(a, e) \in S$, in making their optimal competitive decisions, would always remain in the set S every period.

However, the question remains open in Huggett [1993] if this equilibrium individual state space S might turn out to be trivial, in the sense that every agents common borrowing constraint binds forever. If so, the invariant probability measure of agent types will place all mass on this minimal credit level. The possibility of such a degenerate stationary equilibrium distribution of agents will mean that the model may not be very useful in applications. In particular, since the aggregate relative price an asset depends on the equilibrium distribution, then the possibility of a degenerate equilibrium distribution implies that the aggregate relative price of the asset may not be well-defined.

In this note, we reinforce the proof of Lemma 3 in Huggett [1993] and show that the unique stationary distribution of agents in this model must also be nontrivial, by invoking a simple comparative statics argument. In other words, we establish that Huggetts result of a unique stationary equilibrium distribution of agents must be one that is nondegenerate. A related literature, although in the context of the Brock and Mirman [1972] model, is also concerned with strengthening results on existence and uniqueness of nontrivial stationary distributions [see e.g. Chatterjee and Shukayev, 2008; Kamihigashi, 2006] while relaxing the number of hypotheses required. The enterprise in this paper is related to this literature, but in the context of the Huggett model.

2. HUGGETT'S MODEL

In this section we first revisit the Huggett [1993] model. Then we provide a brief discussion on the notion of an endogenously compact individual state space and its implication for the existence of a unique stationary equilibrium distribution of agents.

In the Huggett model, time is discrete, and each period is indexed by $t \in \mathbb{N} := \{0, 1, ...\}$.[1] The population of agents has mass 1. Each measure zero agent receives a stream of stochastic endowment of consumption good. Let $E = \{e_l, e_h\}$, where $e_h > e_l$, be the set of endowment realizations. Each random sequence $(e_t)_{t\in\mathbb{N}}$ is governed by a given Markov chain (π, π_0) on E, where π is the stochastic matrix and π_0 the initial unconditional distribution on E. $\pi(e'|e) := \Pr\{e_{t+1} = e'|e_t = e\} > 0$, $e', e \in E$, is independent of t, and another agents' realization of e. Let $A := [\underline{a}, +\infty)$ be the space of possible asset levels. The parameter $\underline{a}$ is interpreted as an exogenous borrowing constraint. Denote the product individual state space as $X := A \times E$.

2.1. **An individual's decision problem.** The *individual state* is $x := (a, e) \in X$. The individual takes as given the aggregate price $q > 0$. Suppose in an equilibrium there is a set $S := [\underline{a}, \overline{a}] \times \{e_l, e_h\} \subset X$ that generates Borel σ-algebra $\mathcal{B}(S)$. The dependence of the equilibrium q on the *aggregate state* given by a probability measure ψ on $(S, \mathcal{B}(S))$ is implicit.[2]

Each agent has a common subjective discount factor $\beta \in (0, 1)$, and identical per-period utility $u : \mathbb{R}_+ \to \mathbb{R}$. The function u is strictly increasing, strictly concave, and twice continuously differentiable. Each agent chooses consumption (c) and saving in terms of a single asset (a'). Let the agent's feasible action correspondence be $\Gamma(q) : A \times E \rightrightarrows \mathcal{B}(\mathbb{R}_+ \times A)$, where at each slice of Γ indexed by $(x; q)$, we have a description of the feasible choice set of an agent *currently* named x:[3]

$$\Gamma(x; q) = \left\{(c, a') : a + e \geq c + a'q, \;\; c \geq 0, \;\; a' \geq \underline{a}\right\}.$$

Denote $(x; q) \mapsto v(x; q) \in \mathbb{R}$ as an agent's value function. Each agent's Bellman equation is

$$v(x; q) = \max_{(c,a')\in\Gamma(x;q)} \left\{ u(c) + \beta \sum_{e'\in E} v(a', e'; q)\pi(e'|e) \right\}, \tag{1}$$

with associated optimal decision rule $(x; q) \mapsto \hat{a}(x; q)$, such that at $(x; q)$, $a' = \hat{a}(x; q)$.

2.2. **Compact equilibrium individual state space $S \subset X$.** The notion of a stationary equilibrium is defined in Huggett [1993, p.956]. Given the Markov matrix for the endowment process, $\pi : E \to [0, 1]$, an initial individual state $x \in S$ for each agent, and an optimal decision rule, $(x; q) \mapsto \hat{a}(x; q)$, we can induce a time-invariant probability measure ψ on the measurable space

[1] As is the usual convention, we may drop the explicit time-t subscript on variables, e.g. $x := x_t$ and $x' := x_{t+1}$.

