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Some Implications of Learning for Price Stability

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Keywords

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JEL Classification

E32, D83, D84

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Some Implications of Learning for Price Stability*

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1 Introduction

If expectations themselves are a source of low-frequency drift in macroeconomic data, then important questions arise about the validity of standard monetary policy advice. While a large literature has emerged evaluating the robustness of rational expectations policy advice to learning dynamics, relatively little attention has been paid to the question of optimal policy design conditional on such belief structures. Important exceptions are Gaspar, Smets and Vestin (2007, 2010), Molnar and Santoro (2013), Eusepi, Giannoni, and Preston (2015) and Mele, Molnar, and Santoro (2015) which explore ways in which optimal policy under learning differs from that under rational expectations. While these papers provide insights on the constraints non-rational belief structures place on monetary policy, they all have in common the implicit assumption that, absent disturbances to the economy, price stability is optimal in the long-run. Stated differently, conditional expectations of inflation in these analyses converge to price stability as the forecast horizon extends to the indefinite future.

This paper explores whether price stability should be expected to arise as an implication of optimal policy. Building on Molnar and Santoro (2013), a New Keynesian model is adapted to be consistent with low-frequency drift in beliefs identified in macroeconomic data — see Crump, Eusepi, and Moench (2015), Eusepi, Giannoni, and Preston (2015) and later discussion. The analysis is distinguished from this earlier work by solving for optimal decisions, conditional on the belief structure, and by employing the welfare-theoretic loss function (to a second-order approximation) implied by the microfoundations, which accounts for a distorted steady state arising from monopolistic competition and tax policy. In this environment price stability is, in general, not optimal in the long run, even when a central bank has an inflation target of zero as part of its objectives, and even when fully informed about the nature of agents' expectations formation. Drifting beliefs represent a fundamental constraint on what can be achieved by monetary policy, and optimal policy exhibits an inflation bias of the kind observed under optimal discretion when compared to optimal commitment under rational expectations. In contrast to the inflation bias under discretion, the positive inflation rate under learning is the best a central bank can do — because beliefs are state variables, there is no distinction between commitment and discretion. However, under some special cases, long-run price stability will emerge as an optimal outcome under imperfect knowledge.

The mechanism generating a positive optimal inflation rate reflects two competing tensions. As originally identified by Kydland and Prescott (1977), for given inflation expectations higher household discount factors lead to a deterioration in the short-run output-gap-inflation trade-off. Exploitation of given expectations generates higher equilibrium inflation rates. However, while learning dynamics imply beliefs are slow-moving state variables, they do adjust over time — indeed, they are consistent with policy in the long run. As beliefs are revised in response to a positive surprise in inflation, permanently higher inflation expectations raise present discounted losses. The central bank internalizes the effects of its policy actions on expectations. For a given sensitivity of beliefs to new information, and, therefore, a given rise in beliefs about long-run inflation, as the discount factor rises, welfare declines — it becomes optimal to lower the inflation rate in the long run. At the same time, the less sensitive are beliefs to new information, the more can beliefs be exploited, because surprise inflation induces smaller adverse shifts in the short-run inflation-output trade-off. This raises the equilibrium long-run inflation rate.

Two special cases are observed when this tension is resolved in favour of one or the other effect, which bounds the optimal rate of inflation, and makes tight connection to rational expectations policy advice. When the size of low-frequency variation in beliefs goes to zero, so that beliefs are almost never revised, the optimal inflation rate coincides with optimal discretion; alternatively, with low-frequency variation in beliefs and when households are highly patient, valuing utility in any period almost equally, then price stability obtains giving the optimal commitment solution. As such, a principle contribution of this work is to make perspicuous connections across optimal policy outcomes under alternative belief structures in the canonical New Keynesian model.

A further contribution concerns recent proposals to raise the Federal Reserve's inflation target from 2 percent, to some higher rate. Blanchard, Dell'Ariccia, and Mauro (2010), Ball (2013) and Krugman (2014) all argue a higher inflation target would lower the potential output costs arising from the zero bound on nominal interest rates. However, Ascari, Phaneuf, and Sims (2015) and Coibion, Gorodnichenko, and Wieland (2012) demonstrate in structural New Keynesian models, that the optimal rate of inflation is not much greater than zero. A range of frictions, such as staggered pricing and wage-contracting, deliver significant welfare costs for non-zero rates of inflation, that more than offset the gains from being constrained by the zero lower bound on interest rates less frequently. The current paper suggests beliefs themselves may be a constraint on policy, under which a positive rate of inflation is optimal. Relative to rational expectations, drifting beliefs might warrant up to a 2 percent higher annual inflation target.

2 The Model

This section recapitulates a simple New Keynesian model presented in Eusepi and Preston (2016), which is valid for arbitrary beliefs. A range of assumptions, which are without loss of generality, are made for expositional simplicity, and give focus to long-run outcomes. Further details on the microfoundations can be found in Woodford (2003).

A continuum of households i on the unit interval maximize utility

$$\hat{E}_t^i \sum_{T=t}^{\infty} \bar{C}_T \beta^{T-t} \left[\ln c_T(i) - \chi n_T(i) \right],$$

where $0 < \beta < 1$ and $\chi > 0$, by choice of sequences for consumption, $c_t(i)$, and labor supply, $n_t(i)$, subject to the flow budget constraint

$$c_t(i) + b_t(i) \leq (1 + i_t) \pi_t^{-1} b_{t-1}(i) + W_t n_t(i)/P_t + \Gamma_t(i)$$

and the No-Ponzi condition

$$\lim_{T \to \infty} \hat{E}_t^i \left(\prod_{s=0}^{T-t} (1 + i_{t+s}) \pi_{t+s}^{-1} \right)^{-1} B_T(i) \geq 0.$$

The variable $b_t(i) \equiv B_t(i)/P_t$ denotes real bond holdings (which in equilibrium are in zero net supply), i_t the nominal interest rate, $\pi_t \equiv P_t/P_{t-1}$ the inflation rate, W_t is the hourly wage, $\Gamma_t(i)$ dividends from equity holdings of firms and $\bar{C}_T$ exogenous preference shifter. The operator $\hat{E}_t^i$ denotes agents' subjective expectations, which might differ from rational expectations. The latter is defined by the operator $\mathbb{E}_t$.

A continuum of monopolistically competitive firms maximize profits

$$\hat{E}_t^j \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \left[p_t(j) y_T(j) - W_T n_T(j) \right]$$

by choice of $p_t(j)$ subject to the production technology and demand function $y_T(j) = n_T(j) = (p_t(j)/P_T)^{-\theta} Y_T$ for all $T \geq t$, with the elasticity of demand across differentiated goods satisfying $\theta > 1$; and exogenous probability $0 < \alpha < 1$ of not being able to reset their price in any subsequent period. When setting prices in period t, firms are assumed to value future streams of income at the marginal value of aggregate income in terms of the marginal value of an additional unit of aggregate income today giving the stochastic discount factor $Q_{t,T} = \beta^{T-t}(P_t Y_t)/(P_T Y_T)$.

In a symmetric equilibrium $c_t(i) = c_t = w_t \equiv W_t/P_t = n_t = Y_t$ for all i, $b_t(i) = b_t(j)$ for all i, j, while for all firms changing prices in period t, $p_t(i) = p_t(j)$. To a first-order log-linear approximation, in the neighborhood of a zero-inflation steady state, individual consumption and pricing can be expressed as

$$\hat{c}_t(i) = \hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta)\hat{w}_T - \beta\left(\hat{\imath}_T - \hat{\pi}_{T+1} - \hat{r}_T^n\right)\right] \quad (1)$$

$$\hat{p}_t(j) = E_t^j \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[(1-\alpha\beta)\hat{w}_T + \alpha\beta\hat{\pi}_{T+1}\right] \quad (2)$$

where for any variable z_t, $\hat{z}_t = \ln(z_t/\bar{z})$ the log-deviation from steady state $\bar{z}$, with the exceptions $\hat{p}_t(j) = \ln(p_t(j)/P_t)$, and $\hat{\imath}_t = \ln[(1+i_t)/(1+\bar{\imath})]$. The associated natural rate of interest $\hat{r}_t^n = \bar{c}_t - \hat{E}_t\bar{c}_{T+1}$ is determined by exogenous fluctuations in the propensity to consume, $\bar{c}_t = \ln\left(\bar{C}_t/\bar{C}\right)$, a stationary process.[1] The caret denoting log deviation from steady state is dropped for the remainder. Aggregating across the continuum of households and firms, and imposing market-clearing conditions, the economy is described by the aggregate demand and supply equations

$$x_t = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta)x_{T+1} - \left(i_T - \pi_{T+1} - r_T^n\right)\right] \quad (3)$$

$$\pi_t = \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[\kappa x_T + (1-\alpha)\beta\pi_{T+1}\right] \quad (4)$$

where the output gap is defined as

$$x_t = y_t - y_t^n = w_t$$

the difference between output and the natural rate of output, the level of output determined by a flexible price economy: here $y_t^n = 0$. The aggregate demand equation determines the output gap as the discounted expected value of future wages, with the second term capturing variations in the real interest rate, applied in future periods, due to changes in nominal interest rates and goods price inflation. The aggregate supply curve determines inflation as the discounted future sequence of marginal costs and the inflation rate. The slope of the Phillips curve is measured by $\kappa = (1-\alpha\beta)(1-\alpha)/\alpha$. The model is closed with assumptions on policy expectations, which are developed in subsequent sections.[2]

[1] Relation (1) imposes the equilibrium condition of zero net supply of goverment debt.

