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Keywords
creative destruction, menu cost, new Keynesian, monetary policy

JEL Classification
E31, E58, O33, O41

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The Optimal Inflation Rate under Schumpeterian Growth*

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April 7, 2015

Abstract

In this study, we analyze the relationship between inflation and economic growth. To this end, we construct a model of endogenous growth with creative destruction, incorporating sticky prices due to menu costs. Inflation and deflation reduce the reward for innovation via menu cost payments and, thus, lower the frequency of creative destruction. Central banks can maximize the rate of economic growth by setting their target inflation rate at the negative of a fundamental growth rate that would be realized without price stickiness. The optimal inflation rate, however, may differ from the growth-maximizing inflation rate because of overinvestment in R&D and indeterminacy. Both mechanisms indicate a higher optimal inflation rate than the negative of a fundamental growth rate. Our calibrated model shows that the optimal inflation rate is close to the growth-maximizing inflation rate and that a deviation from the optimal level has sizable impacts on economic growth.

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1 Introduction

Sustained economic growth is one of the goals of central banks.\footnote{The Bank of England Act sets “its objectives for growth and employment.” The Bank of Japan Act and Treaty on European Union aim for “sound development of the national economy” and “sustainable development of Europe based on balanced economic growth and price stability,” respectively. In addition, the Federal Reserve Board considers sustained economic growth as one of its main purposes, although this is not stated in its founding legislation (Federal Reserve System (2005)).} However, there are only a handful theoretical works on the impact of inflation on economic growth. In this study, we fill in this gap by constructing a model of innovation-driven endogenous growth with the feature of sticky prices. More specifically, we combine the standard menu cost model formulated by Sheshinski and Weiss (1977) and the standard quality ladder model formulated by Grossman and Helpman (1991), which involves intentional R\&D investment by firms. We then derive the long-run optimal rate of inflation that should be set by central banks.

Our model demonstrates that the rate of economic growth is maximized when central banks set their target inflation rate at the negative of a fundamental growth rate that would be realized without price stickiness. In the model economy, firms’ output prices are diverted from their optimal level owing to menu costs. Firms do not fine-tune their nominal prices in response to developments in nominal wages. Thus, both inflation and deflation suppress the real profit flows that potential entrants would obtain after succeeding in innovation, which decreases innovation incentives and lowers the real economic growth rate. When the inflation rate equals the negative of a fundamental growth rate, the nominal growth rate is zero and nominal rigidity is virtually eliminated. The rates of both creative destruction and economic growth are maximized.

The optimal rate of inflation may deviate from this negative of the fundamental real growth rate for the following two reasons. One is the possibility of overinvestment due to negative externalities of innovation via the business-stealing effect in Aghion and Howitt (1992). In this case, central banks should discourage firms’ R\&D investment. The other is indeterminacy (multiple equilibria) that occurs around the growth-maximizing rate of inflation. When central banks set the inflation rate at the negative of a fundamental growth rate, a worse equilibrium may be realized such that the nominal growth rate is negative and the real growth rate is lower than the desired level. If one or both of these mechanisms are significant, a higher rate of inflation than the growth-maximizing rate becomes optimal.

The model calibrated to the US economy shows that the cost of suboptimal inflation is sizable. On the balanced growth path, the growth rate is reduced by half at about 10% inflation or deflation. This quantitative impact is strikingly large relative to the existing studies, for example, Jones and Manuelli (1995) and Lucas (2000). The problems of overin-
vestment and indeterminacy do not seem to have significant impacts on the optimal rate of inflation. This inflation rate is very close to the growth-maximizing rate of inflation, which is around \( \approx 2\% \).

In the model, the optimal inflation rate is zero when the price index is measured without adjusting quality improvement. However, in the final part of this paper, we report on an extension of our model to incorporate imitations by rival firms and show that the optimal inflation rate increases. The zero inflation rate is no longer optimal, even when the price index is measured without adjusting quality improvement.

While the focus of this study is on the optimal long-run inflation rate under endogenous growth, in our opinion, another contribution of this study is its presentation of a new basic and tractable model to link monetary policy to economic growth. Our model is based on a creative destruction model, which is appropriate for analyzing reallocation through entry and exit. The model of sticky prices is a device to generate effects of nominal variables on the real economy. Such a model is distinct from a model with money, such as a cash-in-advance (CIA) constraint or money-in-utility, and now serves as a workhorse model for monetary policy analysis (Woodford (2003)). The synthesis of these two main strands of macroeconomics has the potential to widen our research scope in the directions of, for example, firm reallocation and pricing in the presence of product substitution.

The following two studies are most closely related to ours. Chu and Cozzi (2014) construct a model with a CIA constraint and creative destruction to analyze cases in which the Friedman rule does not hold, that is, the optimal nominal interest rate is strictly positive. In their model, the CIA constraint is imposed not only on consumption but also on R&D and the positive nominal interest rate works as a tax on R&D. Because overinvestment of R&D could exist, such a tax policy could improve welfare, depending on parameter values. On the other hand, the indeterminacy problem does not arise in their model.\(^2\)

Bilbiie et al. (2014) develop a model with sticky prices and endogenous product variety to examine the optimal rate of inflation under love for variety. They derive cases in which the optimal inflation rate diverts from zero (at which nominal rigidity vanishes in their model) under some utility functions. However, in their model, monetary policy does not influence economic growth, but only the level of the real economy.\(^3\)

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\(^2\)Studies on the neutrality of money date back to Tobin (1965). He argues that an increase in money supply has a positive impact on the level of economic activity. See also Sidrauski (1967) and Brock (1974). Jones and Manuelli (1995) show that money is not neutral in a CIA model with the AK production if there is nominal rigidity due to tax codes.

\(^3\)As argued by Schmitt-Grohé and Uribe (2010), the optimal inflation rate is approximately zero in standard new Keynesian models. See also Goodfriend and King (1997), Khan et al. (2003), Benigno and Woodford (2005), and Coibion et al. (2012), among others. Our model shows that the endogenous determination of the growth rate modifies the optimal rate of inflation by the fundamental growth rate. Moreover,
Other than these two papers, empirical studies exist on inflation and economic growth. There, the correlation is mostly negative, implying that inflation is harmful for economic growth. However, such a correlation is significant only during high inflation crises and becomes insignificant under low inflation Bullard and Keating (1995), (Barro (1996), Bruno and Easterly (1998), and Ahmed and Rogers (2000)).

The remaining structure of this paper is as follows. Section 2 develops the basic model. Section 3 discusses the optimal rate of inflation. Section 4 conducts numerical simulation and Section 5 examines the robustness of our results. Section 6 concludes.

2 Model

Time is continuous. A representative household consumes and supplies a fixed amount of labor. Firms develop a new product by R&D investment and enter a market. At the same time, firms with an old product exit. A central bank conducts a strict inflation-targeting policy by anchoring an inflation rate at a certain level.

No aggregate uncertainty is present and we focus on the balanced growth path. The growth rate of aggregate consumption is denoted by \( a \), which equals the growth rate of the real wage and real household income. The growth rate of an aggregate price index is denoted by \( \pi \), which takes account of quality improvement. It differs from the growth rate of an aggregate price index that is based on posted observable prices. The growth rate of nominal variables, such as nominal demand, nominal consumption, and nominal wages, is given by

\[
g \equiv a + \pi.
\]

For convenience, we denote initial values in period \( t = 0 \) without time subscripts, for example, \( X_0 \equiv X \).

2.1 Firm Entry

A team of \( h \) researchers is required to obtain an idea for a new product at a Poisson rate of 1. Firms can enter a product market freely, which leads to \( W_t h/P_t \geq V_t \), where \( W_t \), \( P_t \), and \( V_t \) represent nominal wages, an aggregate price index embedding quality improvement, and the expected present value of industry-leading firms, respectively. Equality holds when the entry rate of firms, \( \mu \), is positive. Otherwise, \( \mu \) is zero. We define \( v_t \) by \( v_t \equiv P_t V_t/E_t \), where \( E_t \) represents aggregate nominal demand. Then, the abovementioned free-entry condition

because product quality is improved endogenously, our study is related to quality bias in the measurement of inflation rates.
is rewritten as
\[
\frac{W_t}{E_t} h = \frac{W}{E} h \geq v, \tag{2}
\]
where \( W/E \) is interpreted as labor share because total labor supply is fixed.

The entry rate is given by
\[
\mu = \frac{L_s}{h}, \tag{3}
\]
where \( L_s \) represents the measure of labor employed to generate R&D investment and launch new firms.

### 2.2 Household

A representative household has the following preferences over all versions of \( k \in \{0, 1, \cdots , K_t(j)\} \) of each product line \( j \in [0, 1] \):

\[
U_t = \int_0^\infty e^{-\rho t} \log C_t dt', \tag{4}
\]

\[
\log C_t = \int_0^1 \log \left[ \sum_{k = 0}^{K_t(j)} q(j, k) x_t(j, k) \right] dj, \tag{5}
\]

where \( \rho \) represents the subjective discount rate, \( C_t \) is aggregate consumption, and \( x_t(j, k) \) and \( q(j, k) \) denote the consumption and quality of version \( k \) in product line \( j \), respectively. Quality evolves as
\[
q(j, k) = q^k. \tag{6}
\]

Throughout the study, the following assumption holds in order to guarantee that the market equilibrium has a strictly positive growth rate, where \( L \) is the total labor.\(^4\)

**Assumption 1**

\[
q > 1 + \frac{h \rho}{L}. \tag{7}
\]

The intertemporal optimization of consumption yields \( \Lambda_t = e^{-\rho t}/C_t \), where \( \Lambda_t \) represents the stochastic discount factor (for real income). Thus, we have the discount rate \( r \) of real income as
\[
r = \rho + a. \tag{8}
\]

\(^4q > 1 \) is sufficient for the model to function fully. The equilibrium growth rate is zero if \( q \) is greater than one but violates (7).
2.3 Industry-Leading Firms

2.3.1 Firm Profits

Equation (5) suggests that goods demand is given by

$$x_t(j,k) = \frac{E_t}{p_t(j,k)}$$ (9)

for the highest version $k = K_t(j)$, and zero otherwise, unless the posted price of version $k = K_t(j)$ in product line $j$, $p_t(j,K_t(j))$, is too high. When only the highest versions of products exist in markets, the aggregate price index $P_t$ can be written as

$$\log P_t = \int_0^1 \log \left[ \frac{p_t(j,K_t(j))}{q(j,K_t(j))} \right] dj,$$ (10)

with quality improvement being embedded.

Firms produce one unit of goods using one unit of labor. From equation (9), we can write the real profits of industry-leading firms with $k = K_t(j)$ as

$$\Pi_t(p_t(j,k)) = \frac{p_t(j,k) - W_t E_t}{p_t(j,k)} \frac{E_t}{P_t}.$$ (11)

Profits are zero for other firms with $k < K_t(j)$. The abovementioned equation can be rewritten as

$$\Pi_t(p_t(j,k)) = \frac{p_t(j,k) - W e^{qt} E}{p_t(j,k)} \frac{E}{P e^{qt}},$$ (12)

where the relative price $\xi_t(j,k)$ and the real profit function $\Pi^0(\xi)$ are defined by

$$\xi_t(j,k) \equiv p_t(j,k) e^{-qt}.$$ (13)

$$\Pi^0(\xi) \equiv \frac{\xi - W}{\xi}.$$ (14)

Below, we set expenditure in the initial period as the numeraire, that is, $E = 1$ and $E_t = e^{at}$.
### 2.3.2 Pricing under Menu Costs

Without nominal rigidity, equations (6) and (12) suggest that the optimal relative price of the industry-leading firm in \( j \) equals

\[
\xi^* = qW. \tag{15}
\]

In this case, the industry-leading firm can monopolize the market for its product line, even if the price of the second highest version of the product is set at the lowest level that equals the production cost, that is, \( W \).

