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The elephant in the ground: managing oil and sovereign wealth

CAMA Working Paper 62/2014
October 2014

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JEL Classification

E21, F65, G11, G15, O13, Q32, Q33

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THE ELEPHANT IN THE GROUND:

MANAGING OIL AND SOVEREIGN WEALTH

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Oil exporters typically do not consider below-ground assets when allocating their sovereign wealth fund portfolios, and ignore above-ground assets when extracting oil. We present a unified framework for considering both. Subsoil oil should alter a fund's portfolio through additional leverage and hedging. First-best spending should be a share of total wealth, and any unhedged volatility must be managed by precautionary savings. If oil prices are pro-cyclical, oil should be extracted faster than the Hotelling rule to generate a risk premium on oil wealth. We then discuss how the management of Norway's fund can practically be improved with our analysis.

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First version: December 2013. This version: September 2014

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1. Introduction

Sovereign wealth funds hold some of the largest portfolios in the world, and many have been funded by selling below-ground assets such as oil, natural gas, copper and diamonds ("oil" for short). These funds can comprise a large part of commodity exporters' wealth. Azerbaijan's US$ 34 billion fund accounts for almost half its GDP, Qatar's US$ 170 billion fund accounts for almost two thirds of GDP, Saudi Arabia's US$ 740 billion funds are approximately four-fifths of GDP, Norway's US$ 840 billion fund is nearly one and a half times GDP, and the United Arab Emirates' US$ 1 trillion funds are over two and a half times its GDP (SWF Institute, 2013; IMF, 2013). Commodity sovereign wealth funds around the world hold over US$ 4 trillion in financial assets (SWF Institute, 2013).

The purpose of these funds is to smooth consumption of oil income: across generations because oil windfalls are temporary, and between periods because oil and asset prices are volatile. While such funds are often professionally managed and allocate their assets using modern portfolio theory, their investment strategies do not take due account of oil price volatility and subsoil reserves. Similarly, existing theories of optimal oil extraction do not take into account volatile financial markets. These are important questions for resource exporters, since commodity prices are notoriously volatile and below-ground assets can be worth more than the above-ground fund.

Our objective is therefore to answer four questions about how below-ground resources should influence above-ground portfolios, and vice-versa. First, how should above-ground assets be allocated given a volatile stock of below-ground assets? Second, how quickly should financial and oil wealth be consumed? Third, how does this change if financial markets are incomplete, so that oil shocks cannot be completely hedged in the portfolio? Finally, how should the optimal extraction rate of below-ground assets be affected by risky above-ground assets?

We will show that policy makers should allocate above-ground assets to accommodate below-ground oil stocks, consume a fixed share of total wealth, manage unhedged shocks with precautionary savings, and extract oil more quickly if marginal oil rents are positively correlated with asset markets.

The fund's asset allocation should accommodate subsoil oil with additional leverage and hedging demands, compared to the case without oil. The leverage demand involves holding more of each risky asset by borrowing the safe asset, or going "short". The amount of leverage depends on the ratio of oil to financial wealth, so is reversed as the fund matures and oil reserves are depleted. The hedging demand involves holding more (less) of financial assets that are negatively (positively) correlated with oil price shocks, after adjusting for the correlations between these assets. Oil price volatility can be fully offset by changing the weights of financial assets if markets are complete (i.e., the oil price is driven by the same shocks as asset markets).

If markets are complete then consumption should be a constant share of total wealth, if not then unhedged shocks should be managed with precautionary savings. Total wealth includes both oil and financial assets. Consuming a constant share of total wealth stabilizes both the mean and the variance of spending. On average total wealth will grow at a constant rate, as oil reserves are replaced by financial assets. The variance of total wealth will also grow at a constant rate, as the fund hedges oil price shocks as much as possible. The policy maker will thus only be exposed to residual volatility, which is managed by precautionary saving. If there is no asset price or oil price volatility, total wealth will be constant in accordance with the permanent income hypothesis and Hartwick's (1977) rule. If the oil price is not fully spanned by available financial assets, there will be more residual oil price volatility, more precautionary savings and growth rates will begin higher and vary over time.

Finally, oil should be extracted more quickly if marginal oil rents are positively correlated with the market. The standard Hotelling rule is to extract oil so that marginal oil rents rise at the riskless rate of interest.[1] Extracting faster generates an additional "risk premium" for bearing the exposure to oil price fluctuations. The premium comes from declining extraction costs, which are convex and fall faster than the rate of extraction. The size of the premium depends on oil's correlation with the rest of the market. If they are uncorrelated then all oil price shocks can be diversified away, no risk premium is needed and the optimal rate of extraction will be unaffected.

Our analysis combines three large and previously unrelated strands of literature. First, the allocation of financial assets is described by CAPM equations suitably modified for subsoil oil wealth. This extends the continuous-time problem of optimal consumption-saving and portfolio choice (Merton, 1990).[2] Second, consumption is described by a stochastic Euler equation.[3] This uses precautionary savings to deal with any residual volatility that cannot be managed by diversifying financial assets and varying the oil extraction rate.[4] Third, the optimal rate of oil extraction is described by a stochastic Hotelling rule modified for financial portfolios. Since oil wealth is invested in risky financial assets, it is also risky

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[1] This assumes certain oil prices and asset returns (Hotelling, 1931; Solow, 1974; Gaudet, 2007).

[2] This builds on classic portfolio theory (Tobin, 1958) and mean-variance theory to construct a diversified portfolio based on the co-movement between financial assets (Markowitz, 1952; 1959). As investors have equal information and markets are complete, they hold the market portfolio as used in the CAPM (Sharpe, 1964). Our extension is akin to those dealing with a non-tradable stream of income in the context of university endowments (Merton, 1993; Brown and Tiu, 2012), labor income including endogenous effort (Bodie et al., 1992; Wang et al., 2013), non-tradable and uninsurable income (Svensson and Werner, 1993; Koo, 1998) and non-financial stores of wealth such as housing (Flavin and Yamashita, 2002; Sinai and Souleles, 2005; Case et al., 2005).

[3] See Leland (1968), Sandmo (1970), Zeldes (1986), Kimball (1990), Carroll and Kimball (2008).

[4] This extends earlier work on precautionary saving in safe assets to cope with oil price volatility (Bems and de Carvalho Filho, 2011; van den Bremer and van der Ploeg, 2013).

above the ground. We show that oil price volatility makes extraction more rapid at first, as in earlier studies but with weaker assumptions on extraction costs.[5]

Our analytical results have bearing on funds such as Norway's Government Pension Fund Global (GPFG)[6]. At present the GPFG invests in a highly diversified equity portfolio, chooses the size of that portfolio based on its risk preferences, and spends a fixed 4% of the fund each year.[7] This closely matches the prescription from modern portfolio theory in the absence of oil (Merton, 1971). However, the GPFG's management mandate does not mention oil once (NBIM, 2013), leaving Norway highly exposed to a large and volatile stock of subsoil wealth: the "elephant in the ground".[8] Norway also restricts investments in some assets for social and political reasons, such as tobacco and defense firms, which motivates our consideration of investment restrictions. If Norway were to implement our first-best policy then welfare would be improved by as much as a 15% permanent increase in the fund's dividend.[9]

[5] We require marginal extraction costs to be positive and increasing in the amount extracted but, unlike Pindyck (1980, 1981), we do not require them to be convex which would create extractive prudence. Others treat extraction with stochastic oil prices, growth and capital, but abstract from above-ground financial assets (Gaudet and Khadr, 1991; Atewamba and Gaudet, 1992). Recent empirical evidence suggests that the Hotelling rule holds not at the intensive margin of production from a well, but at the extensive margin of number of wells drilled (Anderson et al., 2014).

[6] At US$840 billion the GPFG is the largest single fund in existence; it was established in 1990 to smooth expenditure financed from oil after a period of fiscal volatility in the 1970s and 1980s. Evaluating governance, accountability and transparency, structure and behavior, the GPFG ranked first on the first two criteria and second overall, behind Alaska's US$45 billion permanent fund (Truman, 2008). It also receives the highest rating on the Linaburg-Maduell Transparency Index (SWF Institute, 2013). It has often been referred to as a "model" for managing sovereign wealth fund assets (Chambers, et al., 2012; Larsen, 2005).

[7] The benchmark is 60% equities, tracking the FTSE Global All Cap Index; up to 5% real estate, tracking the Investment Property Databank's Global Property Benchmark; and up to 40% bonds, of which 70% government and 30% corporate bonds, both tracking Barclays indices.

[8] Norway has proven reserves of nearly 9 billion barrels of oil and 73 trillion cubic feet of natural gas (BP, 2014). At 2013 prices these are worth US$ 945 billion and US$ 777 billion, respectively.

[9] Empirical simulations using the correlation of oil prices with financial assets indicate that Norway's exposure to aggregate oil price volatility can be halved if oil wealth is hedged in the sovereign wealth fund (Gintschel and Scherer, 2008) and that the fund should invest less

In practice, Norway must consider short-sale constraints, transaction costs and time-varying asset price correlations.[10] We incorporate these in a second-best policy that varies only the equity/bond mix and the spending rule. Raising the share of equities from 45% to 60%, and reducing spending to 3% of fund assets would achieve 58% of the first-best welfare improvement. These results provide some theoretical footing for the recent debate in Norway on altering the asset allocation and spending rule of the GPFG.[11] Other important considerations such as pension liabilities and the general equilibrium effects of consumption are beyond the scope of this analysis.

Section 2 extends portfolio theory to allow for below-ground oil wealth with a predetermined path for oil production, dealing with cases where the oil price is completely spanned by asset markets and where it is not. Section 3 derives the optimal path for oil extraction and shows that extraction is brought forward if oil prices are positively correlated with asset markets. Section 4 applies this analysis to the current policy debate around Norway's GPFG. Section 5 concludes.

2. Portfolio allocation and spending for a given path of oil extraction

We first assume that investment restrictions do not exist and markets are complete, so that the properties of oil can be replicated by a bundle of risky financial assets. This gives closed-form solutions for how oil should affect the portfolio allocation and the spending rule of the fund. We then introduce investment restrictions and show that any unhedged risk should be managed by additional precautionary savings.

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aggressively in risky assets as it ages (Scherer, 2009; Balding and Yao, 2011). These studies focus on asset allocation but abstract from optimal consumption-saving decisions or oil extraction.

[10] As noted in a recent report to the Norwegian *Storting* (Parliament) by Norway's Ministry of Finance (Ministry of Finance, 2014a) after a preliminary version of this paper was circulated.

[11] There have been calls to remove oil and gas stocks from the GPFG portfolio (Milne, 2014), and reduce spending to below 3% of fund assets (Olsen, 2012).

Suppose the policy maker chooses consumption C and assets weights w_i, i = 1,.., n, to maximize the expected present value of utility with discount rate $\rho > 0$:

$$J(F,P_O,t)=\max_{C,w_i} E_t\left[\int_t^\infty U\left(C(s)\right)e^{-\rho(s-t)}ds\right], \tag{1}$$

subject to the budget constraint:

$$dF=\sum_{i=1}^{m} w_i(\alpha_i - r)Fdt+(rF+P_O O-C)dt+\sum_{i=1}^{m} w_i F\sigma_i dZ_i, \tag{2}$$

where the value function $J(F,P_O,t)$ depends on the size of the fund F, the oil price P_O, and time t. The quantity of oil O is predetermined and declines exponentially at rate κ, $O(t)=O(0)e^{-\kappa t}$.[12] We abstract from extraction costs. The fund consists of m risky assets, i = 1,.., m, with drift α_i and volatility σ_i and one safe asset, $i = m+1$, with return r and volatility σ_{m+1} =0.[13] The total number of risky and safe assets is $n \equiv m +1$. The fund holds N_i shares of assets, i = 1,.., n, each with price P_i, so that $F=\sum_{i=1}^{n} P_i N_i$. The share of each asset in the fund is defined as $w_i \equiv P_i N_i / F$, so that $F=\sum_{i=1}^{n} w_i F$. The price of each risky asset follows a Geometric Brownian Motion:

$$dP_i=\alpha_i P_i dt+\sigma_i P_i dZ_i, \quad i=1,..,m, \tag{3}$$

where dZ_i is a Wiener process with cov($dZ_i\ dZ_j$) = $[\rho_{ij}]$ for i = 1,.., m. The returns of risky assets have covariance matrix $\Sigma = [\sigma_{ij}] = [\rho_{ij}\ \sigma_i\ \sigma_j]$. We abstract from mean reversion in asset prices and assume constant coefficients in (3).

[12] The results can readily be extended for the case of a constant windfall of finite duration.
[13] Our analysis is partial equilibrium so total wealth comprises only fund and oil wealth. This supposes that the government runs a non-fund balanced budget, so we can abstract from taxes and non-fund income.

The weight of the safe asset in the fund, $w_n = 1 - \sum_{i=1}^{m} w_i$, can be positive or negative corresponding to the weight of the risky portfolio being smaller or larger than one, and to a long position ($w_n > 0$) or short position ($w_n < 0$) in the safe asset. Total holdings of risky assets is called the "portfolio", $(1 - w_n)F = \sum_{i=1}^{m} w_i F$, whose share is denoted by $w \equiv 1 - w_n$.

Preferences exhibit constant relative risk aversion, $U(C) = C^{1-1/\theta} / (1 - 1/\theta), \theta \neq 1$ and $U(C) = \ln(C), \theta = 1$, where θ is the coefficient of intertemporal substitution, $1/\theta$ the coefficient of relative risk aversion, and $1 + 1/\theta$ the coefficient of relative prudence. These are a form of hyperbolic absolute risk aversion preferences, so permit an analytical solution to the asset allocation problem (Merton, 1971).[14]

The country is a small oil exporter that does not affect the oil price. The world oil price follows a Geometric Brownian Motion:

$$dP_O = \alpha_O P_O dt + \sigma_O P_O dZ_O, \tag{4}$$

where the drift in the oil price is not too large, $\alpha_O < r$.[15] The effect of oil reserves on the fund depends on how prices of oil and other assets co-move. Let all risky assets be driven by a common set of shocks (e.g., to demand, supply, technology or the weather), $du \sim$ i.i.d. $N(0, dt)$. The correlation of each asset depends on how it is affected by these shocks, $dZ = \Lambda\, du$, where $\Lambda = [\lambda_{ij}]$ is an invertible $m \times m$ matrix and $dZ = [dZ_1, .., dZ_m]'$ is the vector of Wiener processes driving the returns on risky assets. The Wiener process driving oil returns is expressed as:

$$dZ_O = \lambda_{Oh} du_h + \Lambda_O du = \lambda_{Oh} du_h + M dZ, \tag{5}$$

[14] Epstein-Zin preferences can be used to separate risk aversion and intertemporal substitution (Epstein and Zin, 1989), as has been done in consumption-saving problems (Attanasio and Weber, 1989; Wang et al., 2013). Note that $1/\theta$ also measures intergenerational inequality aversion.