[2] We also assume a cashless economy with no government spending and no government debt. For analyses of fiscal policy under learning dynamics see Eusepi and Preston (2012, 2016). However, including explicitly the fiscal activities of the government does not alter results, at least under the assumption of lump-sum taxation.

2.1 Beliefs

Households have incomplete knowledge about the true structure of the economy. They observe only their own objectives, constraints and realizations of aggregate variables as well as prices that are exogenous to their decision problems and beyond their control. They have no knowledge of the beliefs, constraints and objectives of other agents in the economy: even though their decision problems are identical, they do not know this to be true. The fact agents have no knowledge of other agents' preferences and beliefs, and have imperfect knowledge about the prevailing policy regime, implies that they do not know the statistical processes describing the equilibrium evolution of various prices and policy variables exogenous to their decision problem.

To give emphasis to long-run outcomes, assume natural-rate shocks are iid. Because the model is purely forward looking under rational expectations, equilibrium implies endogenous variables are linear functions of the natural-rate disturbance, with zero mean in deviations from steady state. Assume then that agents have the following forecasting model

$$\begin{aligned} z_t &= a_t + \varepsilon_t \\ a_t &= a_{t-1} + v_t \end{aligned}$$

where

$$z_t = \begin{bmatrix} \pi_t \\ x_t \\ i_t \end{bmatrix} \text{ and } a_t = \begin{bmatrix} a_t^\pi \\ a_t^x \\ a_t^i \end{bmatrix}$$

and $E[\varepsilon_t \varepsilon_t'] = R$ and $E[v_t v_t'] = Q$ give the prior beliefs on the variance of iid primitive disturbances, ε_t, and innovations to low-frequency drift, v_t. This time-varying parameter model implies agents must learn the long-run averages of each series. This is the only source of uncertainty agents face about the true dynamics — therefore called a "shifting endpoint model" by Kozicki and Tinsley (2001). As the prior variance Q goes to zero, the forecasting model converges to rational expectations beliefs.

Beliefs are revised according to a standard Kalman filter recursion. Given emphasis to long-run outcomes, we assume a steady-state Kalman filter, in which agents have priors satisfying

$$Q = g^2 R$$

for a constant gain coefficient $0 < g < 1$. This implies beliefs evolve according to

$$a_t = a_{t-1} + g(z_t - a_{t-1}).$$

Because Q is in general non-zero, beliefs fail to converge to rational expectations equilibrium in the presence of uncertainty. However, as shown by Evans and Honkapohja (2003), beliefs will be ergodically distributed around the underlying rational expectations equilibrium. Forecasts are then determined as

$$\hat{E}_t z_T = a_{t-1}$$

which, combined with the structural equations, provides the "anticipated-utility" solution to the model — see Sargent (1999) and Eusepi and Preston (2016) for discussion. Note, as standard in the learning literature, long-run conditional expectations of any variable are revised only in response to surprise movements in that same variable. See Sargent, Williams, and Zha (2006) and Eusepi, Giannoni, and Preston (2015) for more general analyses.

Four remarks are warranted. First, Crump, Eusepi, and Moench (2015) demonstrate a closely related class of beliefs are consistent with the properties of a large range of professional forecasts.[3] Second, this belief structure delivers analytical tractability, ensuring a linear-quadratic optimal policy problem which can be solved using standard methods. Third, the assumption is less restrictive than might be thought. In fact, the drift term generally imposes the strictest requirements on policy for stability and, more generally, it drives the largest deviations relative to rational expectations predictions. Moreover, the drift itself has a clean economic interpretation representing time variation in the perceived long-run conditional mean of each variable of interest. In the case of inflation, it reflects uncertainty about a central bank's inflation target. Fourth, for more general specifications of monetary policy the evolution of beliefs may depend upon lagged endogenous variables, such as the output gap under the optimal commitment policy. However, the dynamics of the output gap are mean reverting. Hence, in the long run, the only relevant uncertainty concerns low-frequency drift. The assumed belief structure, therefore, captures the variation of interest for making the conceptual points of the paper.

[3] Formally they permit estimates of the long-run conditional mean of any variable to depend on short-run forecast errors of all variables. The Kalman gain matrix in this case cannot be summarized by a single gain coefficient. While the results of Eusepi, Giannoni, and Preston (2015) make clear such belief structures place constraints on equilibrium dynamics under optimal policy, the simpler case is considered here to obtain analytical results.

3 The Policy Problem

The central bank seeks to minimize the expected discounted quadratic loss

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\pi_t^2 + \lambda_x \left(x_t - x^*\right)^2\right] \tag{5}$$

where $\lambda_x \geq 0$ determines the relative weight placed on inflation stabilization versus output gap stabilization, and x^* is the optimal output gap. The period loss function is derived as a second-order approximation to household utility. The assumption of non-rational expectations does not affect these calculations. The optimal output gap is proportional to the difference between the steady state efficient and natural rates of output — which here differ due to monopolistic competition.[4]

Assume the central bank has rational expectations and knows the structure of the economy. Assuming rational expectations serves to emphasize the degree to which imperfect knowledge on the part of households and firms constrains what can be achieved — a less sophisticated central bank presumably faces a more difficult control problem.

3.1 Rational Expectations Equilibrium

Under rational expectations the aggregate demand and supply equations can be given the familiar recursive representation

$$x_t = \mathbb{E}_t x_{t+1} - \left(i_t - E_t \pi_{t+1} - r_t^n\right) \tag{6}$$

$$\pi_t = \kappa x_t + \beta \mathbb{E}_t \pi_{t+1}. \tag{7}$$

The conditions under which relations (3) and (4) are reducible to these expressions are discussed by Preston (2005) and Eusepi and Preston (2016). The central bank seeks to minimize the objective function (5) subject to (6) and (7) by choice of state-contingent sequences $\{x_t, \pi_t, i_t\}$. The following proposition can be stated.

Proposition 1 *Under rational expectations, optimal policy under discretion implies long-run inflation is*

$$\pi^D = \lim_{T \to \infty} \mathbb{E}_t \pi_{t+T} = \frac{\kappa \lambda_x x^*}{\kappa^2 + \lambda_x (1 - \beta)}.$$

Optimal policy under commitment implies long-run inflation is

$$\pi^C = \lim_{T \to \infty} \mathbb{E}_t \pi_{t+T} = 0.$$

[4] See Woodford (2003) for derivations.

Proofs for these standard results can be found in Woodford (2003). The proposition establishes the classic inflation-bias first identified by Kydland and Prescott (1977). The bias is larger the greater the weight given output gap stabilization; the larger is the household discount factor; the larger the slope of the Phillips curve, which in turn is larger the greater the frequency of price adjustment, and the smaller the degree of strategic complementarity in price setting. Under commitment, the welfare loss from staggered price-setting, the model's only friction, is minimized under price stability.