In the presence of nominal rigidity, relative prices can deviate from \( \xi^* \). We assume that firms pay menu costs when they change their prices as much as \( \kappa \) times \( E_t/P_t \) in period \( t \). Menu costs are paid in units of aggregated consumption goods. When firms enter the market in period \( t \), they set their price at \( p_0 \) (relative price \( \xi_0 \)). They do not pay menu costs at entry. The price is left unchanged, unless they pay menu costs. As shown in equation (13), during that period, relative price \( \xi_t \) changes with the rate of \( -g = -a - \pi \) because the real wage increases with the rate of \( a \) and the aggregate price rises with the rate of \( \pi \). In other words, when \( g \) is positive (negative), these goods become cheaper (more expensive) as time passes. Suppose that in period \( t_{i+1} \) for \( i = 0, 1, 2, \ldots \), firms pay menu costs and reset their price at \( p_{i+1} \) (relative price \( \xi_{i+1} \)). The evolution of relative price \( \xi_t \) is described as

\[
\xi_t = \xi_t e^{-g(t-t_i)} \tag{16}
\]

and the expected present value of industry-leading firms at entry, \( V_t \), as

\[
V_t = \sum_{i=0}^{\infty} \left( \int_{t_i}^{t_{i+1}} \Pi_t (\xi_t e^{-g(t'-t_i)}) e^{-(r+\mu)(t'-t)} dt' - \frac{\kappa E_{t_{i+1}}}{P_{t_{i+1}}} e^{-(r+\mu)(t_{i+1}-t)} \right) \tag{17}
\]

using equation (13).

It is optimal for firms to obey the following \( S \)-s rule. First, the maximum price \( S \) satisfies

\[
S = \xi^* = qW. \tag{18}
\]

If firms set a price above \( S \), firms with lower quality \( k < K_t(j) \) can attract consumers by setting the relative price equal to \( W \). Thus, firms lower their prices by paying menu costs if their relative prices exceed \( S \). Second, a minimum relative price \( s \) exists. When \( g > 0 \), relative prices fall as time goes by. Firms start by setting the relative price \( \xi \) at \( S \), and when \( \xi \) reaches \( s \), they pay menu costs and raise their prices at \( \xi = S \). When \( g < 0 \), relative
prices rise as time goes by. Firms set $\xi = s$ initially, and when $\xi$ reaches $S$, they pay menu costs and lower their prices to $\xi = s$. We denote the time interval between $s$ and $S$ by $\Delta$.

Suppose $g > 0$. Then, the minimum relative price $s$ is given by

$$s = Se^{-g\Delta}.$$  \hfill (19)

Industry-leading firms that enter the market in period $t$ set the relative price at $S$. Their value, (17), is described as

$$V_t = \frac{1}{P} \sum_{i=0}^{\infty} \left( \int_{t+i\Delta}^{t+(i+1)\Delta} \Pi_t'(S e^{-g(t'-t)})e^{-(r+\mu)(t'-t)} dt' - \frac{\kappa}{P} e^{a(t+(i+1)\Delta)} e^{-(r+\mu)(t+(i+1)\Delta-t)} \right).$$

Using equation (12) as well as (8), we obtain

$$V_t = \frac{e^{at}}{P} \sum_{i=0}^{\infty} e^{-(r+\mu)i\Delta} \left( \int_0^\Delta \Pi^t_0(S e^{-gt''})e^{-(r+\mu)t''} dt'' - \kappa e^{-(r+\mu)\Delta} \right).$$

We normalize $V_t$ by the real expenditure in $t$ and denote the normalized value of entrants $v_t$ as

$$v = v_t = \frac{V_t}{E_t/P_t} = \frac{1}{1 - e^{-(\rho+\mu)\Delta}} \left( \int_0^\Delta \Pi^t_0(S e^{-gt''})e^{-(\rho+\mu)t''} dt'' - \kappa e^{-(\rho+\mu)\Delta} \right).$$  \hfill (20)

The optimal time interval, $\Delta$, makes its derivative with respect to $\Delta$ zero:

$$\frac{dv}{d\Delta} = 0 = -(\rho + \mu)v + \Pi^0(S e^{-g\Delta}) + \kappa(\rho + \mu).$$  \hfill (21)

Substituting equation (14) into equation (20), we obtain

$$v = \frac{1}{1 - e^{-(\rho+\mu)\Delta}} \left( \int_0^\Delta \frac{S e^{-gt''} - W}{S e^{-gt''}} e^{-(\rho+\mu)t''} dt'' - \kappa e^{-(\rho+\mu)\Delta} \right)$$

$$= \frac{1}{\rho + \mu} - \frac{1}{1 - e^{-(\rho+\mu)\Delta}} \frac{1 - e^{-(\rho+\mu)\Delta}}{\rho + \mu - g} - \frac{\kappa e^{-(\rho+\mu)\Delta}}{1 - e^{-(\rho+\mu)\Delta}}.$$  \hfill (22)
Substituting equation (22) into equation (21), we have

\[ q \kappa = \frac{e^{g\Delta} - e^{-(\rho+\mu-g)\Delta}}{\rho + \mu} - \frac{1 - e^{-(\rho+\mu-g)\Delta}}{\rho + \mu - g}. \]  

Let \( \Delta(g, \mu) \) be the solution to equation (23). The maximized value of an entrant is written as

\[ v = v(g, \mu, \Delta) = \frac{1}{\rho + \mu} \left( 1 - \frac{e^{g\Delta(g, \mu)\Delta}}{q} \right) + \kappa. \]  

Similarly, when \( g < 0 \), we have

\[ s = Se^{g\Delta}. \]  

The firm value is given by

\[ v = \frac{1}{1 - e^{-(\rho+\mu)\Delta}} \left( \int_0^\Delta \Pi^0(Se^{-gt''})e^{-(\rho+\mu)t''} dt'' - \kappa e^{-(\rho+\mu)\Delta} \right) 
   = \frac{e^{-(\rho+\mu)\Delta}}{1 - e^{-(\rho+\mu)\Delta}} \left( \int_{-\Delta}^0 \Pi^0(Se^{-gt''})e^{-(\rho+\mu)t''} dt'' - \kappa \right). \]  

Its derivative with respect to \( \Delta \) yields

\[ \frac{d}{d\Delta} v = 0 = - (\rho + \mu) v + \Pi^0(S e^{g\Delta}), \]  

and we have

\[ v = \frac{1}{\rho + \mu} - \frac{e^{-g\Delta}/q}{1 - e^{-(\rho+\mu)\Delta}} \left( 1 - \frac{e^{-(\rho+\mu-g)\Delta}}{\rho + \mu - g} \right) - \frac{\kappa e^{-(\rho+\mu)\Delta}}{1 - e^{-(\rho+\mu)\Delta}}. \]

From the last two equations, we obtain

\[ q \kappa = \frac{e^{(\rho+\mu-g)\Delta} - e^{-g\Delta}}{\rho + \mu} - \frac{e^{(\rho+\mu-g)\Delta} - 1}{\rho + \mu - g}. \]  

The maximized value of an entrant is written as

\[ v = v(g, \mu, \Delta) = \frac{1}{\rho + \mu} \left( 1 - \frac{e^{-g\Delta(g, \mu)}}{q} \right). \]  

If \( g = 0 \), firms set the relative price at \( \xi^* \), which stays constant over time. They do not
need to reset their prices and never pay menu costs. The firm value \( v_t \) equals

\[
v = \frac{1}{p + \mu} \frac{q - 1}{q}.
\]

The following lemmas present how the optimal choice of an individual firm responds to changes in \( g \) and \( \mu \).

**Lemma 1** For a given \( g \neq 0 \) and \( \mu \geq 0 \),

\[
\frac{d\Delta}{dg} \bigg|_{d\mu=0} < 0 \quad \text{if and only if} \quad g > 0,
\]

\[
\frac{d\Delta}{d\mu} \bigg|_{dg=0} > 0 \quad \text{if and only if} \quad g > 0.
\]

**Proof.** Appendix A.1. ■

If there is no nominal growth \( (g = 0) \), firms can keep the optimal relative price level without any price update. Thus, \( \Delta(0, \mu) = \infty \) for any \( \mu \). The duration of fixed nominal price, \( \Delta \), decreases as \( g \) moves apart from 0. A response of \( \Delta \) to an increase in \( \mu \) is asymmetric between the cases of \( g > 0 \) and \( g < 0 \). More frequent innovations imply that incumbents discount their future profits more. When \( g > 0 \), the relative price of a good moves away from the optimal price under flexible prices so that the cost of waiting to update the price is discounted more under higher \( \mu \). Hence, \( \Delta \) is increasing in \( \mu \). On the other hand, when \( g < 0 \), the relative price approaches the optimal price under flexible prices, so that postponing the price update is beneficial for incumbents. Hence, high probability of creative destruction leads to a decline of the incentive to wait, and accordingly, \( \Delta \) is decreasing in \( \mu \). Note that \( \mu \Delta \) is increasing in \( \mu \) if \( g > 0 \), while it is ambiguous when \( g < 0 \).

**Lemma 2** For any \( g \neq 0 \) and \( \mu > 0 \),

\[
\frac{dv}{d|g|} \bigg|_{d\mu=0} < 0 \quad \text{and} \quad \frac{d(|g|\Delta)}{d|g|} \bigg|_{d\mu=0} > 0.
\]

In addition, for any \( g \) and \( \mu \geq 0 \),

\[
\frac{dv}{d\mu} \bigg|_{dg=0} < 0.
\]

The value of entrants is decreasing in \(|g|\) and \(\mu\). The former effect stems from menu cost payments and the latter stems from the business stealing effect for incumbents.

2.4 Firm Distribution

Because of menu costs, industry-leading firms are heterogeneous with respect to their prices.\(^5\) Relative prices \(\xi\) are distributed in the range between \(s\) and \(S\), or time after the last price change is distributed in the range between 0 and \(\Delta\). We denote the density function of \(\xi(t')\) by \(f(\xi(t'))\), where \(t' \in [0, \Delta]\) and \(\xi \in [s, S]\). Since \(\xi(t')\) changes at the growth rate of \(-g\) for \(\xi \in (s, S)\), the density function should satisfy

\[
f(\xi(t')) = f(\xi(t' - dt'))(1 - \mu dt') = f(\xi(t')(1 + g dt'))(1 - \mu dt')
\]

for small \(dt'\), if firm distribution is stationary. This equation implies that the density at \(t'\) should equal that of \(t' - dt'\) multiplied by the survival probability of firms during interval \(dt'\), that is, \(1 - \mu dt'\). Unless \(g\) is zero, this equation is transformed into

\[
d \log f(\xi(t')) = \frac{\mu}{g} d \log \xi(t'),
\]

where

\[
\xi(t') = \xi(0)e^{-gt'}
\]

and \(1 = \int_0^\Delta f(\xi(t'))dt'\). Therefore, we obtain

\[
f(\xi(t')) = \begin{cases} 
\frac{\mu}{Se^{gt'} - S/e^{gt'}} \left(S e^{-gt'}\right)^{\mu/g} & \text{for } g > 0 \\
\frac{\mu}{Se^{gt'/g} - S} \left(Se^{gt'/g}\right)^{\mu/g} & \text{for } g < 0
\end{cases}
\]

\[
= \frac{\mu}{1 - e^{-\mu \Delta}} e^{-\mu t'} \quad \text{for } g \neq 0.
\]

The distribution is non-uniform when \(\mu > 0\).\(^6\)

When \(g = 0\), there is no price heterogeneity. The relative price of all industry-leading firms remains at \(\xi^*\).

---

\(^5\)Transitional dynamics exist between balanced growth paths because of this heterogeneity of firms, unlike standard creative destruction models, in which the economy jumps onto a balanced growth path.

\(^6\)Non-uniformity of price distribution when \(\mu > 0\) is essential for the non-neutrality of money, as analyzed in Oikawa and Ueda (2015).
2.5 Aggregate Variables

2.5.1 Labor Market

Equation (9) yields aggregate labor demand for production, $L_x$:

$$L_x = \int_0^1 x_t(j, K_t(j))dj = \int_0^1 \frac{E_t}{p_t(j, K_t(j))}dj = \int_0^1 \frac{1}{\xi_t(j, K_t(j))}dj = \int_0^\Delta f(\xi(t')) \frac{1}{\xi(t')} dt'$$

From equations (29) and (30), we have

$$L_x(g, \mu) = \begin{cases} \frac{\mu - g}{S} \frac{1 - e^{-(\mu - g)\Delta}}{1 - e^{-\mu \Delta}} & \text{for } g > 0, \\ \frac{\mu - g}{S} \frac{e^{-\Delta} - e^{-(\mu - g)\Delta}}{1 - e^{-\mu \Delta}} & \text{for } g < 0, \end{cases}$$

and $L_x = 1/\xi^* \text{ if } g = 0$.

The next lemma describes the behavior of labor demand for production for given $g$ and $\mu$.

**Lemma 3** $L_x(g, \mu)$ is increasing in $|g|$. It is increasing in $\mu$ if $g \geq 0$. In addition, it is increasing if $g < 0$ but sufficiently close to zero.