[15] This is a sufficient condition for the present discounted value of oil to be finite, and is consistent with empirical estimates (e.g., van den Bremer and van der Ploeg, 2013).

where $M = \Lambda_O \Lambda^{-1}$. If the fund has unrestricted access to all assets and the instantaneous return on oil can be perfectly replicated ("spanned") by a bundle of traded securities, we get a closed-form solution. The unhedged component of oil prices is then zero, so $\lambda_{Oh} = 0$. Λ_O = [λ_{O1}, .., λ_{Om}] is a vector determining how the oil price responds to the vector of underlying shocks, *du*, and $\mathrm{cov}(dZ_O, dZ) = \Sigma M$. If there is an unspanned unhedged component of the oil price, it has weight $\lambda_{Oh} = \sqrt{1 - \sum_{i=1}^{m-1} \lambda_{Oi}} \neq 0$, Λ_O = [λ_{O1}, .., λ_{Om-1}] and du_h is a residual oil-specific shock, uncorrelated with the asset market shocks, *du* (see appendix A.1).

2.1. Oil can be replicated by a bundle of risky and safe assets

Oil wealth might not be easily tradable, but can be treated as tradable if we have complete markets and its return can be replicated by a synthetic bundle of traded financial assets. Oil price shocks can then be fully offset (i.e., hedged) within the fund as they are fully spanned by the set of financial assets ($\lambda_{Oh} = 0$). We first derive the value of capitalized oil revenues and the dynamics of total wealth.

Proposition 1: *If oil returns are completely spanned by financial assets, then oil wealth can be replicated by a bundle of risky and safe assets, i* = 1, .., *n. The value of this bundle is the capitalized value of oil revenues:*

$$V(P_O, t) = P_O(t) O(t) / \psi, \quad \psi \equiv r + \kappa - \alpha_O + \sum_{i=1}^{m} \beta_i (\alpha_i - r), \tag{6}$$

where $\beta_i = \frac{\sigma_O}{\sigma_i} M_i$ *and* $M_i \equiv [\Lambda_O \Lambda^{-1}]_i$. *Total wealth consists of fund assets and subsoil oil assets,* $W = F + V$, *and behaves according to:*

$$dW = \sum_{i=1}^{m} \overline{w}_i W(\alpha_i - r)dt + (rW - C)dt + \sum_{i=1}^{m} \sigma_i \overline{w}_i W dZ_i, \tag{7}$$

where $\bar{w}_i \equiv \frac{w_i F + \beta_i V}{F + V}$ *for* $i = 1, \ldots, m$.

Proof: See appendix A.1.

The oil price can be replicated if all shocks hitting it also affect at least one risky financial asset.[16] Replication is achieved by linearly combining small exposures to many financial assets in a "replicating bundle", since they are all driven by correlated and normally distributed processes in (5). These exposures, β_i, depend on the correlation of each risky asset with the oil price, and its uniqueness amongst other financial assets. They are chosen so that the bundle matches the variance of oil revenues, and the safe asset in the bundle is chosen so that it matches the drift. Since oil wealth and the bundle then behave identically, the price of the replicating bundle is the value of oil wealth (6). Oil wealth is current oil revenues divided by the effective discount rate ψ, where the latter is the safe return r plus the rate of decline of oil production κ minus the drift in the oil price α_O and the adjustment to compensate risk-averse investors for bearing oil price risk.[17] Oil wealth only reacts to the current oil price because we assume that oil price shocks are permanent.

Any exposure to oil price risk, dZ_O, can thus be artificially constructed by combining oil revenue with an amount of the replicating bundle. If claims to oil can be sold off, the proceeds could be invested in a diversified portfolio and the problem reduces to that in Merton (1990). But we suppose that claims to oil cannot be packaged and sold off, because of political or practical constraints.[18]

[16] Oil prices depend in general equilibrium on more fundamental shocks (Bodenstein et al., 2012).
[17] The value of an uncertain stream of income follows from discounting at the risk-free rate if the probability space is adjusted to a risk-neutral measure using a theorem due to Girsanov (1960).
[18] Payment for temporary exploitation of a well is often made as part of an auction process, but risk-averse extraction firms are usually unwilling to take on all price and production risk.

Using the replicating bundle the problem can be simplified into choosing the net weight of each risky asset, $\bar{w}_i$ for i = 1,.., m, in total wealth, $W = F + V$.

Proposition 2: *If the oil price is spanned by the market, the net weight of each risky asset in total above-ground and below-ground wealth is constant:*

$$\bar{w}_i = \delta_i \bar{w}, \quad i = 1\ldots m, \quad \delta_i \equiv \frac{1}{v}\sum_{j=1}^{m} v_{ij}(\alpha_j - r), \tag{8}$$

and the net weight of all risky assets in above- and below-ground wealth is:

$$\bar{w} \equiv \sum_{i=1}^{m} \bar{w}_i = \theta v, \quad v \equiv \sum_{i=1}^{m}\sum_{j=1}^{m} v_{ij}(\alpha_j - r), \tag{9}$$

where $v_{ij} \equiv [\Sigma^{-1}]_{ij}$*, and the share of safe assets in the total portfolio is* $1 - \bar{w}$*. The weight of each risky asset in the fund is given by:*

$$w_i = \bar{w}_i + \underbrace{\left(\bar{w}_i \frac{V}{F}\right)}_{\text{leverage demand}} + \underbrace{\left(-\beta_i \frac{V}{F}\right)}_{\text{hedging demand}}, \quad \beta_i = \tfrac{\sigma_O}{\sigma_i} M_i, \quad i = 1,..,m. \tag{10}$$

Proof: See appendix A.2.

Sovereign wealth funds should thus be structured so that net exposure to each asset in total above- and below-ground wealth is constant. In line with the Tobin-Markowitz theorem, the problem is separated into two steps: construct the optimal portfolio of risky assets (8) and choose how much of total wealth to allocate to that portfolio (9). To achieve this, the fund needs to allow for subsoil wealth with offsetting leverage and hedging demands for each asset as shown in (10).

The optimal allocation of risky assets in total (above- and below-ground) wealth (8) is independent of preferences and the level of wealth, but depends on the drift and covariance of asset returns. Optimal diversification implies that an asset has a higher weight if it is less correlated with other assets (lower v_{ij}).

The optimal size of the portfolio of risky assets in total wealth (9) is constant. It is proportional to the overall risk-adjusted return of the portfolio v and the willingness to take risk θ (the inverse of the coefficient of relative risk aversion). If there is only one risky asset, (9) reduces to the Sharpe ratio, $\bar{w} = \theta(\alpha_1 - r)/\sigma_1^2$, so the portfolio is proportional to the excess return of the risky asset over the safe asset and willingness to take risk and inversely proportional to the variance of the return on the risky asset. With various risky assets the overall risk-adjusted return is lower if the risky assets are positively correlated with each other, so that there is less scope for fluctuations to offset each other and to hedge oil.

To ensure that net exposure to each financial asset is a constant share of total wealth (8), subsoil oil creates offsetting leverage and hedging demands for each risky asset in the fund (10).[19] Both are proportional to the ratio of oil wealth V to fund wealth F, so are higher if there is still a lot of oil in the ground. As oil reserves are depleted, both types of demand diminish and the asset allocation of the fund approaches its non-oil level, $\bar{w}_i$.[20]

Leverage demand involves holding more of each risky asset in the fund. Each risky asset comprises a fixed share of total wealth (8). Without oil this means holding $\bar{w}_i F$ shares of each asset. Now imagine that oil wealth is the same size as the fund and completely uncorrelated ($\beta_i = 0, \forall i$), so that total wealth is double fund wealth, $W = F + V = 2F$. The fund would then need to hold twice as much of each risky asset, by holding less of (or borrowing more of) the riskless asset. With one risky asset the leverage demand is a version of the Sharpe ratio,

[19] Merton (1990, chapter 21) refers to these components as “wealth” and “substitution” effects.

[20] This assumes that withdrawals from the fund are not so rapacious (i.e., ρ is not too high, cf. (8)) that fund assets fall quicker than oil is extracted and V/F rises over time.

$\theta(\alpha_1 - r)\sigma_1^{-2}(V / F)$.[21] As oil is extracted the leverage demand goes to zero, so the fund should reallocate from risky to safe assets.

Hedging demand offsets exposure to oil price risk. If markets are complete and oil is correlated with only one asset, $dZ_O = \rho_{Ok} dZ_k$, hedging demand is the product of the oil-asset beta and leverage ratio[22], $-\rho_{Ok}\sigma_O / \sigma_k (V / F)$. If oil price risk is purely idiosyncratic ($\rho_{Ok} = 0$), hedging demand is zero. If it is positively correlated with the financial asset ($\rho_{Ok} = 1$), hedging demand is negative. In contrast, if oil and the financial asset are negatively correlated ($\rho_{Ok} = -1$), the fund should hold even more of the risky asset to hedge oil price risk. Again, as oil is extracted and the exposure to price risk falls, the hedging demand goes to zero. Equation (10) generalizes this to multiple risky financial assets.

If all financial asset returns are independent (Λ is diagonal), oil should be hedged by investing more in assets that are negatively correlated (e.g., assets that use oil as an input such as manufacturing and consumer goods industries) and less in assets that are positively correlated (e.g., oil and gas stocks or substitutes like renewable energy). This is especially true if there is still a lot of oil in the ground, so oil price exposure is high. In that case, one should also leverage up all the demands for risky assets that would prevail in the absence of oil.

If all financial asset returns are correlated, the hedging of oil must also take into account the covariance of each risky asset. It is possible that the fund should invest less in assets that are negatively correlated with oil. For example, consider a shock du_G which affects oil and asset A but not others, λ_{OG}, $\lambda_{AG} > 0$ and $\lambda_{iG} = 0$, for all $i \neq A$. All other shocks du_j affect oil and asset A in opposite ways, $\lambda_{Oj} > 0$ and $\lambda_{Aj} < 0$, for all $j = 1,.., m$. In this case, it is possible that oil and asset A are

[21] Mean-variance analysis gives a similar expression (Gintschel and Scherer, 2008; Scherer, 2009).
[22] The slope coefficient of a regression of demeaned asset returns versus demeaned oil returns.

negatively correlated, $\sum_{j=1}^{m} \lambda_{Oj}\lambda_{Aj} < 0$, but the fund should nevertheless invest *less* in asset *A* to offset the exposure to shock *G*. The allocation of all other assets will have to adjust to hedge the effects of the remaining shocks, du_j for $j \neq g$.

In practice one could implement this with a mix of two benchmark indices: the "market index", $\bar{w}_i$, and an "oil hedging index", β_i, constructed to replicate movements in the oil price with traded assets. Over time the mix would shift from the second to the first index as oil is extracted from the ground, as in (10).

Oil wealth also affects precautionary saving and consumption from the fund.

Proposition 3: *If the instantaneous return on oil is spanned by the market, the stochastic Euler equation for the expected growth in the consumption rate is:*

$$\frac{\frac{1}{dt}E_t\left[dC\right]}{C} = \theta(r-\rho) + \frac{1}{2}(1+1/\theta)\sigma_W{}^2\bar{w}^2. \tag{11}$$

The optimal consumption rate is a fixed share of total financial and oil wealth:

$$C = r_W^* W, \quad r_W^* \equiv r + \theta(\rho - r) + \frac{1}{2}\theta\left(1-\theta\right)\left(\frac{\alpha_W - r}{\sigma_W}\right)^2, \tag{12}$$

where the drift and the volatility of total wealth are $\alpha_W \equiv \sum_{i=1}^{m}\delta_i\alpha_i$ *and* $\sigma_W \equiv \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{m}\delta_i\delta_j\sigma_{ij}}$. *Total wealth follows the Geometric Brownian Motion:*

$$dW = \alpha_W^* W dt + \sigma_W \bar{w} W dZ_W, \quad \alpha_W^* \equiv (\alpha_W - r)\bar{w} + r - r_W^*, \tag{13}$$

where aggregate volatility is $dZ_W = \frac{1}{\sigma_W}\sum_{i=1}^{m}\delta_i\sigma_i dZ_i$. *This implies that:*

$$W(t) = W(0)\exp\left[\left(\alpha_W^* - \frac{1}{2}\sigma_W^2\bar{w}^2\right)t + \sigma_W\bar{w}Z_W(t)\right], \quad t \geq 0. \tag{14}$$

The value function takes the form:

$$J(W,t) = \tfrac{\theta}{\theta-1}\exp(-\rho t)\left[\theta\rho - (\theta-1)\eta\right]^{-1/\theta} W^{(\theta-1)/\theta}, \tag{15}$$

where $\eta = r + \theta(\alpha_W - r)^2 / 2\sigma_W^2$.

Proof: See appendix A.3.

Aggregate risk is managed by precautionary saving. This creates a buffer stock of assets by depressing consumption today, as seen from upward tilt of the expected consumption path in the final term of (11). The degree of tilt increases with the coefficient of relative prudence (1 + 1/θ), the riskiness of the portfolio σ_W^2, and the size of the risky portfolio in total above- and below-ground wealth, $\bar{w}$. The buffer is not used to temporarily support consumption when asset prices are low, since asset price shocks are random walks and thus persistent. Its sole function is to compensate future periods for bearing additional risk.

The degree of precautionary saving increases with aggregate risk, since individual risky assets hedge both one another and the oil price. If there is perfect positive or negative correlation between the risky assets, the portfolio can be constructed so that all shocks offset each other and $dZ = 0$ and there is no need for precautionary saving. However, this would not be optimal as the policymaker is willing to accept some risk for a higher return. If markets are complete and the fund is constructed properly, consumption should not be directly affected by oil price shocks, only indirectly through their effect on total wealth (11).