3.2 Learning Dynamics

Under learning dynamics feasible inflation and output gap sequences must satisfy the aggregate demand (3) and aggregate supply (4) relations. Evaluating the forecasts in each expression provides

$$\pi_t = \kappa x_t + \kappa \frac{\alpha\beta}{1-\alpha\beta} a^x_{t-1} + \frac{(1-\alpha)\beta}{1-\alpha\beta} a^\pi_{t-1} \quad (8)$$

$$x_t = -(i_t - r^n_t) - \frac{1}{1-\beta}\left(\beta a^i_{t-1} - a^\pi_{t-1}\right) + (1-\beta) a^x_{t-1} \quad (9)$$

with beliefs evolving according to

$$a^\pi_t = a^\pi_{t-1} + g\left(\pi_t - a^\pi_{t-1}\right) \quad (10)$$

$$a^x_t = a^x_{t-1} + g\left(x_t - a^x_{t-1}\right) \quad (11)$$

$$a^i_t = a^i_{t-1} + g\left(i_t - a^i_{t-1}\right). \quad (12)$$

The optimal policy problem is to minimize the loss (5) subject to the five constraints (8)-(12) by choice of sequences $\{x_t, \pi_t, i_t, a^\pi_t, a^x_t, a^i_t\}$, taking as given initial beliefs a^π_{-1}, a^x_{-1} and a^i_{-1}. Eusepi, Giannoni and Preston (2015) emphasis two properties of optimal policy problems under drifting beliefs. First, there is no distinction between discretion and commitment. Beliefs are state variables. Second, the aggregate demand equation is in general a constraint on the optimal choices of the central bank. Because of the interdependence between interest rates and interest-rate beliefs, it will not be true one can always determine from the aggregate demand equation an interest-rate path that is consistent with any choice of sequences for the output gap and inflation rate.

The first-order conditions characterizing optimality are described in the appendix. They constitute a system of linear rational expectations equations in the variables

$$\left\{x_t, \pi_t, i_t, a^\pi_t, a^x_t, a^i_t, \lambda_{1,t}, \lambda_{2,t}, \lambda_{3,t}, \lambda_{4,t}, \lambda_{5,t}\right\}$$

where $\lambda_{j,t}$ for $j \in [1,5]$ are Lagrange multipliers attached to each of the five constraints, whose solution has the following general properties.

Proposition 2 *If the system composed of the conditions (8)-(12) and the policy problem's first-order conditions admits a bounded solution, then this system has a unique bounded rational expectations equilibrium for all initial conditions* $\left\{a_{-1}^{\pi}, a_{-1}^{x}, a_{-1}^{i}\right\}$. *The unique steady-state implies*

$$\pi^{LR} = \lim_{T\to\infty} \mathbb{E}_t \pi_T = \frac{\lambda_x \kappa x^*}{\kappa^2 \Xi + \lambda_x (1-\beta)} \tag{13}$$

and

$$x^{LR} = \lim_{T\to\infty} \mathbb{E}_t x_T = 0$$

where

$$\Xi = \frac{g\beta + (1-\beta)(1-\alpha\beta)}{g\beta(1-\beta) + (1-\beta)(1-\alpha\beta)} \geq 1.$$

The proof, which appeals to results in Giannoni and Woodford (2010), is in the Appendix. The logic of this result implies that regardless of the sources of uncertainty, long-run conditional expectations converge to steady state provided that the equilibrium is bounded. For this reason, inclusion of different shocks, or more general specifications of statistical processes for these shocks, has no effect on long-run conditional expectations. As shown in the Appendix, for the system to admit a bounded solution it is sufficient for $g \in (0,1)$ to satisfy (i) either $g < 2(1-\beta)$ or $g > \beta^{-1} - \beta$, and, (ii) in case $g > \frac{(1-\alpha\beta)(\lambda_x + \kappa^2)}{\lambda_x(1-\beta)+\kappa^2}$, to also satisfy either $g < 2(1-\alpha\beta)$ or $g > \left(\beta^{-1}+1\right)(1-\alpha\beta)$. These conditions are met when $\beta \leq 1/2$, and generally also when β is large enough. For instance, if $\beta = 0.99$, the system admits a bounded solution for any g other than in the intervals $[0.0200, 0.0201]$ and $[2(1-\alpha\beta), 2.01(1-\alpha\beta)]$.[5] When the above conditions are violated, some variables may not be bounded. In particular, while inflation and the output gap remain close to their optimal steady state under optimal policy, the nominal interest tends to fluctuate more and more as g approaches these thresholds.

3.3 Some Implications

The steady-state of inflation (13) obtained in Proposition 2 shows price stability is in general not optimal under imperfect knowledge. Optimal policy delivers a positive inflation rate, the magnitude of which depends on all model parameters, so long as the optimal output gap is positive. However, under maintained parametric assumptions the optimal inflation rate is no larger than what obtains under optimal discretion. Long-run inflation is endogenous in the

[5] While the restrictions (i) are necessary for a bounded solution, conditions (ii) are sufficient but not necessary.

sense that while it is true that a policy maker attempts to have beliefs evolve in a certain way, in equilibrium different beliefs lead to different inflation outcomes — beliefs constrain what can be achieved by monetary policy. Importantly, in the long-run beliefs are consistent with observed outcomes. There are no systematic errors on the part of agents.

Two limiting cases are revealing. Consider beliefs in which the constant gain is in the neighborhood of zero. Then long-run inflation is given by the expression

$$\lim_{g \to 0} \pi^{LR} = \frac{\kappa \lambda_x x^*}{\kappa^2 + \lambda_x (1 - \beta)}$$

which is precisely the inflation bias observed in this model with rational expectations and under optimal policy with discretion. That this occurs follows immediately from the nature of beliefs. With a gain in the neighborhood of zero, beliefs are approximately never revised — monetary policy cannot influence beliefs. The central bank therefore takes expectations as given, exactly as would be the case if firms had rational expectations.

Alternatively, suppose households are very patient, approximately weighting utility in each period equally. The optimal long-run inflation rate is then given by

$$\lim_{\beta \to 1} \pi^{LR} = 0.$$

Price stability is optimal, and, importantly, the outcome under optimal commitment policy under rational expectations. This establishes an important robustness result. Optimal commitment policies under rational expectations are often criticized on the ground that they rely too heavily on the ability to manage future expectations through announcements.[6] This result indicates price stability may nonetheless be optimal even when a central bank has no influence over future expectations through announced policy actions, and can only influence beliefs through actions, so long as household have a high degree of patience.

3.4 Some Intuition

To provide intuition, Figure 1 plots long-run inflation as a function of beliefs and household patience. Parametric assumptions are: $\lambda_x = 0.017$; $x^* = 0.05$; $\kappa = 0.104$; $\beta = 0.99$; and $\alpha = 0.80$ — standard values in Woodford (2003). In the case of a gain in the neighborhood of zero, the discretionary inflation bias obtains. As is well understood, as the discount factor of households rises, the short-run trade-off between inflation and the output gap worsens. In equilibrium, the long-run inflation rate increases as the central bank nonetheless attempts to achieve the optimal output gap x^*.

[6] See Woodford (2010) for one response to this concern.

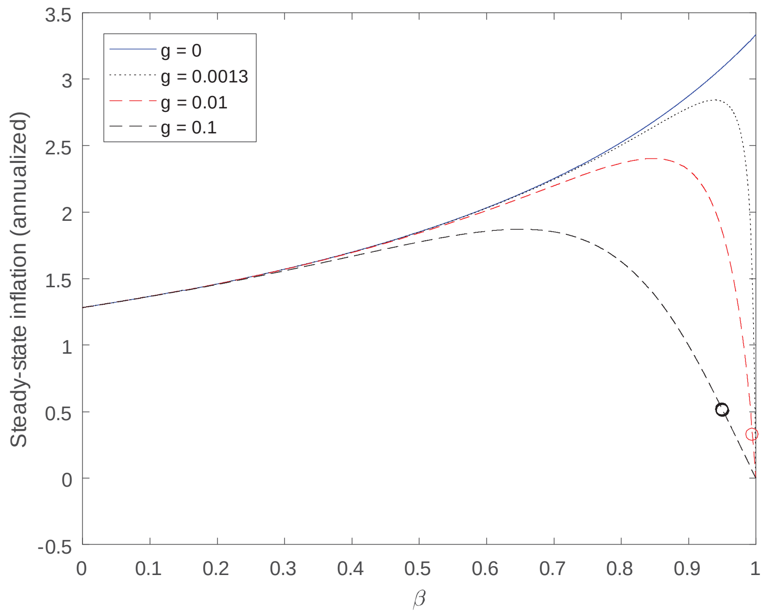

Figure 1: Steady-state inflation as a function of the discount factor for different gains. Circles denote points in the parameter space which fail to satisfy the conditions of proposition 2.

For positive gains, a second effect operates. As before, an higher discount factor raises equilibrium long-run inflation due to the worsening short-run trade-off between inflation and the output gap. However, the central bank internalizes the effects of policy on the evolution of inflation expectations. Higher inflation leads to higher present discounted losses, exacerbated by greater concern about the future at higher discount factors. This leads to lower desired equilibrium inflation in the long run. This second effect is stronger the larger the gain — i.e. the more sensitive beliefs are to inflation. The limiting case of a perfectly patient household would lead to an infinite loss at any positive rate of inflation. There is no advantage to exploiting expectations in the short-run, making price stability optimal.