**Proof.** Appendix A.3.4. 

A higher frequency of creative destruction or a faster nominal change leads to a smaller firm value. A decline in a firm value decreases the real wage through the free-entry condition, which, in turn, decreases the upper bound of relative price $S$ and increases $L_x$. When $g \geq 0$, this impact always dominates other impacts stemming from shifts in the distribution of prices and changes in $\Delta$ described in Lemma 1. When $g < 0$, it is not clear which impact dominates. However, if $g$ is sufficiently close to zero, the same argument holds.

Below, we assume that the monotonicity in $L_x$ stated in Lemma 3 holds in the region of $g$ examined. This is not a strict condition in that $g < -\rho$ implies a negative nominal interest rate ($i = \rho + g$ from equations (1) and (8)). In the numerical analysis in Section 4, we set $\rho = 0.05$ and the monotonic relationship occurs naturally.

2.5.2 Aggregate Price Level and Aggregate Inflation Rate

Here, we derive the relationship between the rate of inflation $\pi$ and the growth rate of nominal variables $g$ for a given creative destruction rate $\mu$.
Equation (10) together with equations (13) and (30) yield the aggregate price level at
\( t = 0 \), \( P \), as
\[
\log P = \int_0^1 \log [p_0(j, k = 0)] \, dj = \int_0^1 \log [\xi_0(j, k = 0)] \, dj = \int_0^\Delta f(\xi(t')) \log [\xi(t')] \, dt'
\]
\[
= \int_0^\Delta \frac{\mu}{1 - e^{-\mu \Delta}} e^{-\mu' t} \log \left[ \xi(0) e^{-\mu t'} \right] \, dt'
\]
\[
= \log \xi(0) + g\Delta \frac{e^{-\mu \Delta}}{1 - e^{-\mu \Delta}} - \frac{g}{\mu}, \tag{33}
\]
as long as \( \mu > 0 \). When \( \mu = 0 \), no new good is invented and \( P = W \).

The inflation rate is calculated using (10) as
\[
\pi dt = \log P_{t+dt} - \log P_t
\]
\[
= \int_0^1 \log \left[ \frac{p_{t+dt}(j, K_{t+dt}(j))}{q(j, K_{t+dt}(j))} \right] \, dj - \int_0^1 \log \left[ \frac{p_t(j, K_t(j))}{q(j, K_t(j))} \right] \, dj.
\]
Changes between \( t \) and \( t + dt \) can be divided into two cases. First, firms change their
prices by paying menu costs with probability \( f(\xi(\Delta)) dt \). Relative prices are given by
\( \xi_t(j, K_{t+dt}(j)) = \xi(0) \) and \( \xi_t(j, K_t(j)) = \xi(\Delta) \). That is, \( p_{t+dt}(j, K_{t+dt}(j)) = \xi(0) e^{g(t+dt)} \)
and \( p_t(j, K_t(j)) = \xi(\Delta) e^{gt} \) from equation (13). Second, new firms enter the market with
the probability of \( \mu dt \). They replace the relative prices of existing goods, \( P \) on aver-
age, with the newly set price, \( p_{t+dt}(j, K_{t+dt}(j)) = \xi(0) e^{g(t+dt)} \). The quality increases as
\( q(j, K_{t+dt}(j)) = q(j, K_t(j)) \cdot q \). Therefore, we have
\[
\pi dt = f(\xi(\Delta)) dt \cdot \log \frac{\xi(0)}{\xi(\Delta)} + \mu dt \cdot \left( \log \frac{\xi(0)}{q} - \log P \right). \tag{34}
\]
Using equations (30) and (33), we transform equation (34) to
\[
\pi = f(\xi(\Delta)) \log \frac{\xi(0)}{\xi(\Delta)} + \mu \log \frac{\xi(0)}{qP}
\]
\[
= \frac{\mu}{1 - e^{-\mu \Delta}} g\Delta - \mu \log q + \mu \left( \frac{g}{\mu} - g\Delta \frac{e^{-\mu \Delta}}{1 - e^{-\mu \Delta}} \right)
\]
\[
= g - \mu \log q, \tag{35}
\]
for any \( g \). Thus, the real growth rate is
\[
a = \mu \log q. \tag{36}
\]
Two Alternative Measures of Inflation Rates The aggregate inflation rate \( \pi \) takes account of quality improvement, and thus, can be interpreted as an accurate cost of living index. The actual consumer price index is intended to capture product turnover and quality changes, but such an attempt is not perfect (see, e.g., Boskin et al. (1996) and Bils (2009)). Thus, we alternatively consider two other measures of inflation rates. The first measure is based on a matched sample: price changes are calculated only when products are unchanged. It is written as

\[
\pi_{\text{matched}} = \begin{cases} f(\xi(\Delta)) \cdot \log(\xi(0)/\xi(\Delta)) & \text{for } g \neq 0 \\ 0 & \text{for } g = 0. \end{cases}
\]

The second measure takes account of product turnover, but quality improvement is unadjusted. When new products replace old ones, price changes are calculated simply by comparing two prices. Then, such an inflation rate is given by

\[
\pi_{\text{quality unadjusted}} = \begin{cases} f(\xi(\Delta)) \cdot \log(\xi(0)/\xi(\Delta)) - \mu \cdot \log(P/\xi(0)) & \text{for } g \neq 0 \\ 0 & \text{for } g = 0. \end{cases}
\]

2.6 Stationary Equilibrium

The aggregate labor demand for R&D, \( L_s \), is simply \( L_s = h \mu \) from equation (3). The labor market-clearing condition is

\[
L = L_x + L_s.
\]

The next proposition presents a unique equilibrium relationship between \( \mu \) and \( g \) to clear the labor market.

**Proposition 1** If \( g \geq 0 \) or \( g < 0 \) but is sufficiently close to 0, then there exists a unique \( \hat{\mu}(g) \) that clears the labor market, and \( \hat{\mu}(g) \) decreases as \( g \) moves away from 0.

**Proof.** Lemma 3 implies that the curve of \( L - L_x(g, \mu) \) is downward sloping in Figure 1. Since \( h \mu \) is upward sloping, \( \hat{\mu}(g) \) that clears the labor market for a given \( g \) is unique unless \( L < L_x(0, g) \). When \( L < L_x(0, g) \), \( \hat{\mu}(g) = 0 \).

In addition, Lemma 3 implies that the curve of \( L - L_x(g, \mu) \) shifts downward in response to an increase in \( |g| \). Hence, \( \hat{\mu}(g) \) is decreasing in \( g \) for \( g > 0 \) and increasing in \( g \) if \( g < 0 \) but is sufficiently close to 0. \( \blacksquare \)

The essence of this proposition is illustrated in Figure 1. From Lemma 3, labor supply for R&D (total labor minus production labor) is downward sloping and shifts downward as \( |g| \) increases. Therefore, the equilibrium \( \hat{\mu}(g) \) is a decreasing function in \( |g| \). Intuitively,
Figures 1: Equilibrium for given $g$.

higher $|g|$ implies more rapid change in relative prices for firms so that they have to revise prices more frequently, causing more menu cost payments. Thus, the faster nominal growth is, the smaller firm values are. Smaller firm values reduce incentives to innovate, which causes slower real growth.

$\dot{\mu}(g)$ is the equilibrium rate of creative destruction for a given $g$, and thus, leads to a real growth rate of $\dot{\mu}(g) \log q$ from equation (36). The next proposition shows the maximum growth rate, which is attained at $g = 0$. In other words, the environment without nominal rigidity leads to the highest rate of creative destruction, and thus, the maximum real growth rate. Although the uniqueness of $\dot{\mu}$ can be shown only for sufficiently large $g$, the maximum growth rate in Proposition 2 is a global maximum.

**Proposition 2** $g = 0$ gives rise to the maximum real growth rate, $\bar{a}$, where

$$\bar{a} \equiv \left(1 - \frac{1}{q}\right)\frac{L}{h} - \frac{\rho}{q} \log q.$$  

**Proof.** From Lemma 3, $L_x$ increases as $g$ moves away from zero, regardless of its sign.
Thus, the curve of $L - L_x$ shifts downward as $|g|$ increases. Hence, the curve $L - L_x(0, \mu) > L - L_x(g, \mu)$ for $g \neq 0$, and thus, $\hat{\mu}(0) > \hat{\mu}(g)$ for $g \neq 0$.

Now, we derive $\bar{a}$. When $g = 0$, the amount of labor not employed by the production sector is

$$L - \frac{1}{qW} = L - \frac{h}{qv} = L - \frac{h(\rho + \mu)}{q - 1}.$$ 

In labor market equilibrium,

$$h\mu = L - \frac{h(\rho + \mu)}{q - 1} \iff \mu = \left(1 - \frac{1}{q}\right) \frac{L - \rho}{h - \frac{\rho}{q}},$$

which leads to equation (38).

This model is closed by a target inflation rate set by the central bank. We assume that the central bank strictly stabilizes the inflation rate at its target $\pi^T$, at least, in the long run. Such strict inflation-targeting policy determines $g$ as equilibrium according to equation (35) or

$$\pi^T = -\hat{\mu}(g) \log q + g \equiv \psi(g).$$

(39)

Clarifying the equilibrium value of $g$ for a given $\pi^T$ is somewhat complicated because of non-monotonicity of $\psi$. Figure 2 illustrates this. For sufficiently large $|g|$, $\hat{\mu}(g) = 0$ and equation (39) is drawn on the 45 degree line. Thus, if the inflation rate is sufficiently far from $-\bar{a}$, the central bank can pin down a specific nominal growth rate as the unique equilibrium. When $|g|$ is sufficiently small, the curve of $\psi$ moves away from the 45 degree line. Since $g = 0$ leads to the maximum real growth rate, a diversion from the 45 degree line is largest at $g = 0$ or $\pi = -\bar{a}$. Thus, $g$ has multiple equilibrium values if $\pi^T$ is around $-\bar{a}$. Specifically, when $\pi^T$ equals $-\bar{a}$, one form of equilibrium is characterized by $g = 0$ and the real growth rate $\bar{a}$ is the highest. However, another form of equilibrium also appears, where $g$ is negative and the real growth rate $a$ is lower than $\bar{a}$. The following proposition summarizes this property.

**Proposition 3** Let $\pi_T \equiv \max_{g<0} \psi(g)$. $\pi_1 \leq -\bar{a}$ exists such that the equilibrium nominal growth, $g$, is uniquely determined if $\pi^T \not\in [\pi_1, \pi_T]$. Multiple equilibria occur if $\pi^T \in [-\bar{a}, \pi_T]$. As long as the equilibrium is unique, the equilibrium value of $g$ is increasing in $\pi^T$.

**Proof.** Appendix A.4
Figure 2: Equilibrium for given $\pi^T$. While $\pi_1^T$ leads to unique equilibrium nominal growth, $g(\pi_1^T)$, $\pi_2^T$ causes indeterminacy of nominal growth.

3 The Optimal Inflation Rate

In this section, we search for the optimal inflation rate. Creative destruction causes negative externalities through the business stealing effect (Aghion and Howitt (1992)). Overly rapid growth harms welfare when potential firms overinvest in R&D. In such a case, the maximized growth rate by setting $\pi^T = -\bar{a}$ (Proposition 2) is not necessarily the optimal rate of inflation. At the same time, a deviation from $\pi^T = -\bar{a}$ is costly because non-zero nominal growth generates both menu cost payments and different markups across firms, increasing inefficiency.\footnote{This type of inefficiency is absent in standard endogenous growth models in which markups are homogeneous across firms. By contrast, it is typical in the new Keynesian literature without real growth in which zero inflation, corresponding to the inflation rate generating $g = 0$ in our model, has been found optimal (Schmitt-Grohé and Uribe (2010)).} Hence, $\pi^T = -\bar{a}$, which is the growth-maximizing and no-rigidity rate of inflation, is not optimal if the benefit from reducing the abovementioned inefficiency overwhelms the burden from menu costs. This feature is discussed in Chu and Cozzi (2014),
who derive a case in which the Friedman rule is not optimal.

It should be noted that we do not consider transitional dynamics between balanced
growth paths when a target inflation rate is changed. Rather, in this study, we compare
distinct balanced growth paths to find the optimal rate of inflation in the long run.