If there is both below and above-ground wealth, (12) indicates that it is optimal to consume a fixed proportion of their sum. The marginal propensity to consume is affected by a higher return on the safe asset through the intertemporal substitution effect (negative as future consumption has become cheaper) and the income effect (positive as lifetime wealth has gone up). The former dominates the latter if the elasticity of intertemporal substitution, θ, exceeds one. It can be seen from (12) that the marginal propensity to consume, r_W^*, then decreases with the return on the safe asset, r, and the average excess return on risky assets, α_W - r; and increases with relative risk aversion, $1/\theta$, and fund volatility, σ_W. The proportion of total wealth consumed each period r_W^* should be less than its expected returns $r_e = \bar{w}\alpha_W + (1-\bar{w})r$, which implies that both consumption and wealth steadily rise over time. The amount depends on prudence, as $r_W^* - r_e = -(1/2)(1+1/\theta)\,\bar{w}^2\sigma_W^2$ where $1+1/\theta$ is the coefficient of relative prudence and we have set $r = \rho$. This is consistent with precautionary savings building up a buffer of assets against future risk (Kimball, 1990), and absolute risk aversion, θ/C, falling as consumption rises.

Without oil or asset price uncertainty ($\alpha_W = r$) and $r = \rho$, we have the Hartwick rule instead of (12). This states that any depletion of below-ground oil wealth must be exactly compensated for by an equal build-up of above-ground financial wealth, so that total wealth does not change over time and consumption is fully smoothed, $W(t) = W(0)$ and $C(t) = rW(0), t \geq 0$ (Hartwick, 1977). With uncertain oil and asset prices and $r = \rho$, we see from (13) how total above- and below-ground wealth evolves over time. It rises due to the premium earned on risky assets, $\alpha_W > r$. It falls (rises) if the intertemporal substitution effect is dominated by the income effect in consumption,[23] with the extent depending on the risk/return tradeoff of total wealth, $-\theta(1-\theta)\left((\alpha_W - r)/\sigma_W\right)^2/2$.

The case without oil resembles the current practice of Norway's GPFG. First, the optimal risky portfolio combines all financial assets based on their risk, hedging potential and return properties, as in (8) if $W = F$. If all investors in the market

[23] That is, if the elasticity of intertemporal substitution θ is less (greater) than unity

have the same information about future asset prices, this is the "market portfolio" (Sharpe, 1964) which is consistent with Norway's use of the FTSE Global All Cap Index as the equity benchmark (at around 7,400 individual stocks, it is a close approximation of the market). Second, the size of the risky portfolio depends on preferences and the overall risk and return of the market, as in (9) if $W = F$. The Ministry of Finance dictates the mix between "risky" equities and "safe" bonds. Originally the mix was 40% equity and 60% bonds but in 2007 it was decided that the fund would accept more risk for a higher return, so the mix was gradually changed to about 60% equity and 40% bonds by 2009 without changing the benchmarks (as specified by the theory). Third, in line with Merton (1990), a fixed share of the fund must be consumed each period, as in (12) if $W=F$. Under Norway's *handlingsregelen* the GPFG releases 4% of accumulated assets for the general budget each year. Of course, Norway has substantial oil reserves and thus these policies cause consumption to vary too much over time. Instead, the asset allocation and spending rule should be expressed in terms of *total* below- and above-ground wealth.

2.2. Oil cannot be replicated by a bundle of traded securities

Due to investment restrictions or incomplete markets it might be impossible to perfectly replicate the oil price with a bundle of traded securities. Now let us suppose that the fund cannot invest in a particular asset, so $\lambda_{Oh} \neq 0$ in (5) and the oil price is not spanned by the (investable) market. In that case, there must be more precautionary saving to cope with residual volatility.[24]

Proposition 4: *If the instantaneous return on the oil price is not spanned by the market, the stochastic Euler equation can be approximated by:*

[24] Earlier work abstracted from risky financial assets and focused at oil price volatility only (an extreme case of incomplete markets) to show that precautionary buffers are needed to cushion against adverse oil price shocks (van den Bremer and van der Ploeg, 2013). Here we allow for risky assets too, but remain within the realm of incomplete markets.

$$\frac{\frac{1}{dt}E_t\left[dC\right]}{C} = \theta(r-\rho) + \frac{1}{2}(1+1/\theta)\left[\sigma_W^2\bar{w}^2 + \lambda_{Oh}^2\sigma_O^2\left(\frac{V}{W}\right)^2\right], \quad (16)$$

where $\bar{w}$ *is defined in proposition 2 and* σ_W *is defined in proposition 3. Total wealth evolves according to:*

$$\begin{aligned} dW = &\left(\sum_{i=1}^{m-1}\bar{w}_i W(\alpha_i - r) + \beta_h(\alpha_h - r) + rW - C\right)dt \\ &+\sigma_M\sum_{i=1}^{m-1}\bar{w}_i W dZ_i + \sigma_O\lambda_{Oh}V du_O \end{aligned} \quad (17)$$

Proof: See appendix A.3.

This states that investment restrictions have both a precautionary and a wealth effect on consumption. Asset weights adjust to find the closest replicating bundle leaving only uncorrelated residual risk (as discussed in appendix A.1.).

The precautionary effect describes the additional savings needed because some oil price risk remains unhedged, as in (16). The first term on the right-hand side is the usual deterministic slope of optimal consumption. The second term captures precautionary saving and is therefore proportional to the coefficient of relative prudence, *CRP* = (1 + 1/θ). The term $\sigma_W^2\bar{w}^2$ inside the square brackets arises from the precautionary saving that is needed under complete markets where all oil price volatility can be fully diversified. It is proportional to the variance of the portfolio of risky assets and the share of risky assets in the fund squared. The other term inside the square brackets is $\lambda_{Oh}^2\sigma_O^2\left(V/W\right)^2$ and arises from the precautionary saving that is required because not all oil price volatility can be fully hedged. Less spanning of the oil price (a higher λ_{Oh}) implies that more precautionary saving is required, especially if oil wealth is volatile and comprises a large share of total wealth. Note that this effect diminishes as oil is extracted.

The wealth effect describes the change in the expected return on total wealth from not investing in a particular asset; see (17). If an asset cannot be held by the fund (cf. asset *h* in (17)), there will still be some exposure to it embodied in the oil price. In the complete markets analysis in section 2.1 this exposure was offset inside the fund, so that the net exposure was a constant share of total wealth. In the incomplete markets case this net exposure cannot be fully offset and will earn a rate of return, changing the expected return on total wealth. Its importance will diminish as oil is extracted.

Some stylized illustrations of the dynamic properties of our oil-CAPM model are presented in online appendix B. We illustrate how the leverage and hedging demands are reversed over time, and how consumption is stabilized as a fraction of total wealth. We also show how investment restrictions alter the asset allocation, and require more precautionary savings.

3. Portfolio allocation and spending with endogenous oil extraction

According to the Hotelling rule, the workhorse of resource economics, the return on keeping oil in situ (the expected capital gains) must equal the return of extracting oil, selling it and getting a return on it (the return on the safe asset) (Hotelling, 1931). This rule dictates the optimal speed of extracting oil from the earth. We show here how to modify this rule if oil and financial asset prices are volatile. Oil extraction should initially be faster than the Hotelling rule if marginal oil rents are positively correlated with the asset market, to generate a higher rate of return on subsoil oil as compensation for the risk of holding it. If the oil price jumps so too should extraction, to make the most of higher prices.[25] Without investment restrictions oil wealth can be hedged by continuously reallocating the

[25] This is because of a change in an arbitrage condition. We assume that oil extraction can be adjusted instantaneously. In practice oil extraction is less flexible than asset portfolios, though we abstract from this.

fund to ensure a constant net exposure to oil. Extraction should not affect consumption directly, only indirectly via the present value of subsoil wealth.

3.1. Optimal rates of oil extraction

Since the data suggest that the oil price is positively correlated with financial assets, we proceed under this assumption. Without loss of generality we also assume that the oil price can be perfectly hedged with a single financial asset *k*, $dZ_O = dZ_k$. The policy maker chooses the consumption rate *C*, the rate of oil extraction *O*, and asset weights w_i, i =1, .., *m* to maximize expected welfare:

$$J(F, P_O, S, t) = \max_{C, w_i, O} E_t \left[\int_s^\infty U\left(C(s)\right) e^{-\rho(s-t)} ds \right], \tag{18}$$

subject to the budget constraint:

$$dF = \sum_{i=1}^{m} w_i (\alpha_i - r) F dt + \left[rF + \Omega(P_O, O) - C \right] dt + \sum_{i=1}^{m} w_i F \sigma_i dZ_i, \tag{19}$$

the Geometric Brownian Motion processes for asset prices (3) and oil prices (4), and the reserve depletion equation:

$$\frac{dS}{dt} = -O(t), \tag{20}$$

where oil rents are revenues less extraction costs, $\Omega(P_O, O) \equiv P_O O - G(O),$ and total extraction costs are increasing in the extraction rate ($G'(\mathrm{O}) > 0$) and convex to ensure a solution ($G''(O) > 0$) (cf., Pindyck, 1984). From the depletion equation (20) cumulative oil extraction cannot exceed initial reserves, $\int_0^\infty O(t) dt \le S_0$.[26]

[26] In practice, oil fields evolve stochastically as new fields are discovered and existing fields becomes more or less economical (e.g., Pindyck, 1978). Extraction costs might be better captured by high upfront investment and small marginal costs. Reserves are also endogenous to exploration effort, but we abstract from these complications here.

Proposition 5: *The optimal path for the expected rate of oil extraction satisfies:*

$$\tfrac{1}{dt}E\left[d\Omega_O\right] = r\Omega_O - \frac{\tfrac{1}{dt}E\left[dJ_F d\Omega_O\right]}{J_F(F,P_O,S,t)}. \tag{21}$$

With quadratic extraction costs, $G(O) = \gamma O^2/2, \gamma > 0,$ *the path for the rate of oil extraction* $O = O(F, P_O, S, t)$ *satisfies*

$$d\Omega_O = r\Omega_O dt - \gamma O_F \sum_{i=1}^{m}\left[(\alpha_i - r)dt + \sigma_i dZ_i\right]w_i F + (1-\gamma O_P)\left[(\alpha_k - r)dt + \sigma_k dZ_O\right]P_O \frac{\sigma_O}{\sigma_k}. \tag{22}$$

Using the partials O_F *and* O_P *from the deterministic solution, the stochastic path for oil extraction can be approximated by (for* $\alpha_O = 0$*):*

$$dO \approx \left(-\tfrac{1}{\gamma}\left(r + \tfrac{\sigma_O}{\sigma_k}(\alpha_k - r)\right)P_O + \left(r + \tfrac{1}{2}\tfrac{\sigma_O}{\sigma_k}(\alpha_k - r)\right)O\right)dt + \tfrac{1}{2}O\sigma_O dZ_O. \tag{23}$$

Proof: See appendix A.4.

The familiar Hotelling rule is captured by the first term on the right-hand side of (21): the expected rate of change in marginal oil rents equals the return on safe assets. Provided $r - \alpha_O > 0$, the rate of oil extraction declines over time.

The stochastic Hotelling rule adds the second term on the right-hand side of (21): the expected rate of change of marginal oil rents must exceed the return on safe assets, if oil and financial asset returns co-move positively. In this case high oil prices drive high marginal oil rents, which are associated with high fund values, *F*, and low marginal utility from an extra dollar in the fund ($\tfrac{1}{dt}E\left[dJ_F d\Omega_O\right] < 0$). As is evident from (23), the level of fund assets does not come into consideration. The higher return compensates for the risk of holding oil in the ground (equal to $-\tfrac{1}{dt}E_t[dJ_F d\Omega_O]/(J_F\,\Omega_O)$). If oil and asset markets are perfectly uncorrelated ($\tfrac{1}{dt}E\left[dJ_F d\Omega_O\right] = 0$), all oil price risk can be diversified and no risk premium is needed. The more correlated oil and asset markets are, the less oil price shocks can be diversified and the higher the risk premium needs to be.

The stochastic Hotelling rule involves extracting oil more quickly than the original deterministic version at first, then reducing the rate of oil extraction (see figure B6 in online appendix B). As the rate of extraction drops, extraction costs fall non-linearly which boosts the rate of return on marginal oil rents (i.e. $\Omega(P_O, O) \equiv P_O O - G(O)$, where $G'(O) > 0, G''(O) > 0$).

Turning to the stochastic properties of oil extraction, (23) indicates that it should be positively correlated with the oil price. A sudden jump in the oil price requires a jump in the extraction rate to make the most of it. The reason is that increasing the extraction rate increases marginal extraction costs (as $G''(O) > 0$), which limits the jump in marginal rents ($\Omega_O(P_O, O) = P_O - G'(O)$). Oil price shocks affect the rate of extraction most when reserves (and in turn *O*) are highest, since this is when the majority of oil remains exposed to volatile prices. As the date of exhaustion approaches, the rate of oil extraction gets closer to what it would be without volatile oil and asset prices. Note that the size of the sovereign wealth fund does not matter, only the properties of the assets in the background.

Our finding that stochastic oil prices increase the oil extraction rate is consistent with earlier studies, but uses a different mechanism. Earlier work ignored financial assets and relied on "extractive prudence" driven by sufficiently convex marginal extraction costs, $G'''(O) > 0$ (Pindyck, 1981).[27] This means it is better to extract oil quickly because once it is above ground and sold it is no longer exposed to risk. Proposition 5 rules out this type of prudence, since it considers quadratic extraction costs ($G'''(O) = 0$). However, in our framework oil rents are

[27] Aggressive oil extraction also occurs if one has convex marginal utility arising from market power (van der Ploeg, 2010).

still exposed to risk above the ground because they must be invested. Hence, oil should be treated as just another part of the total portfolio. The effect of risk on extraction is driven by "extractive risk aversion" ($G''(O)$) rather than extractive prudence ($G'''(O)$) and so requires less onerous restrictions on extraction costs.

3.2. Sovereign wealth funds with endogenous rates of oil extraction

Here we show that with no investment restrictions oil rents can be fully hedged by the fund, regardless of the path of oil extraction. This involves continuously adjusting the asset allocation so that the net exposure to risk remains a constant share of total above- and below-ground wealth. We also establish that the oil extraction path should not affect consumption directly, only through its effect on the expected present value of oil rents.