Figure 2 plots the optimal long-run inflation rate as a function of the gain for different values of the discount factor. For gains in the neighborhood of zero, the rational expectations stabilization bias is observed, regardless of household patience. As the gain rises, higher discount factors imply greater present discounted losses from exploiting the sluggishness of inflation expectations. It becomes optimal to deliver low long-run equilibrium inflation.

Proposition 3 *In general the optimal long-run inflation rate is positive such that*

$$0 = \pi^C \leq \pi^{LR} \leq \pi^D$$

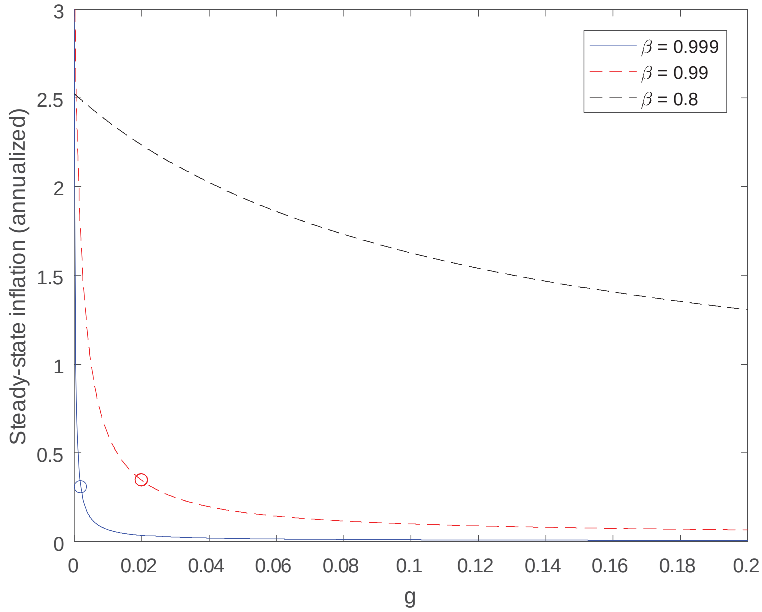

Figure 2: Steady state inflation as a function of the gain for different discount factors. Circles denote points in the parameter space which fail to satisfy the conditions of proposition 2.

and satisfies the limit properties

$$\begin{aligned} \lim_{\beta \to 1} \pi^{LR} &= \pi^{C} \\ \lim_{g \to 0} \pi^{LR} &= \pi^{D}. \end{aligned}$$

The proof follows directly from (13) and proposition 1.

The insights of figure 2 acquire further interest in the light of recent discussion about the optimal inflation rate. Ascari, Phanouf and Sims (2015) and Coibion, Gorodnichenko and Wieland (2011) demonstrate in structural New Keynesian models with rational expectations that the optimal rate of inflation is not much greater than zero. As the average inflation rate rises, various frictions generate costs sufficient to offset the benefits of avoiding the zero lower bound on interest rates. The learning literature adduces evidence for a range of gain coefficients of the order 0.01 and 0.05. And Eusepi and Preston (2011) estimate a gain of 0.0013, using a real business cycle model solved under the anticipated utility approach adopted here. For this estimate of the gain, the optimal inflation rate ranges from zero to about 3 percent on an annual basis. With β equal to 0.99, the optimal inflation rate becomes quite different from zero, equal to about 2 percent. While a serious study of the quantitative relevance of learning dynamics for the optimal inflation rate is beyond the present discussion, this casual empiricism suggests learning might be an important argument in favor of higher inflation targets.

A final observation is warranted. One can interpret the gain as a metric of credibility. In the model agents are attempting to learn, among other things, the long-run inflation objective of the central bank. The smaller the gain, the smaller the drift in beliefs about the long-run inflation target in response to different disturbances. The results indicate that a central bank perceived as more credible would find it optimal to have a higher inflation rate. Exemplifying this interpretation, Carvalho, Eusepi, Moench, and Preston (2015) show, using a model with a state-contingent gain fitted to US data, the 1970s are characterized by a high gain of 0.12, while in the past 20 years the gain has been small and declining to value of about 0.015 in recent years, as the Federal Reserve acquired greater credibility for low and stable inflation: beliefs over this latter period exhibit strong convergence to the central bank's inflation target. Over this time the incentives of the central bank to create, optimally, positive inflation rise with the decline in gain, reflecting increased latitude to respond to economic developments when beliefs are less sensitive to new information.

3.5 Patient central bank

The previous sections demonstrate the presence of an intertemporal trade-off leads to lower long-term inflation when agents are more patient. This begs the question of whether for given household preferences a central bank that is more patient than society will also deliver lower inflation in equilibrium. Is there a "patient central bank" analogue to Rogoff's (1985) conservative central bank?

Suppose the central bank has discount factor $\tilde{\beta}$, which can be different from the household discount factor β. The policy problem becomes to minimize

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \tilde{\beta}^t \left[\pi_t^2 + \lambda_x (x_t - x^*)^2\right]$$

subject to the five constraints (8)-(12) by choice of sequences $\{x_t, \pi_t, i_t, a_t^\pi, a_t^x, a_t^i\}$, taking as given initial beliefs a_{-1}^π, a_{-1}^x and a_{-1}^i. Analogous calculations to the benchmark problem permits deriving the optimal long-run inflation rate to be

$$\pi = \frac{\lambda_x \kappa x^*}{\kappa^2 \Xi\left(\tilde{\beta}; g, \beta\right) + \lambda_x (1 - \beta)}$$

where the function $\Xi\left(\tilde{\beta};g,\beta\right)$ is bounded below by unity and has the properties

$$\lim_{g\to 0}\Xi\left(\tilde{\beta};g,\beta\right)=1$$

and

$$\frac{\partial\Xi\left(\tilde{\beta};g,\beta\right)}{\partial\tilde{\beta}}>0$$

for positive gains. It follows that for given household preferences and beliefs, a patient central banker will give a lower long-term equilibrium inflation rate than observed under discretion.

In the limit of a very patient central banker, $\tilde{\beta}\to 1$, who values each period's loss equally

$$\pi=\frac{\kappa\lambda_x x^*}{\kappa^2\frac{((1-\alpha\beta)+\kappa\alpha\beta)}{(1-\beta)}+\lambda_x\left(1-\beta\right)}.$$

While the optimal inflation rate is always lower than discretion, a patient central bank does not deliver price stability. Interestingly optimal inflation is independent of the gain, which makes sense — a patient central banker who values losses equally finds no advantage in exploiting short-term expectations, no matter how quickly or slowly they adjust. Finally, note that welfare under a patient central bank will be lower than under the benchmark policy, since that policy is the optimal policy for the welfare-theoretic loss function.

4 Discussion

This final section provides brief commentary on various aspects of the optimal policy problem.

4.1 On the Literature

A number of papers have explored optimal policy in the New Keynesian model under learning dynamics. These include Gaspar, Smets and Vestin (2007, 2010), Molnar and Santoro (2013), Eusepi, Giannoni and Preston (2015) and Mele, Molnar, and Santoro (2015). The central difference to these analyses is the assumed loss function. Consistent with the underlying microfoundations, the policy problem assumes a distorted steady state, which implies the central bank seeks to deliver a positive steady-state output gap in equilibrium. While the merits of this assumption are discussed further below, it permits establishing a general property of policy under learning dynamics.

Earlier analyses, such as Sargent (1999) and Molnar and Santoro (2013), demonstrate the household's discount factor is critical to optimal policy outcomes, and specifically, their relation to optimal policy predictions under rational expectations. Sargent (1999), in a model of central bank learning with a rational private sector, shows that if the discount factor is unity then policy delivers the optimal commitment equilibrium. Values less than unity deliver optimal discretion. Molnar and Santoro (2013), which provides the analytical foundations of this analysis, instead study a model of private sector learning. They show that for discount factors less than unity, as the gain goes to zero the optimal discretion equilibrium obtains under learning. However, because they focus on the optimal response to shocks, rather than on the question of long-run inflation outcomes, their linear analysis is unable to consider the role of household patience in delivering optimal commitment — the belief structure analyzed fails to nest the form of expectations under optimal commitment, since such policies are history dependent.