### 3.1 Welfare in Competitive Equilibrium

On a given balanced growth path, welfare at $t = 0$ is obtained as

$$U = \int_{0}^{\infty} e^{-\rho t} \log C_t dt = \frac{\mu \log q}{\rho^2} + \frac{\log C}{\rho}. \tag{40}$$

Welfare increases because either the growth rate of consumption, $a$, or the consumption
level at $t = 0$, $C$, is higher. Since nominal demand consists of nominal consumption and
menu cost payments, we obtain the following resource constraint

$$P_t C_t + \kappa E_t f(\xi(\Delta)) = E_t. \tag{41}$$

Hence, $C$ on a given balanced growth path is solved as

$$C = \frac{1 - \kappa f(\xi(\Delta))}{P}.$$

### 3.2 Social Planner’s Problem

The social planner chooses $\mu$ to maximize welfare

$$\max \int_{0}^{\infty} e^{-\rho t} \log C_t dt$$

subject to

$$\log C_t = \int_{0}^{1} \log q K_t(j) x_t(j) dj,$$

$$L = h\mu + \int_{0}^{1} x_t(j) dj,$$

$$K_{t+\mu dt}(j) = K_t(j) + 1 \text{ with probability } \mu dt.$$

This maximization problem is equivalent to

$$\max \int_{0}^{\infty} e^{-\rho t} (K_t \log q + \log x_t) dt \quad \text{s.t. } \dot{K}_t = \frac{L - x_t}{h}$$

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where we define
\[ K_t = \int_0^1 K_t(j) dj, \]
and substitute \( x_t = x_t(j) \) for all \( j \) because of the concavity of the problem. This optimization leads to the following solution on a balanced growth path,
\[ \mu^* = \frac{L}{h} - \frac{\rho}{\log q}. \tag{42} \]

### 3.3 Underinvestment and Overinvestment in R&D

To clarify the point, we ignore the indeterminacy problem in this subsection and focus on \( g \geq 0 \). Thus, if the central bank sets \( \pi^T = -\bar{a} \), it achieves \( g = 0 \). Under \( g = 0 \), the competitive equilibrium \( \hat{\mu}(g) \) produces
\[ \hat{\mu}(0) = \frac{q - 1}{q} \frac{L}{h} - \frac{\rho}{q}. \tag{43} \]

Compared with the socially optimal growth rate in equation (42), we obtain the following proposition.

**Proposition 4** Suppose \( g \geq 0 \). If
\[ h \leq \frac{L}{\rho(q/\log q - 1)}, \tag{44} \]
then R&D is underinvested and \( \pi = -\bar{a} \) is the second best inflation rate. Otherwise, R&D is overinvested and the second best inflation rate exists in between \([-\bar{a}, \psi(\mu^{-1}(\mu^*))]\).

**Proof.** Provided equation (44), we have \( \hat{\mu}(0) \leq \mu^* \). Suppose that \( g > 0 \) exists, which improves welfare (40). To increase welfare with such \( g \), \( C \) must increase because \( \hat{\mu}(g) < \hat{\mu}(0) \). However, \( \hat{\mu}(0) \leq \mu^* \) indicates that in competitive equilibrium with \( g = 0 \), \( C \) is overly large and \( \mu \) is too small relative to the socially optimal levels. Therefore, \( \hat{\mu}(0) \) produces the highest welfare level in competitive equilibrium. Since there is no way to increase the entry rate by monetary policy, \( g = 0 \) is the second best policy. If equation (44) is not satisfied, R&D is overinvested. By increasing the inflation rate from \(-\bar{a} \), overinvestment is relaxed. The inflation rate \( \pi = \psi(\hat{\mu}^{-1}(\mu^*)) \) serves as the maximum inflation rate because it makes \( \mu = \mu^* \).

When there is underinvestment in R&D, the authorities want to stimulate innovation and can achieve the maximum growth rate under \( g = 0 \). Thus, setting \( g = 0 \) is the second
best policy.\(^8\) On the other hand, if there is overinvestment in R&D, the authorities can direct real growth toward the optimal speed by setting \(g > 0\), imposing R&D tax in the form of menu costs.

3.4 Indeterminacy

As shown in Proposition 3, multiple equilibria around \(\pi = -\bar{a}\) always exist, causing an indeterminacy problem (such as \(\pi^T = -\bar{a}\) in Figure 2). In other words, setting \(\pi^T = -\bar{a}\) may end in a negative nominal growth rate, implying slower growth and larger menu costs than desired by the central bank. If the equilibrium values are sufficiently far apart from one another, this problem may cause significant damage to the economy. Then, the optimal rate of inflation should be higher than \(-\bar{a}\). Because \(\pi^T \geq 0\) never brings about indeterminacy, a positive inflation target is one way to resolve this problem. Another way is to adopt nominal growth-targeting policy instead of inflation-targeting policy because equilibrium is always unique if central banks set \(g\).

4 Numerical Simulation

4.1 Benchmark

Our model is calibrated to the US economy on an annual basis. We assume \(L = 1\) and set \(\rho = 0.05\). During 1977–2012, the average per-capita real GDP growth rate is \(a = 0.0162\) and the average entry and exit rates of establishments is \(\mu = 0.118\). Because of the difficulty in measuring an accurate inflation rate embedding product turnover and quality improvements, we simply assume that the average inflation rate during that period is zero.\(^9\) We call this case the benchmark. For the menu cost parameter, we set \(\kappa = 0.022\) following Midrigan (2011). To derive these numbers as equilibrium in the model, \(h\) and \(q\) are calibrated as 0.6588 and 1.1475, respectively.

Then, we simulate how the equilibrium changes depending on inflation rates. The simulation result is shown in Figure 3. The circle in each graph indicates the benchmark case. When the inflation rate equals \(-2.09\%\), \(g\) equals zero and the maximum real growth rate is achieved at the level of \(\bar{a} = 2.09\%\). In this calibration, welfare is also maximized at this

---

\(^8\) In models in which money plays an explicit role through either CIA or money-in-utility, the optimal nominal interest rate is zero in the case of underinvestment (Chu and Cozzi (2014)). In our model, the optimal nominal interest rate equals \(\rho\) because the optimal inflation rate is \(-\bar{a}\) and, in this case, the real interest rate is \(\rho + \bar{a}\).

\(^9\) The average CPI and per-capita nominal GDP growth rates are 0.0387 and 0.0495, respectively, during the period.
level. If we use the two alternative measures of inflation rates, welfare is maximized when the rates are zero. In other words, posted prices should not change to achieve optimality.

In addition, Figure 3 suggests that a deviation from the optimal level has sizable impacts on economic growth. A 10% diversion from the optimal inflation rate (both inflation and deflation) reduces the real growth rate by about half. Compared with the benchmark real growth rate of 1.62%, the real growth rate can be raised to 2.09% by lowering the inflation target to −2.09%. These quantitative impacts are strikingly large relative to the existing studies, for example, Jones and Manuelli (1995) and Lucas (2000).

4.2 Indeterminacy

Figure 4 shows that two possible equilibria $g$'s exist for the inflation rates around $\pi = -\bar{a}$. The multiplicity in $g$ for given $\pi^T$, however, does not appear to cause a significant problem under the parameter set used in the benchmark simulation because the region of indeterminacy is narrow. It is too narrow to be shown in Figure 3. Equilibrium is unique if $\pi$ is higher than the optimal level of $-\bar{a}$ only by 0.03%.

4.3 Underinvestment and Overinvestment in R&D

Figure 5 demonstrates under what parameter values $g > 0$ improves welfare. A pair of high $q$ and high $h$ tend to generate overinvestment. The benchmark pair is located in the region of underinvestment, as displayed in the circle. It is worth noting that overinvestment is not sufficient to make the optimal $g^*$ deviate from zero because it imposes menu costs on firms and increases price distortion. We obtain $g^* \neq 0$ when overinvestment is sufficiently large, as is shown in the area above the green curve in the figure.

5 Robustness

5.1 Different Parameterization

We check the robustness of our results to different parameter values. First, we use a lower menu cost value of $\kappa = 0.007$ following the work of Levy et al. (1997), who directly measure the size of menu costs. Second, as a benchmark, we assume that the average inflation rate is not 0 but 0.0387. This value corresponds to the average CPI growth rate from 1977 to 2012.

Figure 6 demonstrates the simulation results. The left panel shows that a lower menu cost makes the curve flatter, that is, less sensitive to inflation. In the right panel, the curve
Figure 3: Quantitative impacts of inflation. The circles in the graphs indicate the benchmark.
shifts upward when a higher inflation rate is assumed at the benchmark. This increases the benefit of lowering the inflation rate to the growth-maximizing level.

5.2 Model with Imitations

We extend our model by embedding imitations by rival firms. Such an extension is motivated by Bils (2009), who observes from micro price data in the United States that, even under a period of positive inflation, firms tend to lower their prices gradually before they exit markets. Positive inflation is realized because new products enter the market with higher prices than those of replaced ones.\(^{10}\)

One possible explanation for this movement of prices is gradual imitation (catch-up) by rival firms because imitation by rivals drags down the markups of leading firms.\(^{11}\) If the optimal markup rates for firms decrease over time, the optimal inflation rate should be influenced. Furthermore, imitation affects incentives for R&D, and thus, changes the real economic growth rate. Since inflation and growth are interrelated in the abovementioned

\(^{10}\)For example, after the invention of the iPhone, Apple has kept producing the products while gradually lowering prices because of intensified competition with Android and other types of mobile phones. And, Apple still earns positive profits.

\(^{11}\)Dutton and Thomas (1984) report that unit costs in many industries are reduced by about 20% by doubling cumulative outputs due to learning-by-doing. This learning effect may explain the gradual decline in prices even with a constant markup rate. See also Bahk and Gort (1993).
Figure 5: The region in which non-zero nominal growth improves welfare. The circle indicates the benchmark.

model, the optimal rate of inflation should depend on whether leading firms suffer from imitation by rival firms.

By extending our model, we show that the optimal inflation rate increases. It is now higher than $-\bar{a}$ and the optimal nominal growth rate is no longer zero. An alternative measure of the inflation rate should exceed zero when quality improvement is unadjusted. That is, the zero inflation rate is no longer optimal when the price index is based on posted prices. In addition, we show that, combined with imitations, the impacts of inflation on real growth become larger, even if the speed of imitation is mild.

In Subsection 5.2.1 below, we explain only the essence of our extended model and present simulation results. The details of the model are explained in Appendix B.

5.2.1 Extended Model

We assume that rival firms gradually imitate the most advanced technology of industry-leading firms. Suppose that the second highest quality is at $q^{k-1}$ initially when a new firm enters a market with the newest version $k$ in a product line. Time passed from the most recent innovation is denoted by $\tau$. Through imitations, the second highest quality increases
to $q^k$ as

$$q^{II}(k, \tau) = \min(q^{k-1}e^{\delta \tau}, q^k), \quad (45)$$

where $\delta$ represents the speed of imitation. This suggests that the ratio of the highest quality to the second highest quality is given by

$$q^I(\tau) = \frac{q^k}{q^{II}(k, \tau)} = \max(qe^{-\delta \tau}, 1). \quad (46)$$

Since $q^I(\tau)$ reaches one in a finite time, industry-leading firms can earn positive profits only in a finite duration. At most, this is

$$\tau^* = \frac{\log q}{\delta}. \quad (47)$$

Without menu costs, the optimal price of the industry-leading firm satisfies

$$p^*_t(\tau) = qW_te^{-\delta \tau} \quad (48)$$

while $\tau \leq \tau^*$. The optimal price markup over $W_t$ declines over time.
In this extended model, in the presence of menu costs, the optimal time interval between
$s$ and $S_i$, $\Delta_i = t_i - t_{i-1}$, is time variant and dependent on $i$, where $t_i$ represents the time
when firms revise their price for $i = 1, 2, \ldots$. This feature makes our model somewhat
complicated, and hence, we rely on numerical simulation.

5.2.2 Numerical Simulation

In conducting numerical simulation, we use the same parameter values as before except for
one new parameter, $\delta$. We do not find its plausible value, and so, for the sake of illustration
only, we assume $\delta = 0.002$. Rival firms catch up to leading firms by 0.2% yearly, which we
believe is a mild parameter value. The case without imitations corresponds to that of $\delta = 0$.

Figure 7 illustrates the path of relative prices when $g$ equals either $\delta$ or 0.01. The
figure shows that when $g = \delta$, industry-leading firms do not need to revise their prices.
By keeping their posted prices unchanged, their relative prices fall at the same speed as
rivals make imitations. By contrast, when $g$ is larger than $\delta$, namely $g = 0.01$ in this figure,
the relative price falls faster than the speed of imitation. Thus, firms need to revise their
prices upward occasionally by paying menu costs. The optimal time interval $\Delta_i$ is time
variant. Specifically, the last $\Delta_i$ is the longest. As the markup shrinks owing to imitation,
industry-leading firms lose the incentive to revise their prices. When their relative price hit
the floor of one, they stop production and rival firms produce goods.