Proposition 6: *With complete markets, continuous trading and the rate of oil extraction chosen optimally, oil wealth can be replicated with a bundle comprising the perfectly correlated asset k and the safe asset n, and the value of this bundle evolves according to:*

$$dV(t) + \Omega(t)dt = \left[rV(t) + (\alpha_k - r)\omega_k(O,t)V(t)\right]dt + \omega_k(O,t)V(t)\sigma_k dZ_k(t), \quad (24)$$

where $\omega_k(O,t) = N_k P_k / V$ *is the continuously adjusted share of asset k in the replicating bundle. Total fund and oil wealth evolves according to:*

$$dW = \sum_{i=1}^{m}(\alpha_i - r)\overline{w}_i W + (rW - C)dt + \sum_{i=1}^{m}\sigma_i \overline{w}_i W dZ_i, \quad (25)$$

where

$$\overline{w}_i = w_i\left(F(t)/W(t)\right)\, i \neq k, \quad \overline{w}_k = w_k(t)\left(F(t)/W(t)\right) + \omega_k(O,t)\left(V(t)/W(t)\right). \quad (26)$$

Proof: See appendix A.5.

Oil rents no longer follow the Geometric Brownian Motion as in section 2, but are driven by the drift $\mu_\Omega(P_O,S,t)dt$ and volatility, $\sigma_\Omega(P_O,S,t)dZ_O$. These coefficients depend on the states P_O and S and the optimally chosen rate of oil extraction, which also depends on those states:

$$d\Omega = \mu_\Omega(P_O,S,t)dt + \sigma_\Omega(P_O,S,t)dZ_O. \tag{27}$$

The drift and volatility of oil rents can be replicated by continuously reallocating the bundle of the perfectly correlated risky asset and the safe asset. One must continuously adjust the amount of asset *k* in the bundle so that the instantaneous change in the value of oil rents, $\sigma_\Omega(P_O,S,t)dZ_O$, is matched perfectly by the instantaneous change in the bundle, $\omega_k(O,t)X(t)\sigma_k dZ_k$. The holding of the safe asset is then chosen so that the instantaneous drifts also match.

As before the fund should be managed to ensure that the net exposure to each financial asset is a constant share of total wealth: $\bar{w}_i = \delta_i w_i, i = 1,..,m$ from proposition 4. Any exposure to asset *k* that is embodied in oil, $\omega_k(O,t)$, can be offset by the asset's weight in the fund, $w_k(t)$, so as to ensure that the net weight in total wealth is constant. By rearranging (26) the holdings of each financial asset in the fund can, as before, be split up into a leveraged component and a hedging component for the perfectly correlated asset *k*:

$$w_i = \bar{w}_i\left(\frac{F+V}{F}\right), i \neq k, \quad w_k(t) = \underbrace{\bar{w}_k + \bar{w}_k\left(\frac{V}{F}\right)}_{\text{leveraged demand}} + \underbrace{\left[-\omega_k(O,t)\left(\frac{V}{F}\right)\right]}_{\text{hedging demand}}. \tag{28}$$

As the asset allocation and consumption problem can be expressed in terms of total wealth (25), the results in propositions 2 and 3 hold. The rate of oil extraction does not affect consumption directly, but only through its effect on total wealth. Oil extraction and consumption are thus separated due to judicious management of the fund: the fund allows consumption to be smoothed in line with the permanent income hypothesis; and the fund buffers consumption from oil price volatility by hedging it with traded financial assets. Only the residual volatility of total wealth (the part of oil wealth that cannot be diversified away) must be managed by extra precautionary saving.

4. Policy implications: Norway's Government Pension Fund Global

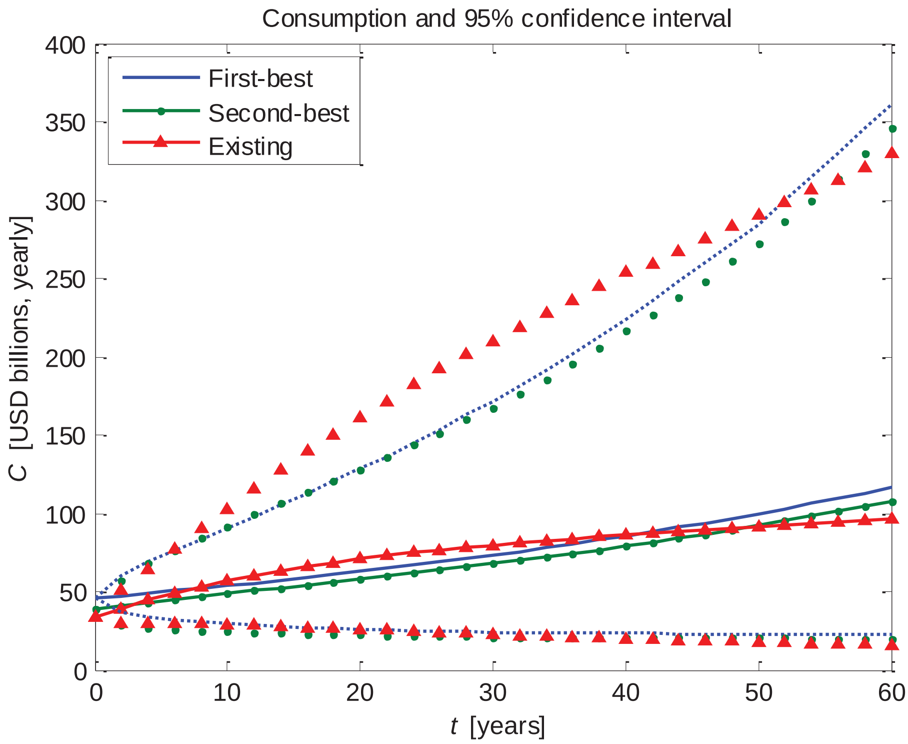
We now turn to the implications for sovereign wealth funds in practice. Currently over thirty countries have commodity sovereign wealth funds. We focus on Norway's GPFG as it is the largest single fund in the world and one of the most transparent. The first-best policy in section 2.1 would improve welfare relative to Norway's current policy by as much as a 15% permanent increase in the fund dividend, or 59% of mainland GDP. A more pragmatic second-best policy alters only the fund's equity-bond mix and the spending rule.[28] The equity-bond mix should then rise from 45% to 60% as oil is extracted, and consumption should fall to below 3% of fund assets over the next thirty years.[29] This policy achieves 58% of the welfare improvement of the first-best policy.[30]

[28] These policy levers are designed to take into account practical and political constraints faced by the fund, as described in a recent report to Norway's *Storting* (Ministry of Finance, 2014a).

[29] This is consistent with recent policy in Norway, with spending falling from nearly 6% of GPFG assets in 2010 to below 3% in 2014. This followed countercyclical spending in 2009 and is below 4% due to concerns of excessive fiscal stimulus (Ministry of Finance, 2014b)

[30] The first-best policy is calculated from closed-form solutions. The second-best policy is calculated using a shooting algorithm to determine the optimal consumption path, and Monte Carlo simulations. The existing policy is calculated using Monte Carlo simulations.

Figure 1: Consumption under Norway's first-best, second-best and existing policies.



4.1. First-best policy: no investment restrictions

Calibrating the model to the Norwegian sovereign wealth fund[31] and assuming that oil production declines exogenously, we find that the first-best case improves welfare relative to Norway's current policy by as much as a 15% permanent increase in the fund's dividend. Welfare is improved in two ways, by making consumption more stable in expectation, and in variance (see figure 1). This involves taking large long and short positions in particular sectors, which is difficult in practice.

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[31] We use the following monthly data from 2011-2013, described in appendix C. Equities: FTSE Global All Cap index, in aggregate and split into ten industries; Bonds: US 10 year Datastream government index; Oil price: Brent crude, current month FOB.

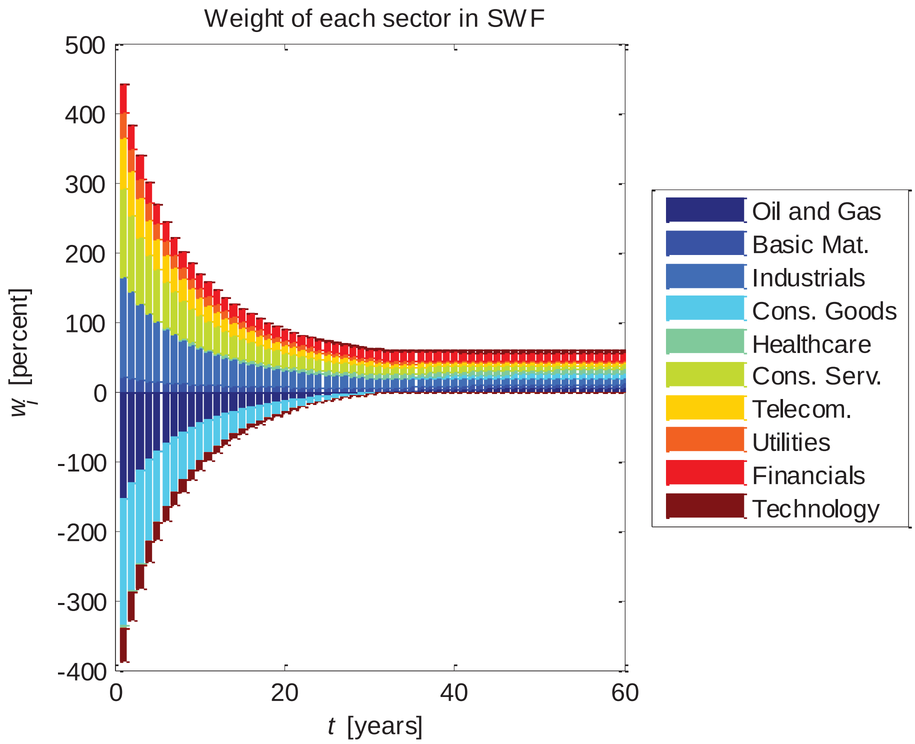
Welfare is improved by making average consumption more stable, seen in the solid lines in figure 1. Norway currently consumes on average 4.0% of fund assets each year, $C = 0.040F$.[32] Spending therefore rises as the fund receives oil revenues, but stabilizes at a greater share of total assets than is optimal. The first-best policy consumes 2.9% of total assets each year, $C = 0.029W$, from the closed-form solution in (12). This stabilizes consumption because total assets change less over time than fund assets, as below-ground wealth is converted to above-ground wealth. Changing the spending rule in this way, without altering the fund's portfolio, gives 58% of the welfare improvement from the first-best policy.

Welfare is also improved by reducing the variance of consumption, as seen in the dotted lines in figure 1. This is done by choosing the fund's portfolio to hedge oil price shocks, so total wealth is less volatile that the fund itself. As consumption is a constant proportion of total wealth, the overall standard deviation of consumption is reduced, from USD 10.3 billion per year after twenty years under the current rule, to USD 7.7 billion under the first-best policy.

To achieve the first-best optimum the sovereign wealth fund must take a combination of large short and long positions in each industry, as illustrated in figure 2. Short positions are taken in the three industries where returns are most correlated with the oil price (Oil and Gas, Consumer Goods, and Technology with correlations of 0.63, 0.0.51 and 0.55 respectively). These are offset by long positions in the other industries, based on the covariance of each industry with oil and with each other, as in (10).

[32] This has the same form as equation (16) if subsoil oil is ignored ($W = F$), as noted in section 2.1.

Figure 2: Weight of each FTSE Global All Cap sector in the fund under the first-best policy.



Implementing this asset allocation is difficult in practice (Ministry of Finance, 2014a). First, it requires large short positions in particular sectors. These positions can exceed 100% of the sovereign wealth fund, and involve taking substantial positions in individual stocks. Such highly leveraged positions expose the country to substantial risk if there are systematic shocks (Das and Uppal, 2004). The size of the fund also means that these positions may become illiquid, so that the assumption of exogenous prices is invalidated.

Second, the short positions assume that the covariance matrix is stable over time. In practice correlations between oil and each sector will vary depending on the source of the shock. A supply shock in one part of the world will have different effects on each asset to a demand shock in another (Kilian, 2009). As these correlations can only be estimated using past data and the size of the hedging positions are so large, there is the potential for large basis risk between oil and the hedging portfolio.

Third, this asset allocation requires the portfolio to be reallocated every period. As oil is extracted both the leverage and the hedging demand for each asset diminishes. Thus, the large short and long positions must be reversed. For a large sovereign wealth fund these transactions are likely to incur additional costs.

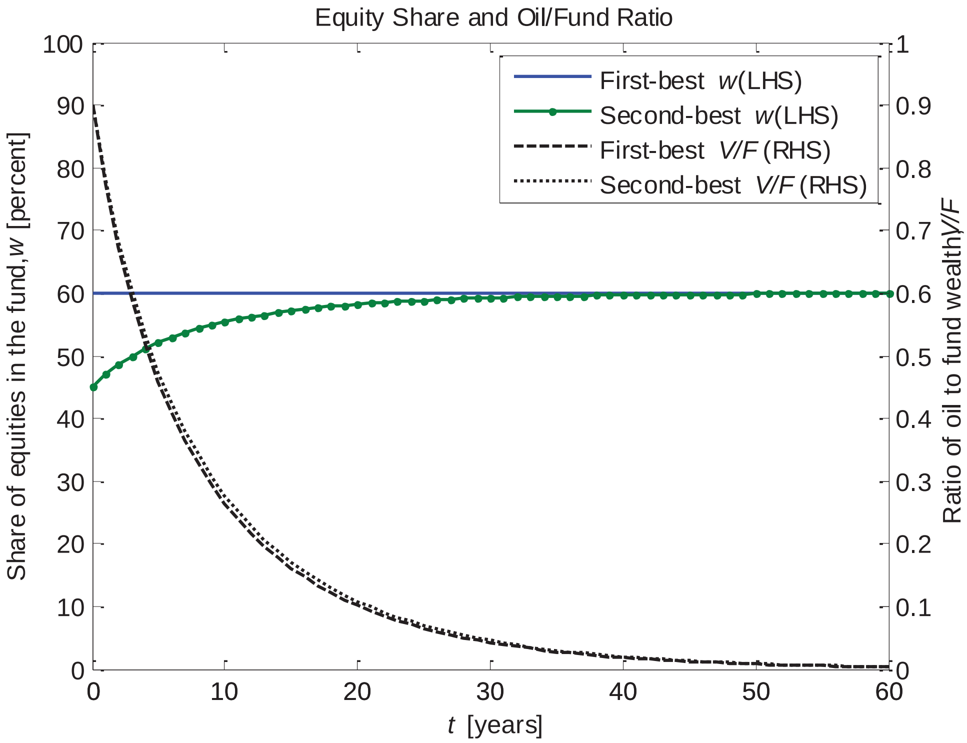
4.2 Second-best policy: varying the spending rule and equity/bond mix

We now consider a second-best scenario where oil extraction is predetermined (e.g., by geology) and all equities are held in the market portfolio, with the policy maker only changing the equity/bond mix and the spending rule. This is transparent, easy to explain, does not require short positions, reduces transaction costs and does not rely on a large, time-varying correlation matrix covering all assets in the market. The equity mix in the fund should then rise from 45% to 60% as oil is extracted, while consumption should fall to below 3% of fund assets. This achieves 58% of the welfare gain from the first-best policy.[33]

The equity share in the fund should rise over time, according to equation (10) and illustrated in figure 3. The hedging demand for the market portfolio, $\beta_M = 0.77$ (estimated using maximum likelihood) outweighs the leverage demand $\bar{w}_M = 0.60$ (from Norway's existing allocation). The fund should thus hold fewer equities and more riskless assets as long as there is a large and risky exposure to the world economy embodied in subsoil reserves. In the very early days of the fund (*V/F* > 4) the equity weight should be zero (or less than zero without short constraints). As oil is extracted, the proceeds should be invested in equities so that the share of equities in the portfolio rises over time. This is based on the

[33] Gintschel and Scherer (2008) just impose short-sale constraints directly. However, this does not address the transactions costs that large funds must face by continuously rebalancing, or the potentially unstable correlations between assets.