This led to subsequent work by Mele, Molnar, and Santoro (2015) which studies the optimal response to disturbances when agents' beliefs nest those that obtain in both the discretion and commitment equilibria under rational expectations. They show that as beliefs converge when the gain goes to zero, optimal policy under learning converges to optimal discretion. While they don't analyze the limiting case of patience near unity, it is likely optimal commitment will obtain so long as the gain coefficient is bounded away from zero. The analysis presented here is identical in spirit. And while our simplifying assumptions restrict attention to questions about the optimal long-run inflation rate, rather than on dynamic responses to shocks, it has the advantage of nesting both optimal commitment and discretion equilibria under rational expectations, while at the same time being a standard linear-quadratic control problem, rather than non-linear. As such it permits clean analytical conditions under which optimal discretion and optimal commitment outcomes will obtain, along with clear intuition for the determinants of long-run inflation. Stated differently, the contribution is to map Sargent's insights into the canonical New Keynesian model with household and firm learning dynamics. Doing so provides a useful frame of reference to interpret results from earlier literature on optimal policy under learning.

4.2 On Beliefs

Despite the simplicity of the specification of beliefs, they are coherent with observed features of aggregate data. There is pervasive evidence that macroeconomic time series exhibit low-frequency movements, well captured by time-varying parameter models. This is true of inflation (see Stock and Watson, 2007; Cogley and Sbordone, 2008; Cogley, Primiceri and Sargent, 2010); output or the output gap (see Stock and Watson, 1989; Cogley and Sargent, 2005; Laubach and Williams, 2003); and nominal interest rates (see Kozicki and Tinsley, 2001; Gurkaynak, Sack and Swanson, 2005). Such models also capture quite well the evolution of short- and long-term surveys forecasts of these same time series. For example Branch and Evans (2006), Edge, Laubach, and Williams (2007), Kozicki and Tinsley (2012) and Crump, Eusepi and Moench (2016).

Yet despite this evidence, most models used for policy evaluation either fail to account for such variation, or provide accounts that are dubitable. Most notably, models deployed for the evaluation of monetary policy, such as the Smets and Wouters (2007) model, attribute low-frequency patterns in data to an exogenously determined drift in the central bank's long-run inflation objectives. While it would be difficult to dispute variation in policy maker preferences for inflation might explain some variation in observed inflation outcomes, its seems equally plausible that other mechanisms are at play. In particular, expectations themselves are likely to be a source of low-frequency variation in data. For example, Carvalho, Eusepi, Moench, and Preston (2015) adduce evidence that drifting long-run inflation expectations are important to explaining the great inflation of the 1970s and subsequent reduction in inflation over recent decades. Uncertainty about long-run properties of the economy, such as potential output and wages, or the central bank's inflation target, represent fundamental constraints on spending, hiring and pricing plans. If agents form inferences about these macroeconomic objectives using statistical methods that permit detection of low-frequency variation, then any decision based on such forecasts will inherit this drift. This paper, along with the earlier discussed literature, contributes to understanding optimal policy under such beliefs.

4.3 On the Lucas Critique

A possible objection to optimal policy analysis under learning dynamics is the Lucas critique. While such exercises are by assumption subject to this critique, there are limits to its relevance. First, any beliefs premised on a statistical filtering problem are subject to the same concerns. Second, despite this, it is not true that beliefs fail to adjust to a new policy regime. While the analysis assumes a fixed gain, so that there is a fixed mapping between short-term forecast errors and long-run beliefs, long-run beliefs are necessarily endogenous to policy through its effects on short-run forecast errors. This limits the extent to which a sophisticated central bank can manipulate beliefs. Indeed, in the long run, beliefs are consistent with equilibrium outcomes. Third, we assume a constant gain. In general the optimal gain will depend on the policy regime in place, and any serious quantitative exercise on optimal policy should account for this. However, the logic of the policy outcomes described here will continue to obtain. And ultimately, even if a central bank could drive the gain to zero, the inflation bias of optimal discretion would obtain.

4.4 On the Welfare Criterion

The sub-optimality of discretion policy identified by Kydland and Prescott (1977) engendered a substantial literature on delegation. To the extent policy makers cannot commit, delegating alternative loss functions, that differ to the true welfare-theoretic loss function, could deliver welfare improvements. This led to Rogoff's (1985) conservative central banker, and other proposals — see Vestin (2000) for a discussion in the context of the New Keynesian policy analysis presented here. Most directly relevant, it led to the proposal that a central bank should be charged with minimizing the loss function (5) with $x^* = 0$. Under such a loss function, the central bank, even though unable to commit, would deliver price stability in the long run.

Putting aside the merits of delegating loss functions — such approaches are likely to be problematic for central bank transparency and communications policy — it is important to recall there is no distinction between discretion and commitment under learning. Beliefs are state variables. It follows that if the central bank does inherit an environment in which there is a distorted steady state (because of monopolistic competition and distortionary taxes for example), then the optimal policy necessarily delivers a positive inflation rate. Delegating a welfare function in which $x^* = 0$ will deliver inferior welfare outcomes if in fact there are economic reasons for a central bank to achieve an a positive output gap on average.

5 Conclusion

This paper demonstrates that price stability is in general not optimal when agents have beliefs that exhibit low-frequency drift. The optimal long-run inflation rate depends on all model parameters, though it is bounded by the optimal long-run inflation rates observed under discretion and commitment when agents have rational expectations. Interestingly, these rational expectations outcomes are delivered as special cases of the optimal policy model under learning. When the size of low-frequency drift is negligible, then optimal inflation coincides with the predictions under discretion. In contrast, for given low-frequency drift, if households are highly patient, price stability is optimal in the long run.

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A Appendix

The optimal policy problem is to minimize the loss (5) subject to the five constraints (8)-(12). The Lagrangian is:

$$\max_{\{\pi_t,x_t,i_t,a_t^\pi,a_t^x,a_t^i\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{c} \frac{1}{2}\left[\pi_t^2 + \lambda_x (x_t - x^*)^2\right] \\ +\lambda_{1,t}\left(-\pi_t + \kappa x_t + \frac{\kappa\alpha\beta}{1-\alpha\beta} a_{t-1}^x + \frac{(1-\alpha)\beta}{1-\alpha\beta} a_{t-1}^\pi\right) \\ +\lambda_{2,t}\left(-x_t - (i_t - r_t^n) - \frac{1}{1-\beta}\left(\beta a_{t-1}^i - a_{t-1}^\pi\right) + (1-\beta) a_{t-1}^x\right) \\ +\lambda_{3,t}\left(-a_t^\pi + a_{t-1}^\pi + g\left(\pi_t - a_{t-1}^\pi\right)\right) \\ +\lambda_{4,t}\left(-a_t^x + a_{t-1}^x + g\left(x_t - a_{t-1}^x\right)\right) \\ +\lambda_{5,t}\left(-a_t^i + a_{t-1}^i + g\left(i_t - a_{t-1}^i\right)\right) \end{array} \right\}.$$

The first-order conditions are:

$$\pi_t - \lambda_{1,t} + g\lambda_{3,t} = 0 \quad (14)$$

$$\lambda_x (x_t - x^*) + \lambda_{1,t}\kappa - \lambda_{2,t} + \lambda_{4,t} g = 0 \quad (15)$$

$$\frac{\kappa\alpha\beta}{1-\alpha\beta}\beta\mathbb{E}_t\lambda_{1,t+1} - \lambda_{4,t} + \beta(1-g)\mathbb{E}_t\lambda_{4,t+1} + (1-\beta)\beta\mathbb{E}_t\lambda_{2,t+1} = 0 \quad (16)$$

$$\frac{(1-\alpha)\beta}{1-\alpha\beta}\beta\mathbb{E}_t\lambda_{1,t+1} + \frac{\beta}{1-\beta}\mathbb{E}_t\lambda_{2,t+1} - \lambda_{3,t} + \beta(1-g)\mathbb{E}_t\lambda_{3,t+1} = 0 \quad (17)$$

$$g\lambda_{5,t} - \lambda_{2,t} = 0 \quad (18)$$

$$-\lambda_{5,t} + \beta(1-g)\mathbb{E}_t\lambda_{5,t+1} - \frac{\beta^2}{1-\beta}\mathbb{E}_t\lambda_{2,t+1} = 0. \quad (19)$$

A.1 Proof of Proposition 2: Steady State

The proof proceeds in two steps. First, compute the steady state rate of inflation to verify the value given in proposition. Second, prove that the optimal policy problem has a unique bounded solution in which dynamics converge to this steady state.