The impacts of inflation are shown in Figure 8, which reveals that imitation decreases
the real growth rate, the creative destruction rate, and welfare. This is because the presence
of imitation decreases the value of entrant firms and, in turn, the incentive to undertake
R&D investment. Thus, welfare decreases, although imitations lower the markup, which
functions to increase efficiency. In addition, the figure shows that the impacts of inflation
on real growth and welfare are amplified by the presence of imitations, even if their speed
is mild.

Welfare is maximized when $g = \delta$. The optimal inflation rate equals $-\bar{a} + \delta$, where $\bar{a}$
represents the real growth rate without price stickiness. Thus, the optimal inflation rate rises
by about $\delta$ (excluding a difference in $\bar{a}$) compared with that in the model without imitations.
According to the bottom right panel, the optimal inflation rate measured without adjusting
quality improvement is more than zero. This implies that the positive inflation rate is
called for even when the price index is based on posted prices. The inflation rate based on
a matched sample remains to support zero inflation.

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6 Conclusion

In this study, we investigated the optimal inflation rate when the real economy grows through creative destruction. We used a menu cost model to generate the link from nominal to real elements. The key linkage between them is that firm values as a reward for innovation are larger when firms do not need to revise their prices frequently. An option for the optimal inflation rate is the negative of the maximum real growth rate, which realizes maximum real growth by eliminating nominal rigidity.

Through qualitative and quantitative analyses, we examined cases in which the optimal inflation rate is different from the growth maximizing rate. We focused on the following two cases. First, because of negative externalities from the destructive aspect of innovation, there may be overinvestment in R&D in the market. Then, the optimal inflation rate that maximizes welfare could divert from the growth-maximizing rate because enlarging menu cost payments by greater nominal growth works as a tax on R&D. Another reason that the optimal inflation rate diverts from the growth-maximizing rate is indeterminacy. Because multiple equilibria exist around the growth-maximizing inflation rate, a central bank might fail to achieve its desired equilibrium. Both cases suggest that a slightly higher inflation rate improves welfare. However, the numerical analyses imply that these factors...
Figure 8: Model with Imitations
are not quantitatively significant. The optimal inflation rate eventually equals the growth-maximizing one.

The current model ignores several factors that could affect the optimal rate of inflation, such as the zero bound of nominal interest rates, downward rigidity of wages, a CIA constraint, and money-in-utility. Incorporation of these factors is for future research themes.

Another important direction for future research is the transitional dynamics between balanced growth paths. Because firms are heterogeneous owing to menu costs, the economy does not jump to a balanced growth path immediately after shocks, unlike standard creative destruction models. Future research is required to study the short-run effects of monetary policy using the model we developed.

References


A Proofs

A.1 Proof of Lemma 1

Lemma A1 Define the following functions over $x \neq 0$ with $y \geq 0$ by

$$h_1(x, y) = y - \frac{1 - e^{-xy}}{x^2},$$
$$h_2(x, y) = y - \frac{e^{xy} - 1}{x^2}.$$

Then, for any $x \neq 0$ and $y \geq 0$, the following relationships hold:

$$h_1(x, y) \geq 0, \quad h_2(x, y) \leq 0,$$
$$\frac{\partial h_1(x, y)}{\partial x} \leq 0, \quad \frac{\partial h_2(x, y)}{\partial x} \leq 0,$$

with equalities only when $y = 0$.

Proof. Proving the signs of the functions is straightforward:

$$h_1(x, y) = \frac{xy - (1 - e^{-xy})}{x^2} \geq 0,$$
$$h_2(x, y) = \frac{xy - (e^{xy} - 1)}{x^2} \leq 0.$$

About the derivative of $h_1$, we have

$$\frac{\partial h_1(x, y)}{\partial x} = \frac{2(1 + e^{-xy})}{x^3} \left[ \frac{xy}{2} - \frac{1 - e^{-xy}}{1 + e^{-xy}} \right].$$

Since $y \geq 0$ and

$$\frac{xy}{2} \geq \frac{1 - e^{-xy}}{1 + e^{-xy}}$$

if and only if $xy \geq 0$,
we have $\frac{\partial h_1(x, y)}{\partial x} \leq 0$.

Last, the partial derivative of $h_2$ is expressed as

$$\frac{\partial h_2(x, y)}{\partial x} = \frac{2(1 + e^{xy})}{x^3} \left[ \frac{xy}{2} - \frac{1 - e^{-xy}}{1 + e^{-xy}} \right].$$

Thus, the sign of $\frac{\partial h_2(x, y)}{\partial x}$ is equivalent to that of $\frac{\partial h_1(x, y)}{\partial x}$. \qed

Proof of Lemma 1
Proof. Suppose $g > 0$. The total differentiation of equation (23) is

$$
\frac{-ge^{\Delta} \left(1 - e^{-(\rho + \mu)\Delta} \right)}{\rho + \mu} d\Delta = \left[ e^{\Delta} \left( \frac{\Delta e^{-(\rho + \mu)\Delta}}{\rho + \mu} - 1 - e^{-(\rho + \mu)\Delta} \right) \right] d\mu
+ \left[ \frac{\Delta e^{\Delta} (1 - e^{-(\rho + \mu)\Delta})}{\rho + \mu} + \frac{\Delta e^{-(\rho + \mu-g)\Delta}}{\rho + \mu - g} - \frac{1 - e^{-(\rho + \mu-g)\Delta}}{(\rho + \mu - g)^2} \right] dg
= e^{-(\rho + \mu-g)\Delta} \left[ \left( \frac{\Delta}{\rho + \mu} - \frac{e^{(\rho+\mu)\Delta} - 1}{(\rho + \mu)^2} \right) \right] d\mu
+ \left[ q\kappa\Delta + \left( \frac{\Delta}{\rho + \mu - g} - \frac{e^{(\rho+\mu-g)\Delta} - 1}{(\rho + \mu - g)^2} \right) \right] dg
\tag{49}
$$

where we substituted equation (23) in the second equality. Functions $h_1$ and $h_2$ are defined in Lemma A1. Owing to Lemma A1, the signs of the coefficients of $d\mu$ and $dg$ are determined uniquely and we have $d\Delta/d\mu > 0$ and $d\Delta/dg < 0$.

Suppose $g < 0$. Similarly, the total differentiation of equation (27) equals

$$
\frac{ge^{-\Delta} \left( e^{(\rho+\mu)\Delta} - 1 \right)}{\rho + \mu} d\Delta = e^{(\rho+\mu-g)\Delta} \left[ \left( \frac{\Delta}{\rho + \mu} - \frac{1 - e^{-(\rho + \mu)\Delta}}{(\rho + \mu)^2} \right) \right] d\mu
+ \left[ -q\kappa\Delta + \left( \frac{\Delta}{\rho + \mu - g} - \frac{e^{(\rho+\mu-g)\Delta} - 1}{(\rho + \mu - g)^2} \right) \right] dg
= e^{(\rho+\mu-g)\Delta} \left[ h_1(\rho + \mu, \Delta) - h_1(\rho + \mu - g, \Delta) \right] d\mu + [-q\kappa\Delta + h_2(\rho + \mu - g, \Delta)] dg
\tag{50}
$$

Again, from Lemma A1, we obtain $d\Delta/d\mu < 0$ and $d\Delta/dg > 0$. 

A.2 Proof of Lemma 2

Proof. The first part of the lemma is proved as follows. Suppose $g_2 > g_1 > 0$. Let $\Delta_2$ and $\Delta_1$ be the optimized duration for each $g$. We first compare $v(g_2, \mu, \Delta_2)$ with $v(g_1, \mu, \Delta_2)$. When $\mu$ and $\Delta$ are common, equation (22) implies that

$$
v(g_2, \mu, \Delta_2) < v(g_1, \mu, \Delta_2) \iff \int_0^{\Delta_2} \Pi_0'Se^{-g_2t} e^{-(\rho + \mu)t} dt < \int_0^{\Delta_2} \Pi_0'Se^{-g_1t} e^{-(\rho + \mu)t} dt
\tag{33}
$$
The latter inequality actually holds true for \( g_2 > g_1 > 0 \) since \( Se^{g_2t} < Se^{g_1t} < S \) for any \( t \in (0, \Delta_2) \). Hence, \( v(g_2, \mu, \Delta_2) < v(g_1, \mu, \Delta_2) \). Moreover, since \( \Delta \)'s are chosen to maximize \( v \)'s, we have

\[
v(g_2, \mu, \Delta_2) < v(g_1, \mu, \Delta_2) \leq v(g_1, \mu, \Delta_1).
\]

Thus, \( dv/dg < 0 \). Using this relationship and equation (24) implies that \(|g|\Delta \) is increasing in \( g \) for a given \( \mu \).

Next, suppose \( g_1 < g_2 < 0 \). Let \( \Delta_1 \) and \( \Delta_2 \) be defined similarly. Analogous to the above argument, equation (26) implies that

\[
v(g_1, \mu, \Delta_1) < v(g_2, \mu, \Delta_1) \leq v(g_2, \mu, \Delta_2).
\]

Thus, \( dv/dg > 0 \). Equation (28) implies that \(|g|\Delta \) is decreasing in \( g \) for a given \( \mu \).

To prove the second part of the lemma, it is sufficient to show that \( \partial v/\partial \mu < 0 \) because of the envelope theorem. Define \( I(t) \) as an indicator function that takes 1 at \( t = 0 \) and 0 otherwise. The value of entering firms can be rewritten as

\[
v_t = \sum_{i=0}^{\infty} \left[ \int_{t_i}^{t_{i+1}} \Pi^0(\xi_{t'}) e^{-(\rho+\mu)(t'-t)} dt' - \kappa e^{-(\rho+\mu)t_{i+1}} \right]
= \int_{t}^{t_{i+1}} \Pi^0(\xi_{t'}) e^{-(\rho+\mu)(t'-t)} dt' + \sum_{i=1}^{\infty} \left[ \int_{t_i}^{t_{i+1}} (\Pi^0(\xi_{t'}) - I(t' - t_i)\kappa) e^{-(\rho+\mu)(t'-t)} dt' \right].
\]

Its derivative with respect to \( \mu \) reads

\[
\frac{\partial v_t}{\partial \mu} = - \int_{t}^{t_{i+1}} (t' - t)\Pi^0(\xi_{t'}) e^{-(\rho+\mu)(t'-t)} dt'
- \sum_{i=1}^{\infty} \left[ \int_{t_i}^{t_{i+1}} (t' - t) \left( \Pi^0(\xi_{t'}) - I(t' - t_i)\kappa \right) e^{-(\rho+\mu)(t'-t)} dt' \right].
\]
Both the first and second terms are negative because
\[
\int_{t_i}^{t_{i+1}} (t' - t) \left( \Pi^0(\xi_{t'}) - I(t' - t_i) \kappa \right) e^{-(\rho + \mu)(t' - t)} dt'
= \int_{t_i}^{t_{i+1}} (t' - t_i + i\Delta) \left( \Pi^0(\xi_{t'}) - I(t' - t_i) \kappa \right) e^{-(\rho + \mu)(t' - t_i + i\Delta)} dt'
= e^{-(\rho + \mu)i\Delta} \int_0^{\Delta} (t'' + i\Delta) \left( \Pi^0(\xi_{t''}) - I(t'') \kappa \right) e^{-(\rho + \mu)t''} dt''
= e^{-(\rho + \mu)i\Delta} \left[ \int_0^{\Delta} t'' \Pi^0(\xi_{t''}) e^{-(\rho + \mu)t''} dt'' + i\Delta \int_0^{\Delta} \Pi^0(\xi_{t''}) e^{-(\rho + \mu)t''} dt'' \right]
= e^{-(\rho + \mu)i\Delta} \left[ \int_0^{\Delta} t'' \Pi^0(\xi_{t''}) e^{-(\rho + \mu)t''} dt'' + i\Delta \left( \int_0^{\Delta} \Pi^0(\xi_{t''}) e^{-(\rho + \mu)t''} dt'' - \kappa \right) \right]
> 0.
\]
Thus, we have \( \partial v / \partial \mu < 0. \]

A.3 Proof of Lemma 3

A.3.1 Distribution Function

Regarding the density function of relative prices, equation (30) can be transformed into
\[
f_{\xi}(\xi | g, \mu) = \begin{cases} 
\frac{\mu}{g} \frac{1}{1 - e^{-\mu \Delta}} \left( \frac{\xi}{S} \right)^{\mu-1} & \text{for } g > 0, \\
\frac{\mu}{-gS} \frac{1}{e^{\mu \Delta} - 1} \left( \frac{\xi}{S} \right)^{\mu-1} & \text{for } g < 0.
\end{cases}
\]  
(51)

The distribution function is written as
\[
F_{\xi}(\xi | g, \mu) = \begin{cases} 
\frac{\left( \frac{\Delta}{S} \right)^{\mu} e^{-\mu \Delta}}{1 - e^{-\mu \Delta}} & \text{for } g > 0, \\
\frac{e^{\mu \Delta} - \left( \frac{\Delta}{S} \right)^{\mu}}{e^{\mu \Delta} - 1} & \text{for } g < 0.
\end{cases}
\]  
(52)

A.3.2 Price Band

As for the price band, \( S \) is determined by
\[
S = qW = \frac{qv}{h}.
\]  
(53)
from equations (2) and (18), where $E = 1$. In addition, $s$ is given by

$$s = \begin{cases} \frac{qv}{h}e^{-s\Delta} & \text{for } g > 0 \\ \frac{qv}{h}e^{s\Delta} & \text{for } g < 0 \end{cases} \tag{54}$$

from equation (24) when $g > 0$ and equation (25) when $g < 0$.