Figure 3: Equity share and the ratio of oil to fund wealth for the first-best and second-best policies.

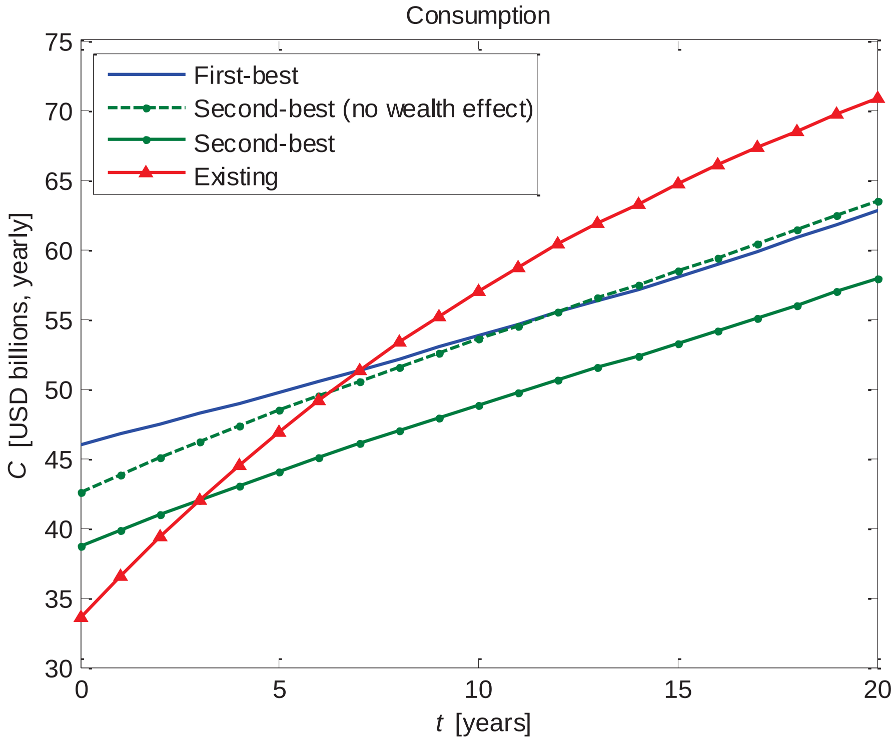


correlation of oil with the market – if it is 50%, then 50% of extraction revenues should be invested in equities.[34] Given the current size of the fund and subsoil reserves, this means rising from 45% today ($V/F\cong1$), to 60% when the fund is exhausted ($V/F=0$).

If the policymaker can only invest in the market portfolio then they will need to do more precautionary savings, as illustrated in figure 4. The first reason is that the FTSE Global All Cap index is not perfectly correlated with the oil price, and so there will remain some unhedged oil price risk. The second reason, which does not appear in our simulation, is that the sovereign wealth fund would face a short-

[34] The correlation between the oil price and the overall equity market will also vary over time, though it will be more stable than a covariance matrix covering all 7,400 assets in the FTSE Global All Cap Index. Varying correlations will alter how quickly the equity share in the fund rises. Future work could account for this using regime-switching (cf. Ang and Bekaert, 2002).

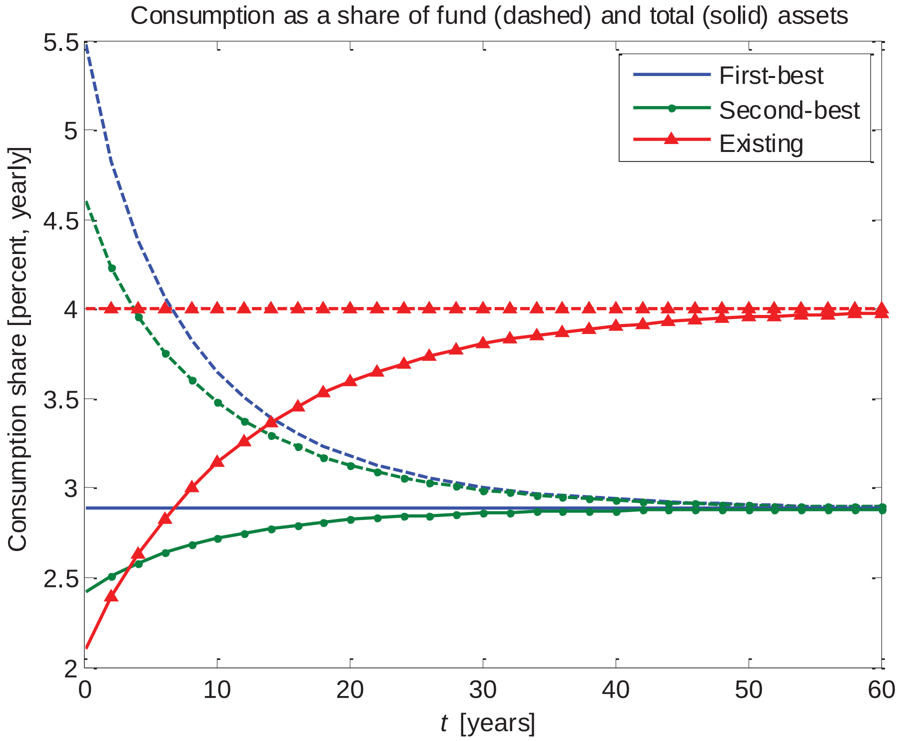
Figure 4: Consumption involves more precautionary savings under the second best policy (zooming in on figure 1).



sale constraint in the early years of the oil boom when $V/F \gtrsim 4$. In both instances the response to additional risk is to delay some consumption to build up a buffer stock of assets. In figure 4 we separate the precautionary savings and the wealth effects, described in equations (16) and (17) respectively. The wealth effects reduces consumption even further because the unspanned component of oil prices earns a low rate of return, $\alpha_h < r$.

Precautionary savings involves consuming a smaller share of total wealth initially, which is illustrated in figure 5. In the first best scenario consumption would initially be USD 46 billion per year, or 2.9% of total wealth. In contrast, in the second-best case a buffer-stock of assets is accumulated by initially consuming USD 39 billion per year, or 2.4% of total wealth. As oil is extracted and subsoil

Figure 5: Precautionary savings involves consuming a smaller share of total wealth to begin with.



reserves become less risky, this spending rule rises towards 2.9% of total wealth. Norway's existing policy involves consumption increasing as a share of total wealth, which could be considered an aggressive form of precautionary savings. Instead of Norway's 4% rule, the second-best policy sees spending as a share of fund assets fall from around 4.6% currently, to 2.9% as subsoil reserves are depleted. However, it is better to express the consumption rule as a share of total rather than fund wealth, because total wealth will be less volatile.

These results are of interest for the recent public debate around Norway's fund. There have been calls for the fund to stop investing in oil and gas stocks.[35] If the aim is to hedge subsoil oil then it should go much further – taking short positions

[35] Due to both hedging (Ministry of Finance, 2008) and environmental (Milne, 2014) reasons.

in oil, gas and other stocks that are positively correlated with oil prices. Alternatively, if the aim is different – such as protecting the environment – then spending should be curtailed to build up a buffer against less diversified risks.

There have also been calls for a change in the spending rule. In 2012 the Norges Bank Governor, Oystein Olsen, argued that spending should be curbed to 3 percent of the fund.[36] This can be justified because tightening the spending rule will make it more sustainable. Reducing spending is also consistent with recent practice in Norway: after fiscal stimulus actual spending declined from over 6 percent of the fund in 2010 to below 3 percent in 2014 (Olsen, 2014).

It is important to note that there are some important considerations that are outside the scope of this analysis. These include absorption constraints and other general equilibrium effects of consuming oil dividends on the Norwegian economy, and other components of national wealth such as pension liabilities. Therefore, this section provides general guidance on the nature and evolution of the equity-bond mix and the spending rule, though the actual levels to be implemented may differ.

5. Concluding remarks

Commodity exporters have two major types of national assets: natural resources below the ground and a sovereign wealth fund above it. Although some attempts to hedge commodity price volatility have been made, from long-term forward agreements in iron ore until 2010 to the purchase of oil options by Mexico in 2008, there is no evidence of systematic coordination of below- and above-ground assets. We have made the case for coordinating these two types of asset by integrating the theories of portfolio allocation, precautionary saving, and optimal oil extraction under oil- and asset-price volatility.

[36] This was motivated by real fund returns averaging less than 4 percent per annum (Olsen, 2012).

Our main findings are as follows. First, commodity exporters should change the allocation of their sovereign wealth fund by leveraging all risky assets and hedging subsoil oil risk. These effects are proportional to the ratio of oil and fund wealth, so unwind as resource reserves are depleted. Second, consumption should be a constant share of total oil and fund wealth. Third, if oil wealth cannot be adequately hedged, then less should be consumed initially in the interests of precautionary savings. Fourth, the rate of oil extraction should be faster if oil prices are volatile and positively correlated with financial markets, generating a higher rate of return on subsoil oil as compensation for the risk it is exposed to.

This is in sharp contrast to sovereign wealth funds in practice. Norway's GPFG invests in the market portfolio without any consideration of oil price risk. It spends up to 4 percent of the fund each year, which allows some buildup of precautionary buffers but does not accommodate the declining oil wealth beneath the ground. If Norway was to implement this theory perfectly then it can improve welfare by as much as a 15% permanent increase in the fund's dividend. However, this is difficult in practice because of short-sale constraints, transaction costs and unstable relationships between assets. To address these practical concerns we put forward a second-best policy which takes the extraction path as given, invests only in the market portfolio, but varies the equity/bond mix and the spending rule. This is transparent, does not require short positions, reduces transaction costs and is easy to implement. The equity mix in the fund then rises from 45% to 60% as oil is extracted while consumption falls to below 3% of fund assets. This achieves 58% of the welfare improvement of the first-best policy and captures some of the elements of Norway's policies.

Our analysis offers a first step towards an integrated approach to managing sovereign wealth funds and natural resources under uncertainty. Future work should allow for the exploration and discovery of new reserves, other components of national wealth, general equilibrium effects of spending resource revenues and more detailed modelling of asset prices. Exploration and discovery might be incorporated by extending Pindyck (1978) to a setting with financial assets – to understand how hedging oil price exposure affects exploration effort. Other components of national wealth might include domestic, non-traded capital[37], pension liabilities and tax revenues that depend on the sectoral composition of the economy. This may point to some benefits from reforming the structure of the economy to make it less vulnerable to commodity price volatility. This would also help understand the general equilibrium effects of spending commodity wealth, including the problems of absorption constraints. Finally, there is scope for modelling oil and asset prices in more detail. We have assumed prices are log-normally distributed, which has the benefit of yielding closed-form solutions. In practice prices exhibit mean reversion, large jumps and time-varying correlations that may all add more detail to these results.

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Appendix A: Proofs

A.1. Proof of proposition 1 (valuing oil with exogenous oil extraction)

Suppose that the investment set contains *m* assets with correlated returns. As their returns are normally distributed they can be expressed as a linear combination of *m* independent shocks, $dZ = \Lambda^* du^*$ where du^* is an *m* x 1 vector. If the oil price is completely spanned by the market then it too will be a linear combination of these *m* independent shocks, $dZ_O = \Lambda_O^* du^*$. Now, let us remove one asset from the investment set. The returns on the remaining *m-1* assets in the investment set can now be expressed as a linear combination of *m*-1 (different) independent shocks, $dZ = \Lambda du$ where *du* is an *m*-1 x 1 vector. If oil returns are expressed in terms of these shocks, there will be a residual component that is not correlated with the market, $dZ_O = \lambda_{Oh} du_h + \Lambda_O du$, as in (5). Thus, while the asset that is removed from the investment set is correlated with other assets, the unhedged component of the oil price is not. The parameters Λ_O, Λ and λ_{0h} can be estimated using principle components analysis, as $\text{cov}(dZ_i, dZ_j) = \sum_{g=1}^{m} \lambda_{ig}\lambda_{jg} \text{var}(du_g) = \sum_{g=1}^{m} \lambda_{ig}\lambda_{jg} dt$ The correlation matrix for *dZ* is $\mathrm{P} = \left(E_V E_A^{1/2} dt^{1/2}\right)\left(E_V E_A^{1/2} dt^{1/2}\right)'$, where E_V is the matrix of eigenvectors, $[E_A^{1/2}]_{ij} = e_i^{1/2}$ is the diagonal matrix of the square roots of eigenvalues, e_i, and $\Lambda = E_V E_A^{1/2}$.

This appendix (A.1.) is concerned with valuing subsoil oil, and so we ignore any investment restrictions that the fund may face. This follows from the assumption that any asset that is outside the investment set can still be observed, and so can be used to value oil wealth. The value thus derived is a market value. In the following appendix (A.2.) we will separate the hedged form the unhedged component when considering oil wealth and the fund together.