In steady state (18) and (19) imply $\lambda_2 = \lambda_5 = 0$. Using relations (15) and (16) imply

$$\lambda_x (x - x^*) = -\lambda_1 \kappa \left(1 + \frac{\alpha\beta^2}{1-\alpha\beta}\frac{g}{1-\beta(1-g)}\right)$$

and using the Phillips curve in steady state provides

$$\lambda_x\left(\frac{(1-\beta)\pi}{\kappa} - x^*\right) = -\lambda_1\kappa\left(1 + \frac{\alpha\beta^2}{1-\alpha\beta}\frac{g}{1-\beta(1-g)}\right).$$

Now (14) and (17) imply

$$\begin{aligned} \pi &= \lambda_1\left(1 - \frac{g}{1-\beta(1-g)}\frac{(1-\alpha)\beta^2}{1-\alpha\beta}\right) \\ &= \frac{\lambda_x\left(x^* - \frac{(1-\beta)\pi}{\kappa}\right)}{\kappa\left(1 + \frac{\alpha\beta^2}{1-\alpha\beta}\frac{g}{1-\beta(1-g)}\right)}\left(1 - \frac{g}{1-\beta(1-g)}\frac{(1-\alpha)\beta^2}{1-\alpha\beta}\right) \end{aligned}$$

and hence

$$\pi\left[\begin{array}{c}\kappa\left(1+\frac{\alpha\beta^2}{1-\alpha\beta}\frac{g}{1-\beta(1-g)}\right)\\+\lambda_x\frac{(1-\beta)}{\kappa}\left(1-\frac{g}{1-\beta(1-g)}\frac{(1-\alpha)\beta^2}{1-\alpha\beta}\right)\end{array}\right]=\lambda_x x^*\left(1-\frac{g}{1-\beta\left(1-g\right)}\frac{\left(1-\alpha\right)\beta^2}{1-\alpha\beta}\right).$$

The steady-state inflation rate is given by

$$\pi=\frac{\lambda_x x^*\left(1-\frac{g}{1-\beta(1-g)}\frac{(1-\alpha)\beta^2}{1-\alpha\beta}\right)}{\kappa\left(1+\frac{\alpha\beta^2}{1-\alpha\beta}\frac{g}{1-\beta(1-g)}\right)+\lambda_x\frac{(1-\beta)}{\kappa}\left(1-\frac{g}{1-\beta(1-g)}\frac{(1-\alpha)\beta^2}{1-\alpha\beta}\right)}$$

as required.

Evaluating the model's equations (8)–(12) in steady state yields

$$\begin{aligned}\pi &= \kappa x+\frac{\kappa\alpha\beta}{1-\alpha\beta}a^x+\frac{\left(1-\alpha\right)\beta}{1-\alpha\beta}a^\pi\\ x &= -\left(i-r^n\right)-\frac{1}{1-\beta}\left(\beta a^i-a^\pi\right)+\left(1-\beta\right)a^x\\ a^\pi &= \pi\\ a^x &= x\\ a^i &= i\end{aligned}$$

or

$$\begin{aligned}\pi &= \kappa x+\frac{\kappa\alpha\beta}{1-\alpha\beta}x+\frac{\left(1-\alpha\right)\beta}{1-\alpha\beta}\pi\\ x &= -\left(i-r^n\right)-\frac{1}{1-\beta}\left(\beta i-\pi\right)+\left(1-\beta\right)x\end{aligned}$$

This implies in turn the steady-state output gap

$$x=\frac{1-\beta}{\kappa}\pi$$

and the steady state interest rate

$$i=\left(1-\beta\right)r^n+\left(1-\beta\left(1-\beta\right)^2\kappa^{-1}\right)\pi,$$

where $r^n=0$ in steady state.

A.2 Proof of Proposition 2: Equilibrium Dynamics

We now show that under optimal policy, the equilibrium dynamics are unique and bounded. Consider the vector of $m=6$ endogenous variables $y_t=\left[\tilde{\pi}_t,\tilde{x}_t,\tilde{\imath}_t,\tilde{a}_t^\pi,\tilde{a}_t^x,\tilde{a}_t^i\right]'$ where the tilde denotes the fact that the variables are expressed in deviations from the steady state characterized above. The $n=5$ structural equations (8)–(12) can be written compactly in the form

$$\begin{bmatrix}1&-\kappa&0&0&0&0\\0&1&1&0&0&0\\-g&0&0&1&0&0\\0&-g&0&0&1&0\\0&0&-g&0&0&1\end{bmatrix}\begin{bmatrix}\tilde{\pi}_t\\\tilde{x}_t\\\tilde{\imath}_t\\\tilde{a}_t^\pi\\\tilde{a}_t^x\\\tilde{a}_t^i\end{bmatrix}=\begin{bmatrix}0&0&0&\frac{(1-\alpha)\beta}{1-\alpha\beta}&\frac{\kappa\alpha\beta}{1-\alpha\beta}&0\\0&0&0&\frac{1}{1-\beta}&1-\beta&-\frac{\beta}{1-\beta}\\0&0&0&1-g&0&0\\0&0&0&0&1-g&0\\0&0&0&0&0&1-g\end{bmatrix}\begin{bmatrix}\tilde{\pi}_{t-1}\\\tilde{x}_{t-1}\\\tilde{\imath}_{t-1}\\\tilde{a}_{t-1}^\pi\\\tilde{a}_{t-1}^x\\\tilde{a}_{t-1}^i\end{bmatrix}+\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}\hat{r}_t^n$$

or

$$\bar{I}y_t = \bar{A}y_{t-1} + \bar{C}\hat{r}_t^n$$

where

$$\bar{I} = \beta \begin{bmatrix} 1 & -\kappa & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ -g & 0 & 0 & 1 & 0 & 0 \\ 0 & -g & 0 & 0 & 1 & 0 \\ 0 & 0 & -g & 0 & 0 & 1 \end{bmatrix}, \quad \bar{A} = \beta \begin{bmatrix} 0 & 0 & 0 & \frac{(1-\alpha)\beta}{1-\alpha\beta} & \frac{\kappa\alpha\beta}{1-\alpha\beta} & 0 \\ 0 & 0 & 0 & \frac{1}{1-\beta} & 1-\beta & -\frac{\beta}{1-\beta} \\ 0 & 0 & 0 & 1-g & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-g & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-g \end{bmatrix}, \quad \bar{C} = \beta \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The $m = 6$ first-order conditions (14)–(19) can be written as

$$\bar{A}_t' \mathbb{E}_t \lambda_{t+1} = \beta^{-1} \bar{I}' \lambda_t - S y_t, \tag{20}$$

where $\lambda_t = [\lambda_{1,t} ..., \lambda_{5,t}]'$ is a vector of n non-predetermined Lagrange multipliers, and S is a diagonal matrix with $[1, \lambda_x, 0, 0, 0, 0]$ on the diagonal. Combining the equilibrium conditions and FOCs, we obtain a system of 11 dynamic equations in 11 variables

$$M \mathbb{E}_t \begin{bmatrix} \lambda_{t+1} \\ y_t \end{bmatrix} = N \begin{bmatrix} \lambda_t \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{C} \end{bmatrix} \hat{r}_t^n \tag{21}$$

where

$$M = \begin{bmatrix} \bar{A}' & S \\ 0 & \bar{I} \end{bmatrix}, \quad \text{and} \quad N = \begin{bmatrix} \beta^{-1}\bar{I}' & 0 \\ 0 & \bar{A} \end{bmatrix}.$$

Given that the dynamic system (21) has six predetermined variables, y_{t-1}, and five non-predetermined variables, λ_t, this system admits a unique bounded solution if five of the generalized eigenvalues of N and M — i.e., the roots of $\det(\mu M - N) = 0$ — lie outside the unit circle while six lie inside. We can show that the characteristic polynomial satisfies

$$\begin{aligned} \det(\phi M - \rho N) &= \frac{-\beta^5}{(1-\beta)^2} \phi^3 \rho^2 (\phi - \rho(1-g)) (\phi\beta(1-g) - \rho) ((1-\beta)\phi + \rho(g+\beta-1)) \\ &\quad \times ((g+\beta-1)\beta\phi + \rho(1-\beta)) \left(\beta a_1 \phi^2 + a_2 \rho\phi + \rho^2 a_1\right) \end{aligned}$$

where

$$\begin{aligned} a_1 &= \left(\left(1 - \beta\frac{1-\alpha}{1-\alpha\beta}\right) g - 1\right) \lambda_x + \kappa^2 \left(g\frac{1}{1-\alpha\beta} - 1\right) \\ a_2 &= \left(1 + \beta\left(1 - g\frac{1-\beta}{1-\alpha\beta}\right)^2\right) \lambda_x + \kappa^2 \left(1 + \beta\left(1 - g\frac{1}{1-\alpha\beta}\right)^2\right) > 0. \end{aligned}$$

This can equivalently be written as the product of 11 factors of the form $(\alpha_i\phi - \beta_i\rho)$ for some complex numbers α_i and β_i. The generalized eigenvalues of the matrix pencil $\mu M - N$ are the quantities $\mu = \phi/\rho = \beta_i/\alpha_i$. The 3 factors ϕ^3 for which $\beta_i = 0$ correspond to the 3 zero eigenvalues of $\mu M - N$. The two factors ρ^2 for which $\alpha_i = 0$ corresponds to the "infinite" eigenvalues of $\mu M - N$. Furthermore, one eigenvalue solves $(\mu_6 - (1-g)) = 0$, or $\mu_6 = 1 - g \in (0,1)$; and one solves $(\mu_7\beta(1-g) - 1) = 0$, or $\mu_7 = \beta^{-1}(1-g)^{-1} > 1$.