### A.3.3 Lemmas

In the following proofs, we rewrite $v(g, \mu, \Delta(g, \mu))$ as $v(g, \mu)$ to simplify notations.

**Lemma A2** Fix $g \geq 0$. $F(\xi | g, \mu_1)$ first-order stochastically dominates $F_\xi(\xi | g, \mu_2)$ for $\mu_1 < \mu_2$.

**Proof.** Suppose $\mu_1 < \mu_2$. Then, the following size relationship holds:

$$S_1 > S_2 \quad \text{and} \quad s_1 > s_2$$

because the Ss price band is given by equations (53) and (54) and Lemmas 1 and 2 suggest that $v$ is decreasing in $\mu$ and $\Delta$ is increasing in $\mu$.

Next, from the first line of equation (52), we know that $F(\xi | g, \mu)$ is non-decreasing in $\mu$ if

$$\frac{1}{g} \left( \log \frac{\xi}{S} - \mu \frac{dS/d\mu}{S} \right) \left( \frac{\xi}{S} \right)^{\frac{\xi}{S}} \geq \frac{de^{-\mu\Delta}}{d\mu} \frac{1}{1 - e^{-\mu\Delta}}. \tag{55}$$

The left-hand side of equation (55) is an increasing function of $\xi$ and the right-hand side is independent of $\xi$. Thus, if the former is larger than the latter at some $\xi$, (55) holds for any $\xi$ larger than that point. Since $s_1 > s_2$, $0 = F(s_1 | g, \mu_1) < F(s_1 | g, \mu_2)$, and thus,

$$F_\xi(\xi | g, \mu_1) \leq F_\xi(\xi | g, \mu_2) \quad \forall \xi,$$

where strict inequality holds for $\xi \in (s_2, S_1)$. Thus, $F_\xi(\xi | g, \mu_1)$ first-order stochastically dominates $F_\xi(\xi | g, \mu_2)$.

**Lemma A3** Fix $\mu \geq 0$. $F_\xi(\xi | g_1, \mu)$ first-order stochastically dominates $F_\xi(\xi | g_2, \mu)$ if $|g_1| < |g_2|$.

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Proof. Since $v(|g_1|, \mu, \Delta(|g_1|, \mu)) > v(|g_2|, \mu, \Delta(|g_2|, \mu))$ from Lemma 2, we find $S_1 > S_2$ in equation (53). As for $s$, using equation (54) with (24) or (28) as well as Lemma 2, we notice that both

$$s = Se^{-\Delta} = \frac{q}{h} \left[ \frac{1}{\rho + \mu} + \kappa \right] e^{-\Delta} - \frac{1}{q(\rho + \mu)}$$

for $g > 0$

and

$$s = Se^{\Delta} = \frac{q}{h(h + \mu)} \left( e^{\Delta} - 1 \right)$$

for $g < 0$

are decreasing in $|g|$. Thus, $s_1 > s_2$.

Now, suppose $g > 0$. From the first line of equation (52), we know that $F_\xi(\xi|g, \mu)$ is increasing in $g$ if

$$\frac{1}{g} \left( -\log \left( \frac{\xi}{S} \right) \frac{\xi}{S} - \mu \frac{dS/dg}{S} \right) \frac{\frac{\xi}{S}}{1 - \left( \frac{\xi}{S} \right)} \geq \frac{de^{-\Delta}}{1 - e^{-\mu\Delta}}.$$

From the same argument used to prove Lemma A2, we can argue $F_\xi(\xi|g_1, \mu) \leq F_\xi(\xi|g_2, \mu)$ if the left-hand side is increasing in $\xi$. Noticing that a function $\frac{x \log x}{x - 1}$ for $x > 0$ is non-decreasing in $x$,

$$-\log \left( \frac{\xi}{S} \right) \frac{\xi}{S} \times \frac{\frac{\xi}{S}}{1 - \left( \frac{\xi}{S} \right)}$$

is non-decreasing in $\xi$. Hence, (56) holds for $\xi \in [s_2, S_1]$ and $F_\xi(\xi|g_1, \mu)$ first-order stochastically dominates $F_\xi(\xi|g_2, \mu)$.

Next, suppose $g < 0$. From the second line of equation (52), we know that $F_\xi(\xi|g, \mu)$ is decreasing in $g$ if

$$\frac{d}{dg} \left( \frac{\xi}{S} \right) \frac{\frac{\xi}{S}}{1} \geq \frac{de^{\mu\Delta}}{dg} \frac{1}{e^{\mu\Delta} - 1}$$

$$\Leftrightarrow \frac{1}{g} \left( \log \left( \frac{\xi}{S} \right) \frac{\xi}{S} + \mu \frac{dS/dg}{S} \right) \frac{\frac{\xi}{S}}{1 - \left( \frac{\xi}{S} \right)} \geq \frac{de^{\mu\Delta}}{dg} \frac{1}{e^{\mu\Delta} - 1}.$$
Therefore, \( F_\xi(g_1, \mu) \) also first-order stochastically dominates \( F_\xi(g_2, \mu) \).

**A.3.4 Proof of Lemma 3**

**Proof.** Suppose \( g_2 > g_1 \geq 0 \). Consider a function \(-\xi^{-1}\). First-order stochastic dominance in Lemma A3 implies

\[
\int -\xi^{-1} F_\xi(d\xi|g_2, \mu) \leq \int -\xi^{-1} F_\xi(d\xi|g_1, \mu) \quad \forall \mu \geq 0
\]

\[
\Leftrightarrow L_x(g_2, \mu) \geq L_x(g_1, \mu) \quad \forall \mu \geq 0,
\]

where we used equation (31) to derive the second line. This is analogous for \( g_2 < g_1 \leq 0 \).

Hence, \( L_x(g, \mu) \) is increasing in \( |g| \).

Next, suppose \( g \geq 0 \). Again, consider \(-\xi^{-1}\). Lemma A2 implies

\[
\int -\xi^{-1} F_\xi(d\xi|g, \mu) \leq \int -\xi^{-1} F_\xi(d\xi|g, \mu_1) \quad \forall g \geq 0
\]

\[
\Leftrightarrow L_x(g, \mu_2) \geq L_x(g, \mu_1) \quad \forall g \geq 0,
\]

Therefore, \( L_x(g, \mu) \) is increasing in \( \mu \) if \( g \geq 0 \).

Next, suppose \( g < 0 \). To show that \( L_x(g, \mu) \) is increasing in \( \mu \) if \( g \) is negative and sufficiently close to zero, we examine whether \( \partial L_x/\partial \mu \) is positive in the limit of \( g \uparrow 0 \).

Equation (32) yields

\[
\frac{\partial L_x/\partial \mu}{L_x} = -\frac{g}{\mu(\mu - g)} - \frac{g \Delta}{\mu \partial \mu} - \frac{\partial S/\partial \mu}{S} + \left( \Delta + (\mu - g) \frac{\partial \Delta}{\partial \mu} \right) \frac{e^{-(\mu - g)\Delta}}{1 - e^{-(\mu - g)\Delta}} - \left( \Delta + \mu \frac{\partial \Delta}{\partial \mu} \right) \frac{e^{-\mu \Delta}}{1 - e^{-\mu \Delta}}.
\]

(57)

The limit of this equation as \( g \uparrow 0 \) equals

\[
\lim_{g \uparrow 0} \frac{\partial L_x/\partial \mu}{L_x} = -\lim_{g \uparrow 0} \frac{\partial \Delta}{\partial \mu} - \lim_{g \uparrow 0} \frac{\partial S/\partial \mu}{S}.
\]

(58)

About the first term on the right-hand side of equation (58), using equation (50), we
have
\[
\lim_{g \to 0} g \frac{\partial \Delta}{\partial \mu} = \lim_{g \to 0} \frac{(\rho + \mu) e^{(\rho + \mu)\Delta}}{(\rho + \mu) - 1} \left[ \left( \frac{\Delta}{\rho + \mu} - \frac{1 - e^{-(\rho + \mu)\Delta}}{(\rho + \mu)^2} \right) - \left( \frac{\Delta}{\rho + \mu - g} - \frac{1 - e^{-(\rho + \mu - g)\Delta}}{(\rho + \mu - g)^2} \right) \right] = 0.
\]

About the second term on the right-hand side of (58), from equation (28), we have
\[
\lim_{g \to 0} \frac{\partial S}{\partial \mu} = \lim_{g \to 0} \frac{\partial v}{\partial \mu} = \lim_{g \to 0} \frac{g \frac{\partial e^{-g\Delta}}{\partial g}}{1 - \frac{e^{-g\Delta}}{q}} - \frac{1}{\rho + \mu} = -\frac{1}{\rho + \mu}.
\]

Therefore, we can verify
\[
\lim_{g \to 0} \frac{\partial L_x}{\partial \mu} = \frac{1}{\rho + \mu} > 0.
\]

**A.4 Proof of Proposition 3**

**Lemma A4** For any \( \mu \geq 0 \),
\[
\lim_{g \to 0} \frac{\partial L_x(g, \mu)}{\partial g} = -\infty.
\]

**Proof.** From equation (32), we have
\[
\frac{\partial L_x}{\partial g} = \frac{1}{\mu - g} - \frac{\Delta + g \frac{\partial \Delta}{\partial g}}{1 - e^{-(\mu - g)\Delta}} + \mu \frac{\partial \Delta}{\partial g} \left( \frac{e^{-(\mu - g)\Delta}}{1 - e^{-(\mu - g)\Delta}} - \frac{e^{-\mu \Delta}}{1 - e^{-\mu \Delta}} \right) - \frac{\partial S}{\partial g}. \tag{59}
\]

To establish the limit of this equation, we examine the limit of each term. First, from equation (50), the second term on the right-hand side becomes
\[
\frac{\partial \Delta}{\partial g} = \frac{e^{-g\Delta} - (e^{(\rho + \mu - g)\Delta} - 1)}{e^{(\rho + \mu)\Delta} - 1} \left[ \frac{\Delta}{\rho + \mu - g} - \frac{e^{(\rho + \mu - g)\Delta} - 1}{(\rho + \mu - g)^2} - q \Delta \right]. \tag{60}
\]

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Substituting equation (27) into equation (60) yields

\[ g \frac{\partial \Delta}{\partial g} = \frac{\rho + \mu}{e^{-g\Delta}} \left( \frac{e^{(\rho + \mu)\Delta}}{\rho + \mu - g} - \frac{e^{(\rho + \mu)\Delta} - 1}{\rho + \mu} \right) - \frac{e^{(\rho + g\mu)\Delta} - 1}{(\rho + \mu - g)^2} \]

\[ = \frac{\rho + \mu}{e^{-g\Delta}} \left[ \Delta e^{-g\Delta} \left( \frac{1}{\rho + \mu - g} - \frac{1}{\rho + \mu} + \frac{e^{-(\rho + \mu)\Delta}}{\rho + \mu} \right) - \frac{1}{(\rho + \mu - g)^2} - \frac{e^{-g\Delta} - e^{-(\rho + \mu)\Delta}}{1 - e^{-(\rho + \mu)\Delta}} \right] \]

\[ = \frac{\frac{g\Delta}{\rho + \mu - g} + \Delta e^{-(\rho + \mu)\Delta}}{1 - e^{-(\rho + \mu)\Delta}} - \frac{\rho + \mu}{(\rho + \mu - g)^2} \frac{1 - e^{-(\rho + \mu - g)\Delta}}{1 - e^{-(\rho + \mu)\Delta}}. \]

Thus, we obtain

\[ \lim_{g \to 0} \frac{\partial \Delta}{\partial g} = -\frac{1}{\rho + \mu} \]

because both \( g\Delta \) and \( \Delta e^{-(\rho + \mu)\Delta} \) converge to zero as \( g \to 0 \). In addition, in the second term, we have

\[ \lim_{g \to 0} \frac{\partial \Delta}{\partial g} = \infty. \]

About the third term on the right-hand side of equation (59), we notice

\[ \lim_{g \to 0} \frac{\partial \Delta}{\partial g} \left( \frac{e^{-(\mu - g)\Delta}}{1 - e^{-(\mu - g)\Delta}} - \frac{e^{-\mu\Delta}}{1 - e^{-\mu\Delta}} \right) \leq 0 \]

since the term in parentheses is always strictly negative for \( g < 0 \).