[2] In Huggett [1993], since the emphasis is on a notion of *recursive stationary equilibrium* where q is constant, we don't have to explicitly carry around the distribution of agent types ψ, as a relevant state variable. Instead, we only make the agents' problems dependent on q as a scalar parameter.

[3] For technical reasons, since A is a continuum, our agent's decision rules $c = c(a, e)$ and $a' = \hat{a}(a, e)$ need to be measurable functions belonging to $\Gamma(q)$. Hence we restrict such selections to only measurable subsets in the image of $\Gamma(q)$. These measurable subsets are in the Borel σ-algebra, $\mathcal{B}(\mathbb{R}_+ \times A)$ generated by $\mathbb{R}_+ \times A \ni (c, a')$.

$(S, \mathcal{B}(S))$ satisfying:

$$\psi(B) = \int_S P(x, B) \mathrm{d}\psi, \qquad \forall B \in \mathcal{B}(S),$$

where $P : S \times \mathcal{B}(S) \to [0, 1]$ is the equilibrium transition probability function.[4]

In Theorem 1, Huggett [1993] provides some sufficient conditions on the model such that given q, the solution to each agent's Bellman equation problem has some nice properties. Specifically, Theorem 1 in Huggett [1993] establishes that the optimal $\hat{a} : X \to [\underline{a}, \infty)$ is *continuous*, is either *strictly increasing* in a, if $a > \underline{a}$, or is *nondecreasing* in a if $a = \underline{a}$.

Theorem 1 and Lemmata 1-3 in Huggett [1993] show that each agent's optimal decision function for credit holdings, $\hat{a} : X \to [\underline{a}, \infty)$, has the typical shape as in Figure 1. In particular, this decision rule has the following properties:

(1) If the current endowment is e_l, and, if the borrowing constraint is not binding, then $\hat{a}(\cdot, e_l)$ is well below the 45°-line in (a, a')-space. That is, an agent who is not currently credit constrained in terms of his asset choice for the following period, and who continues to face a sequence of low endowment realizations, will be reducing his asset level in each corresponding subsequent periods (Lemma 1) until his asset level hits the borrowing constraint, $\underline{a}$, where either the agent is constrained to just borrow $\underline{a}$ in the subsequent period; or

(2) If the agent has high endowment, e_h, the agent will start saving, but there is an asset level, $\bar{a}$, such that the policy function at e_h, $\hat{a}(\cdot, e_h)$, crosses the 45°-line in (a, a')-space. This is proved by Lemma 3, which uses both Lemma 1 and Lemma 2.

Thus, in an equilibrium, if there is to be an endogenous $\bar{a}$, as shown in Lemma 3 in Huggett [1993], which is the smallest fixed point satisfying $\hat{a}(a, e_h) = a$, then it is straightforward to deduce that $S := [\underline{a}, \bar{a}] \times \{e_l, e_h\}$ is an endogenously compact metric space. That is, each agent x beginning in S will always stay within S, or the equilibrium asset decision rule will be $\hat{a} : S \to [\underline{a}, \bar{a}]$.

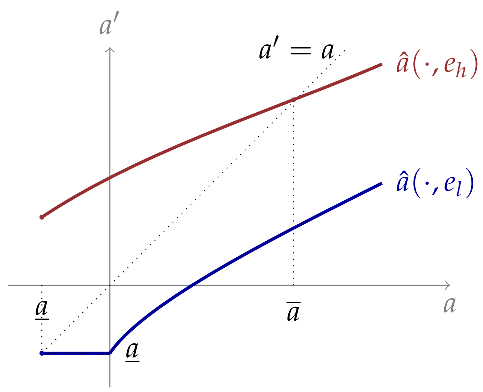


FIGURE 1. Characterization of optimal policy function $\hat{a} : X \to \mathbb{R}$. When $e = e_h$, for each a, $\hat{a}(a, e_h) > \hat{a}(a, e_l)$. If $\hat{a}(a, e) \geq \underline{a}$ binds, then $\hat{a}(\cdot, e)$ is nondecreasing in a.

[4] The details are discussed very nicely in Huggett [1993].

Theorem 2 of Huggett, applying theorem 2 in Hopenhayn and Prescott [1992], provides sufficient conditions for the existence and uniqueness of a unique stationary distribution of agent types, ψ, for a given q. These conditions in turn include the requirement that S is a compact metric space.

3. A MISSING STEP

In this section, we complete the missing step required to ensure that indeed Huggett's endogenous upper bound $\bar{a}$ on assets is nontrivial. That is, we are required to show that $\underline{a} < \bar{a} < \infty$.