Taking equation (5) with $\lambda_{Oh} = 0$, we can express the oil price as:

$$P_O(t) = P_O(0)\exp(-\phi t)\prod_{i=1}^{m}\left[\frac{P_i(t)}{P_i(0)}\right]^{\beta_i}, \tag{A1}$$

with $\phi \equiv -\alpha_0 + \sum_{i=1}^{m}\beta_i\left(\alpha_i - \frac{1}{2}\sigma_i^2\right) + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\beta_i\beta_j\sigma_{ij}$ and $\beta_i \equiv \sigma_O M_i / \sigma_i$, $M_i \equiv \left[\Lambda_O\Lambda^{-1}\right]_i$,

which can be verified by applying Ito's lemma and comparing coefficients with equation (4).

Lemma A1: For an exponentially declining windfall, $O(t) = O(0)e^{-\kappa t}$, if markets are complete then the capitalized value of oil income ("oil wealth") is:

$$V(P_O, t) = P_O(t)O(t)/\psi, \quad \psi \equiv r + \kappa - \alpha_O + \sum\nolimits_{i=1}^{m}\beta_i(\alpha_i - r). \tag{A2}$$

Proof: First, we construct a portfolio that is identical to the capitalized value of oil. Second, we construct another portfolio consisting of the risky and safe financial assets and oil wealth. Third, we show that the posited expression for oil wealth satisfies an arbitrage condition between these portfolios that must hold.

First, we construct a portfolio with value $V(P_1, .., P_m, t)$ which consists of assets 1, .., n and distributes an amount of cash equal to $P_O(t)O(t)$ per unit time. This value evolves according to:

$$dV = (\mu_V V - P_O O)dt + \sigma_V V dZ_V. \tag{A3}$$

With the aid of Ito's lemma the dynamics of the portfolio can be written as:

$$dV = \sum_{i=1}^{m} V_i dP_i + V_t dt + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} V_{ij} dP_i dP_j = \left[\sum_{i=1}^{m}\alpha_i V_i P_i + V_t + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\sigma_{ij} V_{ij} P_i P_j\right]dt + \sum_{i=1}^{m}\sigma_i V_i P_i dZ_i, \tag{A4}$$

where $V_i = \partial V / \partial P_i$ and $\sigma_{ij} = \sigma_i\sigma_j\rho_{ij}$. Comparing coefficients with (A3) gives:

$$\mu_V V - P_O O = \sum_{i=1}^{m} \alpha_i P_i V_i + V_t + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} \sigma_{ij} V_{ij} P_i P_j$$
$$\sigma_V V dZ_V = \sum_{i=1}^{m} \sigma_i P_i V_i dZ_i. \qquad \text{(A5)}$$

Finally, let $dZ_V = \Lambda_V du$. This implies:

$$\sigma_V dZ_V = \sigma_V \Lambda_V du = \Gamma' dZ = \Gamma' \Lambda du, \quad \Gamma \equiv \left[V_1 \sigma_1 P_1 / V, .., V_m \sigma_m P_m / V\right]. \qquad \text{(A6)}$$

Second, we create another portfolio with value $X(t)$ that consists of oil wealth $V(t)$, the risky assets and the safe asset. This portfolio is dynamically constructed, so that short positions offset the long positions, there is no net risk, and the net value of the portfolio is always equal to zero. Hence, the weight of the safe asset in total wealth is $w_r = -w_V - \sum_{i=1}^{m} w_i$, where w_V is the weight of oil in total wealth. The return to this portfolio is:

$$\begin{aligned} dX &= w_V \left(\frac{dV + P_O O dt}{V} \right) + \sum_{i=1}^{m} w_i \left(\frac{P_i}{P} \right) + w_r r dt \\ &= \left[w_v (\mu_V - r) + \sum_{i=1}^{m} w_i (\alpha_i - r) \right] dt + w_V \sigma_V dZ_V + \sum_{i=1}^{m} w_i \sigma_i dZ_i \qquad \text{(A7)} \\ &= \left[w_v (\mu_V - r) + \sum_{i=1}^{m} w_i (\alpha_i - r) \right] dt + w_V \Gamma' \Lambda du + \Psi \Lambda du, \end{aligned}$$

where the second equality follows from (A3), the third equality from (A6) and $\Psi \equiv [w_1 \sigma_1, .., w_m \sigma_m]'$.

Suppose that the weights in this new portfolio are dynamically constructed so that there is no risk: $w_V \Gamma' \Lambda du + \Psi \Lambda du = 0$ and the last two terms in the last equality of (A7) vanish. The weights that would achieve this are $w_i = -(V_i / V) P_i w_V, i = 1, .., m.$ Arbitrage dictates that such a constructed portfolio must have a zero expected excess return over the risk-free rate:

$$0 = w_V(\mu_V - r) + \sum_{i=1}^{m} w_i(\alpha_i - r), \qquad V(\mu_v - r) = \sum_{i=1}^{m} V_i P_i(\alpha_i - r). \tag{A8}$$

Combining (A8) with (A5) gives the following optimality condition for the portfolio:

$$\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\sigma_{ij}P_iP_jV_{ij} + \sum_{i=1}^{m} rP_iV_i - rV + V_t + P_O O = 0. \tag{A9}$$

Third, we note that the proposed capitalized value of oil income and associated partials:

$$\begin{aligned}
V(P_O,t) &= \frac{1}{\psi}P_O(0)\exp(-\phi t)\prod_{i=1}^{m}\left[\frac{P_i(t)}{P_i(0)}\right]^{\beta_i} \mathrm{O}(0)\exp(-\kappa t),\\
V_i &= \frac{\beta_i V}{P_i}, \qquad\qquad V_t = -\phi V,\\
V_{ii} &= \frac{\beta_i(\beta_i - 1)V}{P_i^2}, \quad V_{ij} = \frac{\beta_i\beta_j V}{P_iP_j}, \; j = 1,..,m, \quad i = 1,..,m,
\end{aligned} \tag{A10}$$

indeed satisfy (A9) by substitution. Therefore, the capitalized value of oil income is given in Lemma A1.□

The result in equation (6) is given in Lemma A1. The instantaneous rate of change in the value of oil income is found by applying Ito’s lemma to this equation to give:

$$P_O O dt + dV = \left[r + \sum_{i=1}^{m}\beta_i(\alpha_i - r)\right]Vdt + \sigma_O V dZ_O. \tag{A11}$$

The result in equation (7) follows from substituting (A11), the evolution of fund assets given by (2), and the decomposition of oil returns in (5) with $\lambda_{O,h} = 0$, into the expression for total wealth, $dW = dF + dV$:

$$dW = \sum_{i=1}^{m} (\alpha_i - r)(w_i F + \beta_i V) + (rW - C)dt + \sum_{i=1}^{m} w_i \sigma_i F dZ_i + \sum_{i=1}^{m} \beta_i \sigma_i V dZ_i . \qquad \text{(A12)}$$

A.2. Proof of proposition 2 (asset allocation with exogenous oil extraction)

This appendix derives the optimal portfolio weights in a sovereign wealth fund in the presence of oil, with and without investment restrictions based on Merton (1990). We begin by restricting investment in asset *m*, so that $\lambda_{Oh} \neq 0$ and the fund will hold *m*-1 securities. We can summarize the properties of the unspanned component of the oil price under this restriction as:

$$dP_h = \alpha_h P_h dt + \sigma_h P_h du_h , \qquad \text{(A13)}$$

noting that while *m* was a traded asset that was correlated with all other assets, *h* is just the unspanned component of the oil price and is uncorrelated with other assets.

Let the value function be $J(F,V,t) = E_t\left[\int_t^{\infty} U(C(s)) e^{-\rho(s-t)} ds\right]$, where F is above-ground wealth and *V* is below-ground wealth. Above-ground wealth is accumulated according to (2), and below-ground wealth evolves according an equation akin to (A11), where we keep track of *h* in the latter replacing *m*:

$$dF = \sum_{i=1}^{m-1} w_i F(\alpha_i - r)dt + (rF + P_O O - C)dt + \sum_{i=1}^{m-1} w_i F \sigma_i dZ_i$$

$$dV + P_O O dt = \left(r + \sum_{i=1}^{m-1} \beta_i (\alpha_i - r) + \beta_h (\alpha_h - r) \right) V dt + \sigma_O V \left(M dZ + \lambda_{O,h} du_O \right). \qquad \text{(A14)}$$

The Hamilton-Jacobi-Bellman equation is:

$$\max_{w_i, C} \left[U(C) e^{-\rho t} + \tfrac{1}{dt} E_t \left[dJ(F,V,t) \right] \right] = 0, \qquad \text{(A15)}$$

where we have:

$$
\begin{aligned}
\tfrac{1}{dt}E_t\left[dJ\right] = J_t + J_F\left(\sum_{i=1}^{m-1} w_i F\left(\alpha_i - r\right) + rF + P_O O - C\right) \\
+J_V\left(\left(r + \sum_{i=1}^{m-1}\beta_i\left(\alpha_i - r\right) + \beta_h\left(\alpha_h - r\right)\right)V - P_O O\right) \\
+\frac{1}{2}J_{FF}F^2\sum_{i=1}^{m-1}\sum_{j=1}^{m-1} w_i w_j \sigma_{ij} + \frac{1}{2}J_{VV}V^2\left[\sum_{i=1}^{m-1}\sum_{j=1}^{m-1}\beta_i\beta_j\sigma_{ij} + \sigma_O^2\lambda_{O,0}^2\right] \\
+J_{VF}VF\sum_{i=1}^{m-1}\sum_{j=1}^{m-1} w_i\beta_j\sigma_{ij}.
\end{aligned}
\tag{A16}
$$

The first-order conditions with respect to C and w_i are:

$$
U'\left(C\right)e^{-\rho t} - J_F = 0 \Rightarrow J_F = U'\left(C\right)e^{-\rho t} \tag{A17}
$$

$$
J_F F\left(\alpha_i - r\right) + J_{FF}F^2\sum_{j=1}^{m-1} w_j\sigma_{ij} + J_{FV}FV\sum_{j=1}^{m-1}\beta_j\sigma_{ij} = 0. \tag{A18}
$$

Equation (A18) can be solved to give the optimal weights in the fund:

$$
\begin{aligned}
w_i &= -\frac{J_F}{FJ_{FF}}\sum_{j=1}^{m-1} v_{ij}\left(\alpha_j - r\right) - \frac{J_{FV}}{J_{FF}}\frac{V}{F}\beta_i \\
&= \frac{C/F}{\partial C/\partial F}\theta\sum_{j=1}^{m-1} v_{ij}\left(\alpha_j - r\right) - \frac{\partial C/\partial V}{\partial C/\partial F}\frac{V}{F}\beta_i.
\end{aligned}
\tag{A19}
$$

To proceed we must find an expression for the partial derivatives in (A19). If markets are complete then these can be found analytically from (12), $\partial C/\partial F = \partial C/\partial V = \partial C/\partial W = r_W^*$. If markets are incomplete then there is no analytical solution, but we can approximate the partials from the complete markets case, or alternatively assume that consumption is a linear function of total wealth. We thus only obtain the leading-order effect of investment restrictions, which is all that is of interest here. With and without investment restrictions we obtain,

$$w_i = \frac{W}{F}\theta\sum_{j=1}^{m-1} v_{ij}\left(\alpha_j - r\right) - \frac{V}{F}\beta_i. \qquad \text{(A20)}$$

If we define the share of each asset in total wealth to be $\bar{w}_i W \equiv w_i F + \beta_i V$ then rearranging (A20) gives the results in equations (8) and (10). The leading-order effect of restricting investment comes through modifying β_i (see appendix B).

A.3. Proof of propositions 3 and 4 (consumption with exogenous oil extraction)

Here we extend Merton (1990) to derive the optimal consumption rate from a fund in the presence of oil, with and without investment restrictions. If markets are complete we can find a closed form for the value function $J(F,V,t)$, using $J_W = J_F = J_V$. Substituting the first-order conditions (A17) and (A18) into the Hamilton-Jacobi-Bellman equation in (A15) gives the optimality condition for the value function:

$$0 = \tfrac{1}{\theta-1}\exp(-\theta\rho t)\,\mathrm{J}_W^{1-\theta} + J_t + rWJ_W - \frac{J_W^2}{J_{WW}}\frac{(\alpha_W - r)^2}{2\sigma_W^2}. \qquad \text{(A21)}$$

A closed-form solution to this stochastic partial differential equation exists and takes form in equation (15), which can be confirmed using Ito's lemma.

The result in equation (12) follows from substituting the value function in (15) into equation (A17).

The Euler equation, describing the expected rate of change of consumption, is found by applying Ito's lemma to (A17):

$$\frac{\tfrac{1}{dt}E_t[dJ_F]}{J_F} = \frac{C\,''(C)}{U'(C)}\frac{\tfrac{1}{dt}E_t[dC]}{C} - \rho + \frac{1}{2}\frac{CU\,'''(C)}{U'(C)}\frac{\tfrac{1}{dt}E_t[dC^2]}{C}. \qquad \text{(A22)}$$

On the left-hand side of this equation we have the expected rate of change of the marginal utility of assets. Using Ito's lemma we have:

$$dJ_F(F,V,t) = J_{FF}dF + J_{FV}dV + J_{Ft}dt + \tfrac{1}{2}J_{FFF}dF^2 + \tfrac{1}{2}J_{FVV}dV^2 + J_{FFV}dFdV. \quad \text{(A23)}$$

In addition the derivative of (A15) with respect to F is:

$$\begin{aligned}
0 = J_{tF} &+ J_{FF}\left(\sum_{i=1}^{m-1} w_i F\left(\alpha_i - r\right) + rF + P_O O - C\right) \\
&+ J_F\left(\sum_{i=1}^{m-1} w_i\left(\alpha_i - r\right) + r\right) \\
&+ J_{VF}\left(\left(r + \sum_{i=1}^{m-1}\beta_i\left(\alpha_i - r\right) + \beta_h\left(\alpha_h - r\right)\right)V - P_O O\right) \\
&+ \frac{1}{2}J_{FFF}F^2\sum_{i=1}^{m-1}\sum_{j=1}^{m-1} w_i w_j \sigma_{ij} + J_{FF}F\sum_{i=1}^{m-1}\sum_{j=1}^{m-1} w_i w_j \sigma_{ij} \\
&+ \frac{1}{2}J_{VVF}V^2\left[\sum_{i=1}^{m-1}\sum_{j=1}^{m-1}\beta_i\beta_j\sigma_{ij} + \sigma_O^2\lambda_{Oh}^2\right] \\
&+ J_{VFF}VF\sum_{i=1}^{m-1}\sum_{j=1}^{m-1} w_i\beta_j\sigma_{ij} + J_{VF}V\sum_{i=1}^{m-1}\sum_{j=1}^{m-1} w_i\beta_j\sigma_{ij}.
\end{aligned} \quad \text{(A24)}$$