About the last term on the right-hand side of equation (59), using equation (28), we have

\[ \lim_{g \to 0} \frac{\partial S/\partial g}{S} = \lim_{g \to 0} \frac{\partial v/\partial g}{v} = \lim_{g \to 0} \left( \frac{\Delta + g \frac{\partial \Delta}{\partial g}}{1 - e^{-g\Delta}} \right) = \infty. \]

Thus, the limit of equation (59) becomes

\[ \lim_{g \to 0} \frac{\partial L_x/\partial g}{L_x} = -\infty. \]

Since \( L_x > 0 \) for \( g = 0 \), \( L_x(g, \mu) \) is decreasing in \( g \) if \( g \) is negative and sufficiently close to zero.

**Proof of Proposition 3**
Proof. Suppose $g \geq 0$. Because of Proposition 1, the right-hand side of equation (39) increases monotonically with $g$. Therefore, as long as the intersection occurs in the region of $g \geq 0$, the equilibrium value of $g$ is unique given $\pi$ and it is increasing in $\pi$.

Suppose $g < 0$. We want to show that $\psi(g)$ is not monotonic with $g$ and $\psi(g)$ is decreasing if $g$ is sufficiently close to zero. If $\psi(g)$ is monotonic with $g$, from equation (39), $d\mu/dg$ must satisfy

$$\frac{d\mu}{dg} = -\frac{\partial L_x/\partial g}{h + \partial L_x/\partial \mu} < \frac{1}{\log q}, \quad (62)$$

where the first equality stems from the labor market-clearing condition, $L - L_x = h\mu$. Recall that $\lim_{g \to 0} \partial L_x/\partial g = -\infty$ by Lemma A4 and $\lim_{g \to 0} \partial L_x/\partial \mu$ is positive and finite, as shown in the proof of Lemma 3. Thus, we have

$$\lim_{g \to 0} -\frac{\partial L_x/\partial g}{h + \partial L_x/\partial \mu} = \infty,$$

which violates equation (62). Therefore, $\psi(g)$ is not monotonic. It is downward sloping when $g$ is sufficiently close to zero.

Since $\hat{\mu}(g) \geq 0$ by definition, equation (39) suggests $\psi(g) \leq g$. Thus, an upper bound of $\psi(g) < 0$ exists for $g < 0$. Let $\pi_2 \equiv \max \psi(g)$. Then, multiple equilibria occur if $\pi \in [\pi, \pi_2]$.

Next, we show that $\hat{\mu}(g) = 0$ for sufficiently small $g < 0$, which implies that $\psi(g)$ is on the 45 degree line and $\pi$ determines a unique $g$.

As $g \to -\infty$, equation (27) is approximated as

$$q\kappa \simeq e^{-g\Delta} \frac{e^{(\rho+\mu)\Delta} - 1}{\rho + \mu}.$$

This suggests that $\Delta \to 0$ and $e^{-g\Delta} \simeq q\kappa/\Delta$. Substituting the latter equation into equation (28) yields $v = (1 - \kappa/\Delta)/(\rho + \mu)$, which falls below zero because $\Delta \to 0$. Then, the free-entry condition (2) must hold with strict inequality, implying that $\hat{\mu}(g) = 0$. Because $\hat{\mu}(g)$ is zero when $g \to -\infty$, the function of $\psi(g)$ is upward sloping with the 45 degree slope.

Thus, similar to a case of $g > 0$, when $g$ is negative and sufficiently far from zero, the equilibrium value of $g$ is unique for a given $\pi$ and it is increasing in $\pi$. In other words, $\pi_1 \leq -\bar{a}$ exists, which yields unique equilibrium for $\pi \notin [\pi_1, \pi_2]$.

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B Model with Imitations

B.1 Industry-Leading Firms

Without nominal rigidity, the optimal relative price of the industry-leading firm satisfies

$$\xi^*(\tau) = qW e^{-\delta \tau}$$  \hspace{1cm} (63)$$

while $$\tau \leq \tau^*$$.

In the presence of menu costs, the expected present value of industry-leading firms $$V_t$$ is described as

$$V_t = \sum_{i=0}^{i_{\text{max}}-1} \int_{t_i}^{t_{i+1}} \Pi_\nu (\xi_i e^{-g(t'-t_i)}) e^{-(r+\mu)(t'-t)} dt' + \sum_{i=0}^{i_{\text{max}}-1} \kappa (E_{t_{i+1}}/P_{t_{i+1}}) e^{-(r+\mu)(t_{i+1}-t)} ,$$

where $$i_{\text{max}}$$ represents the maximum number of times that firms reset their prices by paying menu costs and $$t_{i_{\text{max}}+1} - t (\leq \tau^*)$$ represents the maximum firm life. It is optimal for firms to obey the following $$S$$s rule. We denote time passed since firm entry by $$t' - t$$. Then, the maximum price $$S(t')$$ satisfies

$$S(t') = \xi^*(t') = qW e^{-\delta (t'-t)} .$$  \hspace{1cm} (64)$$

Regarding firms’ pricing strategies, the following three cases can be considered. The first case is $$0 < \delta < g$$, in which the imitation process is slow. Prices are always sufficiently lower than $$S(t')$$, that is, $$\xi_{t'} \leq S(t')$$ for all $$t'$$. Then, firms set $$\xi_{t'} = S(t')$$ and reset prices when $$\xi_{t'} = s(t')$$. The second case is $$0 < g < \delta$$, in which the imitation process is fast. Prices fall below $$S(t')$$ immediately after price revision, if firms set $$\xi_{t'} = S(t')$$. Thus, firms set $$\xi_{t'} = s(t')$$ and reset prices when $$\xi_{t'} = S(t')$$. The third case is $$g < 0 < \delta$$, in which firms set $$\xi_{t'} = s(t')$$ and reset prices when $$\xi_{t'} = S(t')$$. The second and third cases are essentially the same.

Suppose $$g > \delta$$. In period $$t_i$$, industry-leading firms reset their prices at $$S(t_i)$$; the relative price $$\xi_t$$ declines and reaches $$s(t_{i+1})$$ in period $$t_{i+1}$$ and the firms reset their prices. After the last period of price revision $$t_{i_{\text{max}}}$$, firms keep their prices unchanged until their relative price $$\xi$$ falls to $$W$$ and they stop production in period $$t_{i_{\text{max}}+1}$$, which satisfies

$$qW e^{-\delta (t_{i_{\text{max}}}-t)} e^{-g(t_{i_{\text{max}}+1}-t_{i_{\text{max}}})} = W .$$  \hspace{1cm} (65)$$
The firm value is described as
\[
V_i = \sum_{t=0}^{i_{\text{max}}} \int_{t_i}^{t_{i+1}} \Pi^0(qW e^{-\delta(t_{i} - t)}) e^{at} e^{-(\rho + \mu)(t_{i} - t)} dt' - \sum_{i=0}^{i_{\text{max}}-1} \frac{\kappa E}{P} e^{at_{i+1}} e^{-(\rho + \mu)(t_{i+1} - t)}. \tag{66}
\]

The first-order condition with respect to \( t_i \) for \( i = 0, 1, \ldots, i_{\text{max}} \) is
\[
\frac{\partial V_i}{\partial t_i} = 0 = -\Pi^0(qW e^{-\delta(t_{i} - t)}) \frac{E}{P} e^{at} e^{-(\rho + \mu)(t_{i} - t)} + \Pi^0(qW e^{-\delta(t_{i} - t)} e^{\gamma(t_{i} - t_{i-1})}) \frac{E}{P} e^{at} e^{-(\rho + \mu)(t_{i} - t)}
\]
\[
+ (g - \delta) \int_{t_i}^{t_{i+1}} qW e^{-\delta(t_{i} - t)} e^{\gamma(t' - t)} \Pi^0(qW e^{-\delta(t_{i} - t)} e^{\gamma(t' - t)}) \frac{E}{P} e^{at'} e^{-(\rho + \mu)(t' - t)} dt'
\]
\[
+ \kappa(\rho + \mu) \frac{E}{P} e^{at} e^{-(\rho + \mu)(t_{i} - t)},
\]
which leads to
\[
0 = 1 - \frac{e^{\gamma(t_{i} - t_{i-1})}}{q} + \frac{g - \delta}{q} \frac{e^{\gamma(\rho + \mu)(t_{i+1} - t_{i})} - 1}{g - \rho - \mu} + \frac{\kappa(\rho + \mu)}{g} e^{-\delta(t_{i} - t)}.
\]

This equation suggests that the duration, \( \Delta_i = t_i - t_{i-1} \), is time variant and dependent on \( i \):
\[
0 = 1 - \frac{e^{\gamma \Delta_i}}{q} - \frac{g - \delta}{q} \frac{e^{-(\rho + \mu - g)\Delta_{i+1}} - 1}{\rho + \mu - g} + \frac{\kappa(\rho + \mu)}{g} e^{-\delta \sum_{j=1}^{i} \Delta_j},
\]
where \( \Delta_{i_{\text{max}}+1} \) satisfies
\[
qe^{-\delta \sum_{j=1}^{i_{\text{max}}} \Delta_j} e^{-g \Delta_{i_{\text{max}}+1}} = 1 \tag{67}
\]
from equation (65).

Then, the maximum and minimum prices, \( S(t_i) \) and \( s(t_i + \Delta_{i+1}) \), are given by
\[
S(t_i) = qW e^{-\delta \sum_{j=1}^{i} \Delta_j}, \tag{68}
\]
\[
s(t_i + \Delta_{i+1}) = S(t_i) e^{-g \Delta_{i+1}} = qW e^{-\delta \sum_{j=1}^{i} \Delta_j} e^{-g \Delta_{i+1}}, \tag{69}
\]
respectively.

The firm value can be rewritten as
\[
v_i = 1 \rho + \mu \left( 1 - e^{-(\rho + \mu) \sum_{j=1}^{i_{\text{max}}+1} \Delta_j} \right) - \sum_{i=0}^{i_{\text{max}}-1} \frac{1}{q} e^{-(\rho + \mu - \delta) \sum_{j=1}^{i} \Delta_j} \left( 1 - e^{-(\rho + \mu - g) \Delta_{i+1}} \right)
\]
\[- \sum_{i=0}^{i_{\text{max}}} \kappa e^{-(\rho + \mu) \sum_{j=1}^{i} \Delta_j} \tag{70}
\]
As long as \( i = 0, 1, \cdots, i_{\text{max}} \), the firm value is positive even if the firm pays menu costs. This suggests the following inequality condition:

\[
0 < \int_{t_{i_{\text{max}}}}^{t_{i_{\text{max}}+1}} \frac{qe^{-\delta(t_{i_{\text{max}}}-t)}e^{-g(t'_{i_{\text{max}}})}}{qe^{-\delta(t_{i_{\text{max}}}-t)}e^{-g(t'_{i_{\text{max}}})}} - 1 \ e^{-(\rho+\mu)(t'_{i}-t)} \ dt' \ e^{-(\rho+\mu)(t_{i_{\text{max}}}-t)},
\]

which leads to

\[
0 < \frac{1}{\rho + \mu} \left( 1 - e^{-(\rho+\mu)\Delta_{i_{\text{max}}+1}} \right) - \frac{1}{\rho + \mu - g} e^\delta \sum_{j=1}^{i_{\text{max}}} \Delta_j \left( 1 - e^{-(\rho+\mu-g)\Delta_{i_{\text{max}}+1}} \right) - \kappa \ 
\]

\[
= \frac{1}{\rho + \mu} \left( 1 - e^{-(\rho+\mu)\Delta_{i_{\text{max}}+1}} \right) - \frac{1}{\rho + \mu - g} e^{-g\Delta_{i_{\text{max}}+1}} \left( 1 - e^{-(\rho+\mu-g)\Delta_{i_{\text{max}}+1}} \right) - \kappa. \ (71)
\]

To derive the last line, we use equation (67). If firms reset their prices more than \( i_{\text{max}} \) times, the right-hand side of the equation becomes negative.