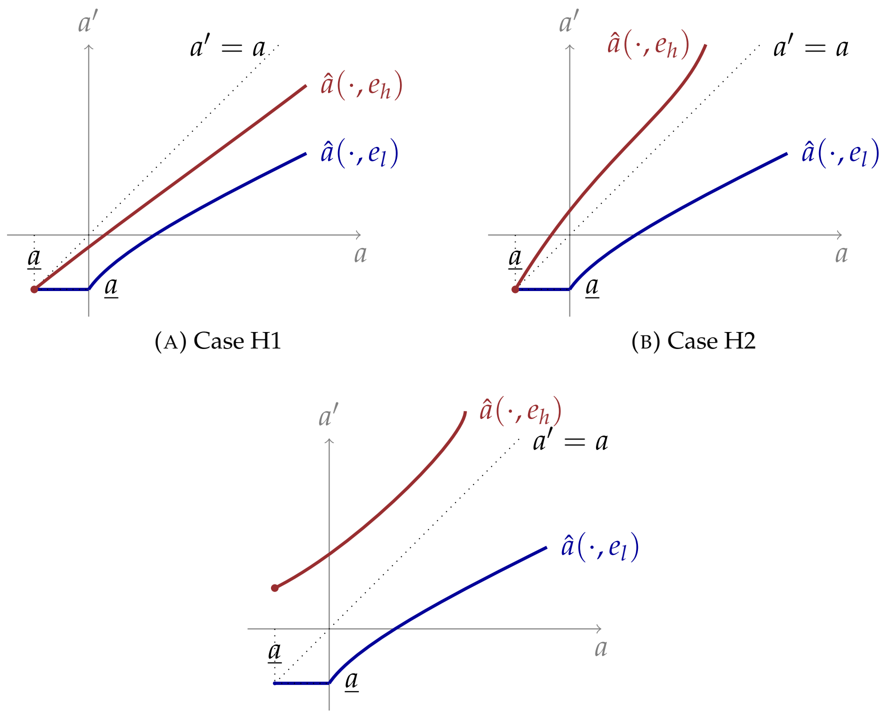
The aim (in Huggett's Lemma 3) is to show that there exists a fixed point $\bar{a}$ satisfying $\hat{a}(a, e_h) = a$. Moreover, we would like to show that it is a nontrivial fixed point: $\bar{a} > \underline{a}$. A contrary hypothesis to this would have three possible cases:

H1. $\hat{a}(a, e_h) < a$ for $a > \underline{a}$ and $\hat{a}(a, e_h) = a$ for $a = \underline{a}$,

H2. $\hat{a}(a, e_h) > a$ for $a > \underline{a}$ and $\hat{a}(a, e_h) = a$ for $a = \underline{a}$, and

H3. There is no a such that $\hat{a}(a, e_h) = a$.

These three contrary hypotheses are depicted by the typical graphs in Figure 2. In establishing the result on the existence of an endogenously compact S by contradiction in Lemma 3, Huggett [1993] made only the contrary hypothesis (H3) that there is no a such that $\hat{a}(a, e_h) = a$.



(A) Case H1

(B) Case H2

(C) Case H3

FIGURE 2. In proving Lemma 3 in Huggett [1993], suppose there is no a ($> \underline{a}$) such that $\hat{a}(a, e_h) = a$. *A priori* there may be three possible cases for the component function $\hat{a}(\cdot, e_h)$ that would satisfy this hypothesis. We can rule out cases H1 and H2.

It turns out, that only one of these contrary hypotheses is possible (i.e. H3), as was assumed in Huggett [1993]. However, it remains to be shown that this *must* be the only case, as we now show in the following lemma.

Lemma. *The decision $\hat{a}(a,e)$ is strictly increasing in e for all $a \geq \underline{a}$.*

Proof. First, we show the case that a current individual state is $(\underline{a}, e)$. An optimal consumption decision $c(\underline{a}, e)$ for an agent currently named $(\underline{a}, e)$ must satisfy the first-order condition

$$u_c[c(\underline{a},e)] \geq \beta q^{-1}\mathbb{E}\left\{ u_c[c(\hat{a}(\underline{a},e),e')] \middle| e \right\}, \quad \text{with equality if } \hat{a}(\underline{a},e) > \underline{a}. \tag{2}$$