The next two eigenvalues solve $(1-\beta)\mu_8 + (g+\beta-1) = 0$ or $\mu_8 = \frac{1-\beta-g}{1-\beta}$, and $(g+\beta-1)\beta\mu_9$ $+ (1-\beta) = 0$, or $\mu_9 = \frac{1-\beta}{(1-\beta-g)\beta}$. As discussed further below, the dynamic system (21) admits a unique bounded solution provided that one of the eigenvalues μ_8, μ_9 is inside the unit circle while the other is outside the unit circle. Note that $\mu_9 = \frac{1}{\mu_8 \beta}$. If the first satisfies $|\mu_8| < 1$, then the second satisfies $|\mu_9| = \left|\frac{1}{\mu_8\beta}\right| > 1$. If $|\mu_8| > \beta^{-1}$, then $|\mu_9| = \left|\frac{1}{\mu_8\beta}\right| < 1$. However if $1 < |\mu_8| < \beta^{-1}$, then $|\mu_9| = \left|\frac{1}{\mu_8\beta}\right| > 1$, in which case both μ_8 and μ_9 are explosive roots. The eigenvalues μ_8 and μ_9 are on both sides of the unit circle provided that $\mu_8 = \frac{1-\beta-g}{1-\beta}$ satisfies either $|\mu_8| < 1$ or $|\mu_8| > \beta^{-1}$, or equivalently if g satisfies either (i) $|1-\beta-g| < 1-\beta$ or (ii) $|1-\beta-g| > \beta^{-1}(1-\beta) = \beta^{-1} - 1$. Suppose first that $0 < g \leq 1-\beta$. Condition (i) is then automatically satisfied. Alternatively, suppose that $g > 1-\beta$, then condition (i) can be rewritten as $g < 2(1-\beta)$, and condition (ii) can be rewritten as $g > \beta^{-1} - \beta$. So, μ_8 and μ_9 are on both sides of the unit circle provided that g satisfies either $g < 2(1-\beta)$ or $g > \beta^{-1} - \beta$.

The last two eigenvalues of $\mu M - N$ solve the characteristic equation

$$q(\mu) \equiv \beta\mu^2 + \frac{a_2}{a_1}\mu + 1 = 0,$$

where the polynomial $q(\mu)$ has the property $q(0) = 1 > 0$. The polynomial $q(\mu)$ has one real root μ_{10} inside the unit circle and one root μ_{11} outside if either $q(1) < 1$ or $q(-1) < 1$. We have $q(1) < 1$ if and only if $g < \frac{(1-\alpha\beta)(\lambda_x+\kappa^2)}{\lambda_x(1-\beta)+\kappa^2}$; $q(-1) < 0$ if $g > \frac{(1-\alpha\beta)(\lambda_x+\kappa^2)}{\lambda_x(1-\beta)+\kappa^2}$ and g satisfies either $g < 2(1-\alpha\beta)$ or $g > (\beta^{-1}+1)(1-\alpha\beta)$.

The matrix polynomial $\mu M - N$ has thus six generalized eigenvalues inside the unit circle $(0, 0, 0, 1-g, \mu_8, \mu_{10})$, and five eigenvalues outside the unit circle $(\mu_9, \mu_{11}, \beta^{-1}(1-g)^{-1}, \infty, \infty)$, provided that g satisfies either $g < 2(1-\beta)$ or $g > \beta^{-1} - \beta$, and, in case $g > \frac{(1-\alpha\beta)(\lambda_x+\kappa^2)}{\lambda_x(1-\beta)+\kappa^2}$, it satisfies either $g < 2(1-\alpha\beta)$ or $g > (\beta^{-1}+1)(1-\alpha\beta)$. Given that the dynamic system (21) has six predetermined variables, y_{t-1}, and five non-predetermined variables, λ_t, this system admits a unique bounded solution, if g meets the above conditions. As the first three columns of the matrix $\bar{A}$ are all zero, initial conditions for $\pi_{t-1}, x_{t-1}, i_{t-1}$ are irrelevant. It suffices to specify initial conditions for the three variables $a_{t-1}^{\pi}, a_{t-1}^{x}, a_{t-1}^{i}$.

B Additional Results Not For Publication

The matrices M and N of the characteristic polynomial $\det(\phi M - \rho N)$ are:

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta^2 \frac{1-\alpha}{1-\alpha\beta} & \frac{\beta}{1-\beta} & \beta(1-g) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\alpha\kappa\beta^2}{1-\alpha\beta} & \beta(1-\beta) & 0 & \beta(1-g) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\beta^2}{1-\beta} & 0 & 0 & \beta(1-g) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta & -\beta\kappa & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta & \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\beta g & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\beta g & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta g & 0 & 0 & \beta \end{bmatrix}$$

$$N = \begin{bmatrix} 1 & 0 & -g & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\kappa & 1 & 0 & -g & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -g & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{(1-\alpha)\beta^2}{1-\alpha\beta} & \frac{\kappa\alpha\beta^2}{1-\alpha\beta} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\beta}{1-\beta} & \beta(1-\beta) & -\frac{\beta^2}{1-\beta} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta(1-g) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta(1-g) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta(1-g) \end{bmatrix}.$$

The characteristic polynomial

$$\det(\phi M - \rho N) = \begin{vmatrix} -\rho & 0 & g\rho & 0 & 0 & \phi & 0 & 0 & 0 & 0 & 0 \\ \kappa\rho & -\rho & 0 & g\rho & 0 & 0 & \phi\lambda_x & 0 & 0 & 0 & 0 \\ 0 & -\rho & 0 & 0 & g\rho & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta^2\phi(1-\alpha)\eta & \frac{\beta\phi}{1-\beta} & \psi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha\kappa\beta^2\phi\eta & \beta\phi(1-\beta) & 0 & \psi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\beta^2\phi}{1-\beta} & 0 & 0 & \psi & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta\phi & -\kappa\beta\phi & 0 & -\beta^2\rho(1-\alpha)\eta & -\alpha\kappa\beta^2\rho\eta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta\phi & \beta\phi & \beta\frac{\rho}{\beta-1} & \beta\rho(\beta-1) & \frac{\beta^2\rho}{1-\beta} \\ 0 & 0 & 0 & 0 & 0 & -g\beta\phi & 0 & 0 & \delta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -g\beta\phi & 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -g\beta\phi & 0 & 0 & \delta \end{vmatrix}$$

where $\eta = \frac{1}{1-\alpha\beta}$, $\psi = \beta\phi - \rho - g\beta\phi$, $\delta = \beta(\phi - \rho + g\rho)$.