Substituting (A18) and (A23) into (A24) gives:

$$0 = \tfrac{1}{dt}E_t\left[dJ_F\right] + J_F r. \quad \text{(A25)}$$

We also have:

$$\tfrac{1}{dt}E_t\left[\left(dC\right)^2\right] = C_F^2\,\tfrac{1}{dt}E_t\left[\left(dF\right)^2\right] + C_V^2\,\tfrac{1}{dt}E_t\left[\left(dV\right)^2\right] + 2C_V C_F\,\tfrac{1}{dt}E_t\left[dVdF\right], \quad \text{(A26)}$$

where:

$$\tfrac{1}{dt}E_t\left[\left(dF\right)^2\right] = F^2\sum_{i=1}^{m-1}\sum_{j=1}^{m-1} w_i w_j \sigma_{ij},$$

$$\tfrac{1}{dt}E_t\left[\left(dV\right)^2\right] = V^2\left(\sum_{i=1}^{m-1}\sum_{j=1}^{m-1} \beta_i\beta_j\sigma_{ij} + \sigma_O^2\lambda_{Oh}^2\right), \tag{A27}$$

$$\tfrac{1}{dt}E_t\left[dVdF\right] = VF\sum_{i=1}^{m-1}\sum_{j=1}^{m-1} w_i\beta_j\sigma_{ij}.$$

Combining (A26) and (A27) we get:

$$\begin{aligned}\frac{1}{dt}E_t[dC^2] &= C_W^2\left[\sum_{i=1}^{m-1}\sum_{j=1}^{m-1}(w_iF+\beta_iV)(w_jF+\beta_jV)\sigma_{ij} + \lambda_{Oh}^2\sigma_O^2V^2\right] \\ &= C_W^2\left(\sum_{i=1}^{m-1}\sum_{j=1}^{m-1}\overline{w}_i\overline{w}_j\sigma_{ij}W^2 + \lambda_{Oh}^2\sigma_O^2V^2\right) \\ &= C_W^2\left(\overline{w}^2\sigma_W{}^2W^2 + \lambda_{Oh}^2\sigma_O^2V^2\right),\end{aligned} \tag{A28}$$

where we have used the approximation $C_W \approx C/W$, $C_W \approx C_F \approx C_V$. The stochastic Euler equations in (11) and (16) follow from substituting (A25) and (A28) into (A22). The version in equation (11) assumes complete markets, so $\lambda_{Oh} = 0$ and $C_W = C/W$ from (12). The version in equation (16) assumes incomplete markets, so $\lambda_{Oh} \neq 0$ and we use the approximation $C_W \approx C/W$ to separate out leading-order effects.

A.4. Proof of proposition 5 (endogenous oil extraction)

The Hamilton-Jacobi-Bellman equation for the problem in (18), (19), (3), (4) and (20) is:

$$0 = \max_{C, w_i, O} \left\{ U(C) e^{-\rho t} + \tfrac{1}{dt} E_t \left[dJ(F, P_O, S, t) \right] \right\},$$

$$\begin{aligned} \tfrac{1}{dt} E_t \left[dJ(F, P_O, S, t) \right] = J_F & \left[\sum_{i=1}^{m} w_i (\alpha_i - r) F + rF - C + P_O O - G(O) \right] \\ & + J_P \alpha_O P_O - J_S O + J_t + \frac{1}{2} J_{FF} F^2 \sum_{i=1}^{m} \sum_{j=1}^{m} w_i w_j \sigma_{ij} \\ & + \frac{1}{2} J_{PP} \sigma_O^2 P_O^2 + J_{FP} \sigma_O P_O F \sum_{i=1}^{m} w_i \sigma_i \rho_{iO}. \end{aligned} \tag{A29}$$

The first-order conditions are:

$$U'(C) e^{-\rho t} = J_F, \tag{A30}$$

$$J_F F(\alpha_i - r) + J_{FF} F^2 \sum_{j=1}^{m} w_j \sigma_{ij} + J_{FP} F \sigma_i \sigma_O \rho_{iO} P_O = 0 \quad \forall i, \tag{A31}$$

$$J_F \left[P_O - G'(O) \right] - J_S = 0. \tag{A32}$$

Upon differentiation of (A29) with respect to the state variables (cf. envelope condition), we get:

$$\tfrac{1}{dt} E\left[dJ_F \right] + J_F \left[r + \sum_{i=1}^{m} w_i (\alpha_i - r) \right] + J_{FF} F \sum_{i=1}^{m} \sum_{j=1}^{m} w_i w_j \sigma_{ij} + J_{FP} \sigma_O P_O \sum_{i=1}^{m} w_i \sigma_i \rho_{iO} = 0, \tag{A33}$$

$$\tfrac{1}{dt} E\left[dJ_S \right] = 0, \tag{A34}$$

$$\tfrac{1}{dt} E\left[dJ_P \right] + J_F O + J_P \alpha_O + J_{PP} \sigma_O^2 P_O + J_{FP} F \sum_{i=1}^{m} w_i \sigma_i \sigma_O \rho_{iO} = 0. \tag{A35}$$

Upon substitution of (A31) into (A33), we get:

$$\tfrac{1}{dt} E\left[dJ_F \right] = -r J_F. \tag{A36}$$

Equation (A34) states that oil is extracted so that the marginal utility of an extra barrel of oil in the ground is always constant. Equation (A36) requires that in expectation the marginal utility of assets (or of consumption from (A30)) must fall at the riskless rate of interest. Applying Ito's lemma to (A30) and combining it with (A36) gives the Euler equation:

$$\frac{\frac{1}{dt}E_t\left[dC\right]}{C}=\left[\frac{-U'(C)}{CU''(C)}\right](r-\rho)-\left[\frac{CU'''(C)}{U''(C)}\right]\frac{\frac{1}{dt}E_t\left[dC^2\right]}{C^2}. \tag{A37}$$

Applying Ito's lemma to (A32) gives rise to:

$$\frac{dJ_S}{J_S}=\frac{dJ_F}{J_F}+\frac{d\Omega_O}{\Omega_O}+\frac{dJ_F d\Omega_O}{J_F\Omega_O},\quad \Omega_O=P_O-G'(O). \tag{A38}$$

Combining (A34), (A36) and (A38) yields the expected Hotelling rule (21) of proposition 5. The extraction path is thus affected by the marginal utility of wealth and marginal oil rents. Marginal oil rents are a function of the oil price and marginal extraction cost. Both the marginal utility of wealth and the rate of oil extraction will be a function of the four state variables, $J_F(F,P_O,S,t)$ and $O(F,P_O,S,t)$. Application of Ito's lemma to both yields:

$$dJ_F=-rJ_F dt+J_{FF}F\sum_{i=1}^{m}w_i\sigma_i dZ_i+J_{FP}P_O\sigma_O dZ_O, \tag{A39}$$

$$dO=\mu_O(F,P_O,S,t)dt+O_F F\sum_{i=1}^{m}w_i\sigma_i dZ_i+O_P P_O\sigma_O dZ_O, \tag{A40}$$

where we have used (A36) and $\mu_O(F,P_O,S,t)=E_t\left[dO\right]/dt$ is the yet to be determined expected rate of oil extraction. Applying Ito's lemma to $\Omega_O=P_O-G'(O)=P_O-\gamma O$ gives:

$$\begin{aligned}d\Omega_O&=dP_O-G''(O)dO-\frac{1}{2}G'''(O)dO^2\\&=\left[\alpha_O P_O-\gamma\mu_O(F,P_O,S,t)\right]dt-\gamma O_F F\sum_{i=1}^{m}w_i\sigma_i dZ_i+(1-\gamma O_P)P_O\sigma_O dZ_O,\end{aligned} \tag{A41}$$

where we have used $G'''(O) = 0$ for quadratic extraction costs. Multiplying (A39) and (A41) gives:

$$\frac{dJ_F d\Omega_O}{J_F \Omega_O} = -\frac{\gamma O_F F}{J_F \Omega_O} \left\{ \sum_{i=1}^{m} w_i \sigma_i \left[J_{FP} \sigma_O P_O \rho_{iO} + J_{FF} F \sum_{j=1}^{m} w_j \sigma_j \rho_{ij} \right] \right\} dt + \frac{(1-\gamma O_P) P_O \sigma_O}{J_F \Omega_O} \left(J_{FP} P_O \sigma_O + J_{FF} F \sum_{i=1}^{m} w_i \sigma_i \rho_{iO} \right) dt. \tag{A42}$$

This can be simplified by substituting in the optimal asset weight condition in (A31) for all assets (the first term on the right-hand side) and for the perfectly correlated asset *k* ($\rho_{kO} = 1$, the second term) to give:

$$\frac{dJ_F d\Omega_O}{J_F \Omega_O} = \frac{1}{\Omega_O} \left[\gamma O_F F \sum_{i=1}^{m} w_i (\alpha_i - r) - (1-\gamma O_P) \left(\frac{\alpha_k - r}{\sigma_k} \right) P_O \sigma_O \right] dt. \tag{A43}$$

Substituting (A41) and (A43) into (21) gives the stochastic Hotelling rule in (22). To gain further intuition we approximate the partial derivatives in (22) with their deterministic counterparts.

Lemma A2: If all prices are deterministic then, $\dot{\Omega}_O = r\Omega_O$. If the oil price is also without drift, $\alpha_O = 0$, then the date of exhaustion is, $T = -\frac{1}{r}\ln(\gamma O(0)/P_O(0))$, and the optimal rate of extraction (to a leading order approximation) is:

$$O(t) = \sqrt{2\tfrac{r}{\gamma} S(t) P_O(t)}. \tag{A44}$$

Proof: The deterministic Hotelling rule comes from setting $E_t[dJ_F d\Omega_O] = 0$ in equation (21), giving, $\dot{\Omega}_O = r\Omega_O$. Using $\Omega_O = P_O - \gamma O$ this corresponds to, $\dot{O} = rO + \frac{1}{\gamma}(\alpha_O - r)P_O(0)e^{\alpha_O t}$, which has the solution:

$$O(t) = O(0)e^{rt} + \tfrac{1}{\gamma} P_O(0)\left(e^{\alpha_O t} - e^{rt}\right). \tag{A45}$$

We can never have $\alpha_O \geq r$ as price growth would delay extraction indefinitely. Provided initial marginal oil rents are positive, $\Omega_O(0) > 0$, and, $\alpha_O < r$, then the extraction rate remains finite. The optimal initial extraction rate must satisfy, $S(t) = \int_t^T O(\tau) d\tau$, and the date of exhaustion *T* must satisfy *O(T)* = 0. The date of exhaustion only has an explicit solution for $\alpha_O = 0$, which we assume with some empirical confidence. This gives:

$$T = -\tfrac{1}{r} \ln(1 - R), \tag{A46}$$

$$S(0) = -\tfrac{1}{r\gamma} P_O(0)\left(\ln(1-R) + R\right), \tag{A47}$$

where R is the small parameter: $0 < R = \gamma O(0) / P_O(0) < 1$. As (A47) only implicitly defines the initial rate of extraction, $O(0) = f(S(0), P_O(0))$, we use asymptotic methods to find a series-solution and study the leading-order effect. Using $-\ln(1-R) = \sum_{n=1}^{\infty} \tfrac{1}{n} R^n$ we get:

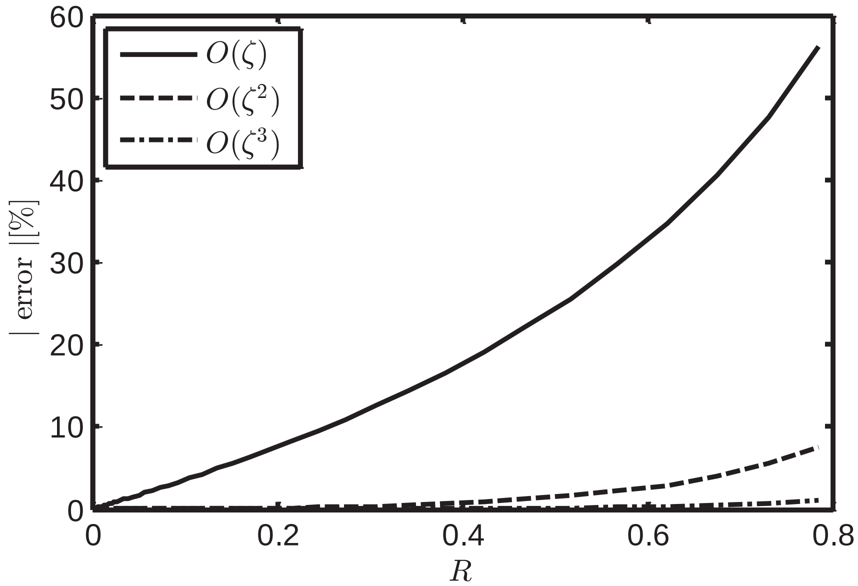
$$\frac{r\gamma S(0)}{P_O(0)} = \sum_{n=2}^{\infty} \frac{R^n}{n}. \tag{A48}$$

This can be inverted to give,

$$O(t) = S(t)\left[\sqrt{2}\xi(t)^{-1} - \tfrac{2}{3} + \tfrac{1}{9\sqrt{2}} \xi(t) + \tfrac{2}{135} \xi(t)^2 + \tfrac{1}{540\sqrt{2}} \xi(t)^3 + o(\xi(t)^4)\right], \tag{A49}$$

where $\xi(t) = \sqrt{r\gamma S(t) / P_O(t)}$, and the coefficients stem from the series inversion so are independent of parameters. The approximation error is quantified by comparison with the numerically exact solution in figure A1. The numerical simulations are calculated using the series solution up to the third order to avoid unnecessary approximation. To the leading order this yields the relationship in (A44). □

Figure A1: Approximation error for deterministic *O(t)* at various orders of approximation.