Consider $(\underline{a}, e) = (\underline{a}, e_l)$. We then perturb e_l to $e_l + \Delta e =: e_h$. We want to show that $\hat{a}(\underline{a}, e_h) > \hat{a}(\underline{a}, e_l)$. By Theorem 1 in Huggett [1993], $\hat{a}(\underline{a}, e_l) = \underline{a}$. Then (2) evaluated at $(\underline{a}, e) = (\underline{a}, e_l)$ is

$$u_c[c(\underline{a},e)] > \beta q^{-1}\mathbb{E}\left\{ u_c[c(\hat{a}(\underline{a},e),e')] \middle| e \right\}. \tag{3}$$

Suppose e_l increases to $e_l + \Delta e =: e_h$, but, suppose $\hat{a}(\underline{a}, e_l + \Delta e) = \hat{a}(\underline{a}, e_l) = \underline{a}$. By Lemma 1 in Huggett [1993], this is consistent with the result $u_c[c(\underline{a}, e_l)] = v_a(\underline{a}, e_l) \geq v_a(\underline{a}, e_h) = u_c[c(\underline{a}, e_h)]$. So either the LHS of (3) declines or remain constant, and by strict concavity of u, $c(\underline{a}, e_l) \leq c(\underline{a}, e_h)$. But since u is strictly concave, the agent would prefer to also shift some of the increase in e towards the next period, and across next-period states. That is, in the RHS of (3), for each fixed $e' \in E$, $u_c[c(\hat{a}(\underline{a}, e), e')]$ must fall. From the agent's budget constraint, this implies that $\hat{a}(\underline{a}, e)$ must increase in e. That is, if $e_h > e_l$, then $\hat{a}(\underline{a}, e_h) > \hat{a}(\underline{a}, e_l) = \underline{a}$, so then (2) would hold with equality at $(\underline{a}, e_h)$. Contradiction.

Second, consider $a > \underline{a}$. By Theorem 1 in Huggett [1993], $\hat{a}(\cdot, e)$ is strictly increasing and continuous in $a > \underline{a}$. By Lemma 1 in Huggett [1993], $\hat{a}(a, e_l) < a$ for $a > \underline{a}$. Using these facts, and since we have shown $\hat{a}(\underline{a}, e_h) > \hat{a}(\underline{a}, e_l) = \underline{a}$, then there exists some a such that $\hat{a}(a, e_h) > a > \hat{a}(a, e_l) \geq \underline{a}$, and for all a, $\hat{a}(a, e_h) > \hat{a}(a, e_l) \geq \underline{a}$. □

4. Discussion

The hypotheses H1 (Figure 2.A) and H2 (Figure 2.B) can thus be ruled out since $\hat{a}(\cdot, e_h)$ must be above the 45°-line in (a, a')-space at the point $\underline{a}$. Thus, we now can rule out any possible trivial equilibrium individual state space as well. The idea is that now, we can proceed to just assume one case [as in Huggett, 1993, Lemma 3] – that $\hat{a}(a, e_h) > a$ for all a – so that there is no fixed point for $\hat{a}(a, e_h)$ in (a, a')-space, but then arrive at a contradiction. The conclusion would have to be that there is an $a^* > \underline{a}$ that is the fixed point, and we can take the least fixed point to be $a^* = \bar{a}$, the endogenous upper bound on assets. Finally, given these results, Theorem 2 of Huggett follows to establish existence and uniqueness of a stationary distribution of agent types, which is now further guaranteed to be nondegenerate.

References

Aiyagari, S. R. (1994): "Uninsured Idiosyncratic Risk and Aggregate Saving," *The Quarterly Journal of Economics*, 109(3), 659–84. Cited on page(s): [2]

Brock, W. A., and L. J. Mirman (1972): "Optimal economic growth and uncertainty: The discounted case," *Journal of Economic Theory*, 4(3), 479–513. Cited on page(s): [2]

Chatterjee, P., and M. Shukayev (2008): "Note on positive lower bound of capital in the stochastic growth model," *Journal of Economic Dynamics and Control*, 32(7), 2137–2147. Cited on page(s): [2]

Heathcote, J. (2005): "Fiscal Policy with Heterogeneous Agents and Incomplete Markets," *Review of Economic Studies*, 72(1), 161–188. Cited on page(s): [2]

Hopenhayn, H. A., and E. C. Prescott (1992): "Stochastic Monotonicity and Stationary Distributions for Dynamic Economies," *Econometrica*, 60(6), 1387–406. Cited on page(s): [2], [5]

Huggett, M. (1993): "The risk-free rate in heterogeneous-agent incomplete-insurance economies," *Journal of Economic Dynamics and Control*, 17(5-6), 953–969. Cited on page(s): [2], [3], [4], [5], [6]

Kamihigashi, T. (2006): "Almost sure convergence to zero in stochastic growth models," *Economic Theory*, 29(1), 231–237. Cited on page(s): [2]