Expanding and rearranging terms, we obtain:

$$
\begin{aligned}
& \det\left(\phi M-\rho N\right) \\
= & \frac{\beta^2\rho^2}{\left(1-\beta\right)^2}\delta\phi^3\psi\left(g\rho\beta^2-\delta\beta+\delta\right)\left(g\phi\beta^2+\psi\beta-\psi\right) \\
& \times\left(\left(\psi+g\beta^2\phi\eta-g\alpha\beta^2\phi\eta\right)\left(\delta-g\beta^2\eta\rho+g\alpha\beta^2\eta\rho\right)\lambda_x+\kappa^2\left(\psi-g\alpha\beta^2\phi\eta\right)\left(g\alpha\eta\rho\beta^2+\delta\right)\right) \\
= & \frac{\beta^4\phi^3\rho^2}{\left(1-\beta\right)^2}\left(\phi-\rho\left(1-g\right)\right)\left(\beta\phi\left(1-g\right)-\rho\right)\left(\left(1-\beta\right)\phi+\rho\left(g+\beta-1\right)\right)\left(\left(g+\beta-1\right)\beta\phi+\rho\left(1-\beta\right)\right)p
\end{aligned}
$$

where

$$
\begin{aligned}
p &= \left(\beta\phi\left(1-g\right)+g\beta^2\phi\eta\left(1-\alpha\right)-\rho\right)\left(\beta\left(\phi-\rho\left(1-g\right)\right)-g\beta^2\eta\left(1-\alpha\right)\rho\right)\lambda_x \\
&\quad +\kappa^2\left(\beta\phi-\rho-g\beta\phi-g\alpha\beta^2\phi\eta\right)\left(g\alpha\eta\rho\beta^2+\beta\left(\phi-\rho\left(1-g\right)\right)\right) \\
&= -\beta\left(\beta a_1\phi^2+a_2\rho\phi+\rho^2a_1\right)
\end{aligned}
$$

and $a_1=\lambda_x\left(g\eta\left(\alpha-1\right)\beta+g-1\right)+\kappa^2\left(-1+\left(1+\alpha\beta\eta\right)g\right)$, $a_2=\lambda_x\left(1+\beta\left(g\eta\left(\alpha-1\right)\beta+\left(g-1\right)\right)^2\right)$ $+\kappa^2\left(1+\beta\left(g\alpha\eta\beta+g-1\right)^2\right)$, or

$$
\begin{aligned}
a_1 &= -\left(1-\frac{1-\beta}{1-\alpha\beta}g\right)\lambda_x-\kappa^2\left(1-g\frac{1}{1-\alpha\beta}\right) \\
a_2 &= \left(1+\beta\left(1-g\frac{1-\beta}{1-\alpha\beta}\right)^2\right)\lambda_x+\kappa^2\left(1+\beta\left(1-g\frac{1}{1-\alpha\beta}\right)^2\right)>0.
\end{aligned}
$$

It is useful to rewrite the polynomial p as $p=-\beta a_1\rho^2q\left(\frac{\phi}{\rho}\right)$, where the polynomial

$$
q\left(\mu\right)\equiv\beta\mu^2+\frac{a_2}{a_1}\mu+1
$$

has the following properties:

$$
q\left(0\right)>0
$$

and

$$
\begin{aligned}
q\left(1\right) &= \beta+\frac{a_2}{a_1}+1 \\
&= \frac{\frac{g}{\left(1-\alpha\beta\right)^2}\left(\left(1-\beta\right)^2\left(g\beta-\alpha\beta+1\right)\lambda_x+\kappa^2\left(g\beta-\beta-\alpha\beta+\alpha\beta^2+1\right)\right)}{\left(\left(1-\beta\frac{1-\alpha}{1-\alpha\beta}\right)g-1\right)\lambda_x+\kappa^2\left(g\frac{1}{1-\alpha\beta}-1\right)} \\
&= \frac{\frac{g}{\left(1-\alpha\beta\right)}\left(\left(\lambda_x\left(1-\beta\right)^2+\kappa^2\right)\beta g+\left(1-\beta\right)\left(1-\alpha\beta\right)\left(\lambda_x\left(1-\beta\right)+\kappa^2\right)\right)}{\left(\lambda_x\left(1-\beta\right)+\kappa^2\right)g-\left(1-\alpha\beta\right)\left(\lambda_x+\kappa^2\right)}
\end{aligned}
$$

is negative if and only if $\left(\lambda_x\left(1-\beta\right)+\kappa^2\right)g<\left(1-\alpha\beta\right)\left(\lambda_x+\kappa^2\right)$. So we have $q\left(1\right)<0$ if and only if

$$
g<\frac{\left(1-\alpha\beta\right)\left(\lambda_x+\kappa^2\right)}{\lambda_x\left(1-\beta\right)+\kappa^2}.
$$

Moreover,

$$\begin{aligned} q(-1) &= \beta - \frac{a_2}{a_1} + 1 \\ &= 1 + \beta + \frac{\left(1 + \beta\left(1 - g\frac{1-\beta}{1-\alpha\beta}\right)^2\right)\lambda_x + \kappa^2\left(1 + \beta\left(1 - g\frac{1}{1-\alpha\beta}\right)^2\right)}{\left(1 - \frac{1-\beta}{1-\alpha\beta}g\right)\lambda_x + \kappa^2\left(1 - g\frac{1}{1-\alpha\beta}\right)} \\ &= \frac{num}{-\frac{1}{1-\alpha\beta}\left(\left(\lambda_x(1-\beta) + \kappa^2\right)g - (1-\alpha\beta)\left(\lambda_x + \kappa^2\right)\right)} \end{aligned}$$

where the numerator

$$\begin{aligned} num &= \left[\beta\left(1 - g\frac{1-\beta}{1-\alpha\beta}\right)^2 + (1+\beta)\left(1 - \frac{1-\beta}{1-\alpha\beta}g\right) + 1\right]\lambda_x \\ &\quad +\kappa^2\left[\beta\left(1 - g\frac{1}{1-\alpha\beta}\right)^2 + (1+\beta)\left(1 - g\frac{1}{1-\alpha\beta}\right) + 1\right]. \end{aligned}$$

The first term in square brackets is positive since $0 < \frac{1-\beta}{1-\alpha\beta} < 1$. The second term in square brackets equals $\frac{\beta}{(1-\alpha\beta)^2}g^2 - \frac{3\beta+1}{1-\alpha\beta}g + 2(\beta+1)$, a quadratic polynomial in g that admits two roots: $2(1-\alpha\beta)$ and $\left(\beta^{-1}+1\right)(1-\alpha\beta)$. It is positive if $g < 2(1-\alpha\beta)$ or if $g > \left(\beta^{-1}+1\right)(1-\alpha\beta)$. The denominator is negative if and only if $\left(\lambda_x(1-\beta)+\kappa^2\right)g > (1-\alpha\beta)\left(\lambda_x+\kappa^2\right)$. So we have $q(-1) < 0$ if

$$g > \frac{(1-\alpha\beta)\left(\lambda_x+\kappa^2\right)}{\lambda_x(1-\beta)+\kappa^2},$$

and g satisfies either $g < 2(1-\alpha\beta)$ or $g > \left(\beta^{-1}+1\right)(1-\alpha\beta)$. This is a sufficient but not necessary condition for $q(-1)$ to be negative.

A necessary condition for $q(-1)$ to be negative involves $g > \frac{(1-\alpha\beta)\left(\lambda_x+\kappa^2\right)}{\lambda_x(1-\beta)+\kappa^2}$, and the entire numerator to be positive. This numerator,

$$num = \frac{\beta\left(\lambda_x(1-\beta)^2+\kappa^2\right)}{(1-\alpha\beta)^2}g^2 - \left(\lambda_x(1-\beta)+\kappa^2\right)\frac{1+3\beta}{1-\alpha\beta}g + (2\beta+2)\left(\lambda_x+\kappa^2\right)$$

is a quadratic polynomial in g that admits the two roots:

$$\begin{aligned} g_1 &= \frac{(1-\alpha\beta)\left((1+3\beta)\left(\lambda_x(1-\beta)+\kappa^2\right) - \sqrt{\left(\lambda_x(1-\beta)+\kappa^2\right)^2(1-\beta)^2 - 8\beta^3(1+\beta)\lambda_x\kappa^2}\right)}{2\beta\left(\lambda_x(1-\beta)^2+\kappa^2\right)}, \\ g_2 &= \frac{(1-\alpha\beta)\left((1+3\beta)\left(\lambda_x(1-\beta)+\kappa^2\right) + \sqrt{\left(\lambda_x(1-\beta)+\kappa^2\right)^2(1-\beta)^2 - 8\beta^3(1+\beta)\lambda_x\kappa^2}\right)}{2\beta\left(\lambda_x(1-\beta)^2+\kappa^2\right)}. \end{aligned}$$

The numerator is thus positive provided that $g < g_1$ or $g > g_2$. In the special case that $\lambda_x = 0$, this reduces to $q = \frac{1-\alpha\beta}{2\beta}(1+3\beta \pm (1-\beta))$, so that $q_1 = 2(1-\alpha\beta)$, $q_2 = (1-\alpha\beta)\left(\beta^{-1}+1\right)$; the numerator is again positive provided that $g < g_1$ or $g > g_2$.