Next, suppose \( g < \delta \). In period \( t_i \), industry-leading firms reset their prices at \( s(t_i) \); the relative price \( \xi_i \) reaches \( S(t_{i+1}) \) in period \( t_{i+1} \) and the firms reset their prices. After the last period of price revision \( t_{i_{\text{max}}} \), firms keep their prices unchanged. When their relative price increases to \( S(t_{i_{\text{max}}+1}) \), they stop production. This suggests

\[
s(t_i) = S(t_{i+1})e^{\delta(t_{i+1}-t_i)} = qW e^{-\delta(t_{i+1}-t_i)} e^{g(t_{i+1}-t_i)}. \ (72)
\]

Similarly, the maximum and minimum prices, \( S(t_{i+1}) \) and \( s(t_i) \), are given by

\[
S(t_{i+1}) = qW e^{-\delta \sum_{j=1}^{i+1} \Delta_j}, \ (73)
\]

\[
s(t_i) = S(t_{i+1})e^{g\Delta_{i+1}} = qW e^{-\delta \sum_{j=1}^{i+1} \Delta_j} e^{g\Delta_{i+1}}, \ (74)
\]

respectively.

The firm value can be written as

\[
v_t = \frac{1}{\rho + \mu} \left( 1 - e^{-(\rho+\mu)\sum_{j=1}^{i_{\text{max}}+1} \Delta_j} \right) - \sum_{i=0}^{i_{\text{max}}} \frac{1}{\rho + \mu - g} e^{-(\rho+\mu-\delta)\sum_{j=1}^{i} \Delta_j + (\delta-g)\Delta_{i+1}} \left( 1 - e^{-(\rho+\mu-g)\Delta_{i+1}} \right)
\]

\[- \sum_{i=0}^{i_{\text{max}}} \kappa e^{-(\rho+\mu)\sum_{j=1}^{i} \Delta_j}. \ (75)
\]

As long as \( i = 0, 1, \cdots, i_{\text{max}} \), the firm value is positive even if the firm pays menu costs.
This suggests the following inequality condition:

\[ 0 < \int_{t_{i_{\text{max}}}}^{t_{i_{\text{max}}+1}} \frac{qe^{-\delta(t_{i_{\text{max}}+1}-t)\Delta_j(t_{i_{\text{max}}+1}-t')}}{qe^{-\delta(t_{i_{\text{max}}+1}-t'}}} e^{-(\rho+\mu)(t'-t)} dt' - \frac{1}{q} e^{\delta \sum_{j=1}^{i_{\text{max}}+1} \Delta_j - g \Delta_{i_{\text{max}}+1} (1 - e^{-(\rho+\mu)\Delta_{i_{\text{max}}+1}})} \]

which reduces to

\[ 0 < \frac{1}{\rho + \mu} (1 - e^{-(\rho+\mu)\Delta_{i_{\text{max}}+1}}) - \frac{1}{\rho + \mu - g} e^{\delta \sum_{j=1}^{i_{\text{max}}+1} \Delta_j - g \Delta_{i_{\text{max}}+1}} (1 - e^{-(\rho+\mu-g)\Delta_{i_{\text{max}}+1}}) - \kappa \]

If firms reset their prices more than \( t_{i_{\text{max}}} \) times, the right-hand side of the equation becomes negative.

When \( g = \delta \), the firm value is given by

\[ v_t = \int_t^{t+\tau^*} \frac{qe^{-\delta(t'-t)\Delta_j} - 1}{qe^{-\delta(t'-t)}} e^{-(\rho+\mu)(t'-t)} dt' = \left[ \frac{1}{\rho + \mu} e^{-(\rho+\mu)(t'-t)} + \frac{1}{q} \frac{1}{\rho + \mu - g} e^{-(\rho+\mu-g)(t'-t)} \right] t + \tau^* \]

\[ = \frac{1 - e^{-(\rho+\mu)\tau^*}}{\rho + \mu} - \frac{(1 - e^{-(\rho+\mu)\tau^*})/q}{\rho + \mu - g} \]  

(76)

### B.2 Firm Distribution

Denote time passed since firm entry by \( \tau \in [0, \sum_{j=1}^{i_{\text{max}}+1} \Delta_j] \). Then, the relative price \( \xi(\tau) \) evolves

\[ \xi(\tau) = \xi_i e^{g(\tau-t_i)} \]  

(77)

where \( t_i = t + \sum_{j=1}^{i} \Delta_j \), \( i = 0, 1, \ldots, i_{\text{max}} \), and

\[ \xi_i = \begin{cases} 
S(t_i) = qW e^{-\delta \sum_{j=1}^{i} \Delta_j} & \text{for } g > \delta \\
S(t_i) = qW e^{-\delta \sum_{j=1}^{i+1} \Delta_j} e^{g \Delta_{i+1}} & \text{for } g < \delta.
\end{cases} \]

After industry-leading firms stop production, that is, \( \tau > \sum_{j=1}^{i_{\text{max}}+1} \Delta_j \), firm rivals produce goods at the relative price of \( W \) until new products are invented.

We define the density function of \( \xi(\tau) \) by \( f(\xi(\tau)) \). Because firms exit at the rate of \( \mu \), this obeys the exponential probability density function,

\[ f(\xi(\tau)) = \mu e^{-\mu \tau}. \]  

(78)
B.3 Aggregate Variables

The aggregate price level at \( t = 0 \) is given by

\[
\log P = \int_0^1 \log[p_0(j, k = 0)] \, dj \\
= \sum_{i=0}^{i_{\text{max}}} \int_{t_i}^{t_{i+1}} \mu e^{-\mu \tau} \log[\xi_i e^{-g(\tau-(t_i-t))}] \, d\tau + \int_{\sum_{j=1}^{j_{\text{max}}+1} \Delta_j}^{\infty} \mu e^{-\mu \tau} \log W \, d\tau.
\]

When \( g > \), we have

\[
\log P = \sum_{i=0}^{i_{\text{max}}} \int_{\sum_{j=1}^{j_{i+1}} \Delta_j}^{\sum_{j=1}^{j_{i+1}} \Delta_j} \mu e^{-\mu \tau} \log q \log W e^{-g \sum_{j=1}^{j_{i+1}} \Delta_j} e^{-g(\tau-\sum_{j=1}^{j_{i+1}} \Delta_j)} \, d\tau + e^{-\mu \sum_{j=1}^{j_{\text{max}}+1} \Delta_j} \log W
\]

\[
= \left(1 - e^{-\mu \sum_{j=1}^{j_{\text{max}}+1} \Delta_j}\right) \log(qW) + \sum_{i=0}^{i_{\text{max}}} e^{-\mu \sum_{j=1}^{j_{i+1}} \Delta_j} (1 - e^{-\mu \Delta_i+1}) (g - \delta) \sum_{j=1}^{i+1} \Delta_j
\]

\[
- \frac{g}{\mu} \left\{1 - e^{-\mu \sum_{j=1}^{j_{\text{max}}+1} \Delta_j} \left(1 + \mu \sum_{j=1}^{j_{\text{max}}+1} \Delta_j\right)\right\} + e^{-\mu \sum_{j=1}^{j_{\text{max}}+1} \Delta_j} \log W. \tag{79}
\]

When \( g < \), we have

\[
\log P = \left(1 - e^{-\mu \sum_{j=1}^{j_{\text{max}}+1} \Delta_j}\right) \log(qW) + \sum_{i=0}^{i_{\text{max}}} e^{-\mu \sum_{j=1}^{j_{i+1}} \Delta_j} (1 - e^{-\mu \Delta_i+1}) (g - \delta) \sum_{j=1}^{i+1} \Delta_j
\]

\[
- \frac{g}{\mu} \left\{1 - e^{-\mu \sum_{j=1}^{j_{\text{max}}+1} \Delta_j} \left(1 + \mu \sum_{j=1}^{j_{\text{max}}+1} \Delta_j\right)\right\} + e^{-\mu \sum_{j=1}^{j_{\text{max}}+1} \Delta_j} \log W. \tag{80}
\]

When \( g = \), we have

\[
\log P = \log q - (1 - e^{-\mu \tau^{*}}) \frac{\delta}{\mu} + \log W, \tag{81}
\]

using \( \tau^{*} = \log q/\delta \).

We compute \( \pi \). When \( g > \), the inflation rate is given by

\[
\pi = \sum_{i=1}^{i_{\text{max}}} f(\xi_i) \cdot \log \left(\frac{S(t_i)}{s(t_i)}\right) + \int_{\sum_{j=1}^{j_{\text{max}}+1} \Delta_j}^{\infty} \mu e^{-\mu \tau} g d\tau - \eta \cdot \log (qP/\xi_0),
\]
which becomes

\[ \pi = \sum_{i=1}^{i_{\text{max}}} \mu e^{-\mu \sum_{j=1}^{i_{\text{max}}} \Delta_j} \log \left( \frac{qW e^{-\delta \sum_{j=1}^{i_{\text{max}}} \Delta_j}}{qW e^{-\delta \sum_{j=1}^{i_{\text{max}}} \Delta_j} e^{-g \Delta_j}} \right) + ge^{-\mu \sum_{j=1}^{i_{\text{max}}+1} \Delta_j} - \eta \log \left( \frac{P}{W} \right) \]

\[ = \sum_{i=1}^{i_{\text{max}}} \mu e^{-\mu \sum_{j=1}^{i_{\text{max}}} \Delta_j} (g - \delta) \Delta_i + ge^{-\mu \sum_{j=1}^{i_{\text{max}}+1} \Delta_j} \]

\[ - \mu \left( 1 - e^{-\mu \sum_{j=1}^{i_{\text{max}}+1} \Delta_j} \right) \log(qW) - \mu \sum_{i=0}^{i_{\text{max}}} e^{-\mu \sum_{j=1}^{i_{\text{max}}} \Delta_j} (1 - e^{-\mu \Delta_{i+1}}) (g - \delta) \sum_{j=1}^{i} \Delta_j \]

\[ + g \left\{ 1 - e^{-\mu \sum_{j=1}^{i_{\text{max}}+1} \Delta_j} \left( 1 + \mu \sum_{j=1}^{i_{\text{max}}+1} \Delta_j \right) \right\} - \mu e^{-\mu \sum_{j=1}^{i_{\text{max}}+1} \Delta_j} \log W \]

\[ + \mu \log W, \]

that is,

\[ a = \mu \left( 1 - e^{-\mu \sum_{j=1}^{i_{\text{max}}+1} \Delta_j} \right) \log q \]

\[ + \sum_{i=1}^{i_{\text{max}}} \mu e^{-\mu \sum_{j=1}^{i_{\text{max}}} \Delta_j} (g - \delta) \left( (1 - e^{-\mu \Delta_{i+1}}) \sum_{j=1}^{i} \Delta_j - \Delta_i \right) \]

\[ + \mu ge^{-\mu \sum_{j=1}^{i_{\text{max}}+1} \Delta_j} \sum_{j=1}^{i_{\text{max}}+1} \Delta_j. \]  \hspace{1cm} (82)

When \( g < \delta \), we have

\[ a = \mu \left( 1 - e^{-\mu \sum_{j=1}^{i_{\text{max}}+1} \Delta_j} \right) \log q \]

\[ + \sum_{i=1}^{i_{\text{max}}} \mu e^{-\mu \sum_{j=1}^{i_{\text{max}}} \Delta_j} (g - \delta) \left( (1 - e^{-\mu \Delta_{i+1}}) \sum_{j=1}^{i+1} \Delta_j - \Delta_{i+1} \right) \]

\[ + \mu ge^{-\mu \sum_{j=1}^{i_{\text{max}}+1} \Delta_j} \sum_{j=1}^{i_{\text{max}}+1} \Delta_j. \]  \hspace{1cm} (83)

When \( g = \delta \), we have

\[ a = \mu \log q + \delta q^{i/\mu}. \]  \hspace{1cm} (84)

Welfare at \( t = 0 \) is given by

\[ U_0 = \frac{a + \rho \log C}{\rho^2}. \]
From the resource constraint, $C$ is given by

$$C = \frac{1 - \kappa \sum_{i=1}^{i_{\text{max}}} f(t_i)) E}{P}$$

$$= \frac{1 - \kappa \sum_{i=1}^{i_{\text{max}}} e^{-\mu \sum_{j=1}^{j_i} \Delta_j}}{P} E. \quad (85)$$