From lemma A2 we have the following approximations for the partial derivatives required to approximate the stochastic Hotelling equation:

$$\frac{\partial O}{\partial F} = 0, \qquad \frac{\partial O}{\partial P_O} = \frac{1}{2}\frac{O}{P_O} = \sqrt{\frac{rS}{2\gamma P_O}}. \tag{A50}$$

An approximation of the stochastic Hotelling rule comes from substituting these partials into (22) to give:

$$d\Omega_O = r\Omega_O dt + (\mathrm{P}_O - \gamma \tfrac{1}{2} O)\frac{\sigma_O}{\sigma_k}[(\alpha_k - r)dt + \sigma_k dZ_O]. \tag{A51}$$

So, a positive shock to the oil price, dZ_O, increases marginal oil rents. The expected path of oil extraction comes from combining equations (A41) and (A51), and setting $\alpha_O = 0$ as in Lemma A2, to give:

$$\begin{aligned} \tfrac{1}{dt} E[dO] &= \mu_O(F, P_O, S, t) \\ &= -\tfrac{1}{\gamma}\Big(r + \tfrac{\sigma_O}{\sigma_k}(\alpha_k - r)\Big)P_O + \Big(r + \tfrac{1}{2}\tfrac{\sigma_O}{\sigma_k}(\alpha_k - r)\Big)O. \end{aligned} \tag{A52}$$

The approximate stochastic path of oil extraction in (23) is found by substituting (A52) into (A40) and solving the initial value problem numerically subject to the exhaustion condition $O(t=T)=S(t=T)=0$.

A.5. Proof of proposition 6 (asset allocation and endogenous oil extraction)

Let there be a traded asset k, which is perfectly correlated with oil, $dZ_O = dZ_k$. The first part of the proposition states that oil rents can be replicated with a bundle containing N_k shares of asset k and N_r shares of the safe asset, $X \equiv N_k P_k + N_r P_r$. This bundle must yield a continuous dividend exactly equal to the optimal oil rents Ω. This replicating bundle can be constructed as follows.

We begin in discrete time with sample period h before moving to continuous time by $h \to 0$ following Merton (1990, p. 125). Construct a bundle so that at every time t the number of shares $N_k(t)$ and $N_r(t)$ are chosen and held until time $t + h$. At the same time a dividend is declared exactly equal to $\Omega(t)$, which is paid continuously throughout the period h. The bundle will start period t at current prices with value $N_k(t-h)\, P_k(t) + N_r(t-h)\, P_r(t)$, as the number of shares in each asset has been chosen in the previous period. At time t the dividend and new shares are chosen to preserve the value of the bundle, $-\Omega(t)h = \sum_{i=k,r}\left[N_i(t) - N_i(t-h)\right]P_i(t)$. The same must be true at $t + h$, so that:

$$
\begin{aligned}
-\Omega(t+h)h &= \sum_{i=k,r}\left[N_i(t+h) - N_i(t)\right]P_i(t+h) \\
&= \sum_{i=k,r}\left[N_i(t+h) - N_i(t)\right]\left[P_i(t+h) - P_i(t)\right] + \sum_{i=k,r}\left[N_i(t+h) - N_i(t)\right]P_i(t).
\end{aligned} \tag{A53}
$$

Taking the limit as $h \to 0$ we get $-\Omega dt = \sum_{i=k,r} dN_i dP_i + dN_i P_i$ assuming all variables are continuous. This describes the rate at which shares have to be sold to finance the dividend.

Equation (24) of proposition 6 combines this expression for the dividends with the path for the replicating bundle. By Ito's lemma the replicating bundle must satisfy:

$$
\begin{aligned}
dX + \Omega dt &= \sum_{i=k,r} \left(N_i dP_i + dN_i dP_i + dN_i P_i \right) + \Omega dt \\
&= \sum_{i=k,r} \left(N_i dP_i \right) \\
&= \omega_k X (\alpha_k - r) dt + rXdt + \omega_k X \sigma_k dZ_k .
\end{aligned}
\tag{A54}
$$

where $\omega_k(t) \equiv N_k(t)P_k(t)/X(t)$ is the weight of the risky asset k in the replicating bundle. The weights $\omega_k(t)$ must be updated continuously to match the stochastic path of oil rents described by (22). As oil wealth and the replicating bundle have the same properties they must also have the same value, by arbitrage $X = V$ giving the result in (24). We have focused on an expression for $dV(t) + \Omega(t)dt$. The explicit expression for V(t) can be found using contingent claims analysis (Merton, 1990) for the case when oil rents follow the general Ito process $d\Omega(t) = a(.)\Omega dt + s(.)\Omega dZ_O$ when $a(.)$ and $s(.)$ are not constants. The value of oil rents must equal that of the replicating bundle, $V(t) = X(t)$, because both share exactly the same properties (made possible by the perfect correlation between asset k and oil). Equation (25) of proposition 6 states that the policy maker's problem can be summarized in terms of total wealth. Total wealth is given by $W(t) = F(t) + V(t)$. Combining equations (19) and (A54) gives equation (25) in proposition 6. When expressed in terms of total wealth this problem is reduced to the standard Merton (1990) analysis. The weight of the asset k in the fund adjusts continuously so that the net weight of oil in total wealth is constant. The weight of all other assets in the fund remain constant, as given in equation (26).

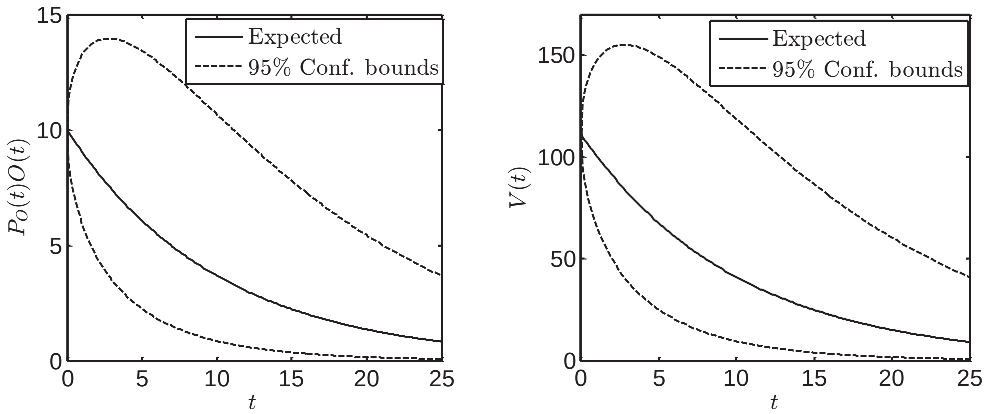
Online appendix B: Illustrations of oil-CAPM with exogenous extraction

B.1. Exogenous oil extraction without investment restrictions

Using the results from sections 2.1 and 2.2 we illustrate how a sovereign wealth fund should be affected by oil, depending on whether or not it has access to hedging assets. For the sake of simplicity we suppose that there is a riskless asset, *R*, and two risky assets: *1* which is uncorrelated with the oil price, and *2* which is perfectly negatively correlated with the oil price.[38] Figure B1 illustrates the exogenously declining path of oil rents. If the sovereign wealth fund can invest in both assets *1* and *2* then there will be a leverage demand for both plus a hedging demand for *2*, as illustrated in figure B2. Without oil a quarter of the fund is invested in each risky asset (as they are identical except for their correlation with oil) and the remainder in the riskless one, $\bar{w}_1 = \bar{w}_2 = 0.25, \bar{w}_R = 0.5$. Including oil introduces an additional leverage demand for each risky asset which begins large but falls as oil is extracted, $+\bar{w}_i V / F$.

Figure B1: Exogenous oil rents and the value of oil

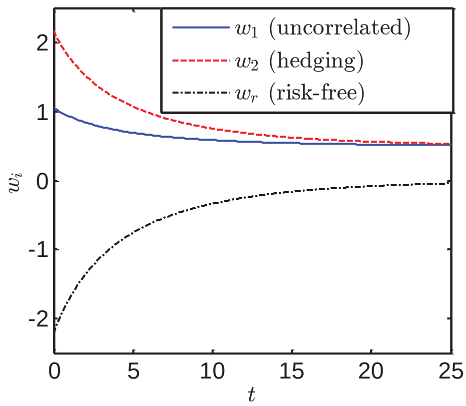
a. Oil revenues ***b. Value of oil wealth***



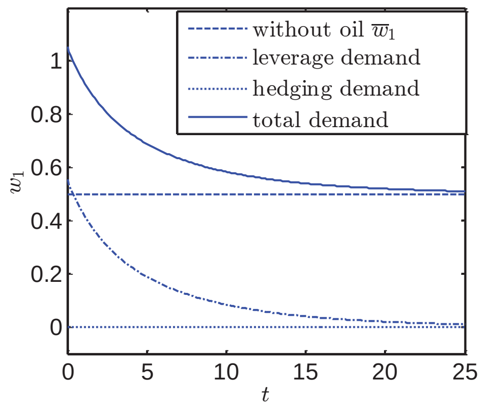
[38] The following illustrative figures assume that *F(0)=100; r=ρ=0.03; θ = 0.5;* $P_i(0) = 1$*;* $\alpha_i = 0.07$, $\sigma_i = 0.02$, $\rho_{ij} = 0$ for *i,j=[A,B]; S(0)=100; O(0) = 10; κ=0.1;* $\alpha_O = 0$ and $\sigma_O = 0.25$.

Figure B2: Asset shares without investment restrictions

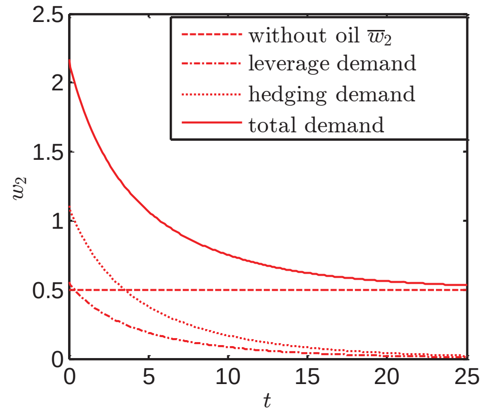
a. Asset weights in the fund



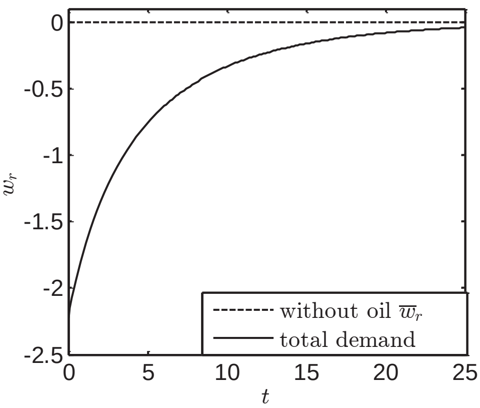
b. Weight breakdown: uncorrelated asset ρ_{O1}=0



c. Weight breakdown: hedging asset ρ_{O2}=-1



d. Weight breakdown: risk-free asset

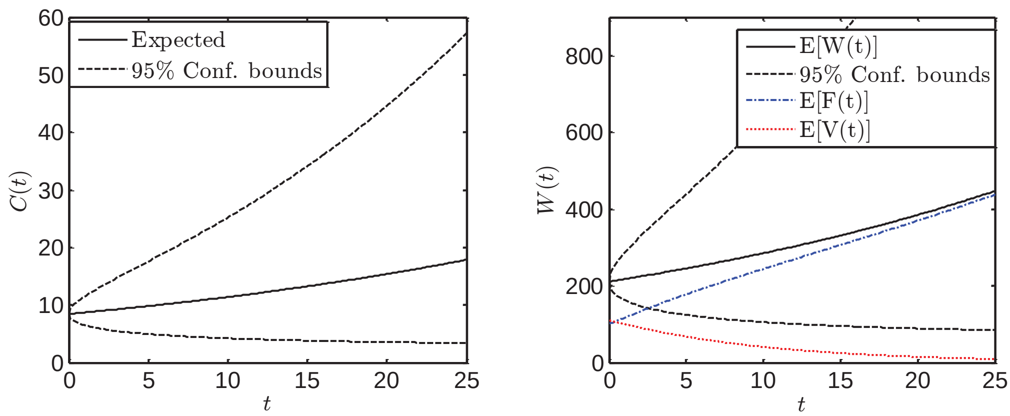


This happens because the fund should also hold a "share" of subsoil wealth in each risky asset. Finally there will also be a hedging demand for asset *2*, because it is negatively correlated with the oil price. Relatively more of asset *2* should be held as it performs well when the oil price falls, and vice versa.

To buy enough of asset *2* to properly hedge the oil price the policy maker would initially need to borrow ("short") the risk-free asset. This too would fall as the exposure to oil prices beneath the ground diminishes.

Figure B3: Optimal consumption and wealth accumulation without investment restrictions

a. Optimal consumption path ***b. Optimal consumption path***



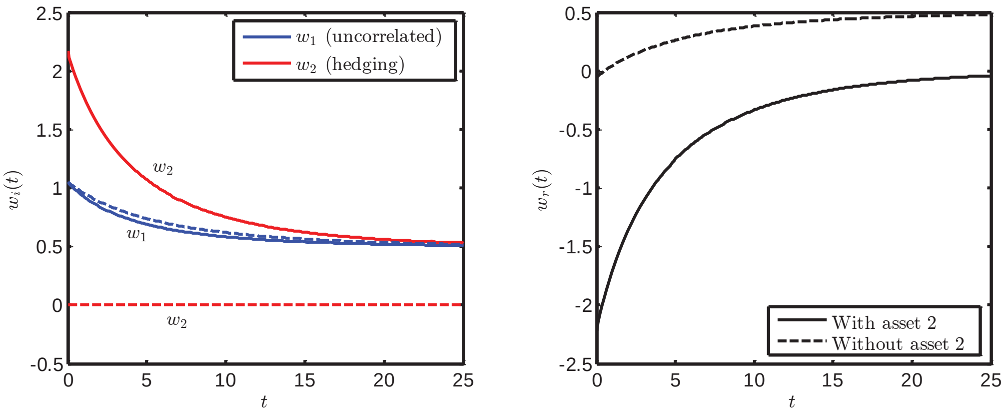
Consumption should be a fixed share of total wealth, rather than just above- or below-ground assets (figure B3). This insulates consumption from oil revenues as they vary, both with production and with oil price shocks. The oil revenues are invested in the sovereign wealth fund according to the Hartwick rule, and consumption grows stably with total wealth to incorporate precautionary savings.

B.2. Exogenous oil extraction with investment restrictions

If the fund is prevented from investing in a particular asset, in our case *2*, then the allocation between all other assets will change (figure B4). We first note that the optimal weight of asset *1* stays much the same because *1* is uncorrelated with *2*, from equation (8). The only difference comes from the change in the drift of the fund *F* because of the precautionary and wealth effects described in proposition 4. Therefore, all the wealth that would have been invested in *2* is now held in the riskless asset (by no longer borrowing it). As it is no longer possible to buy the hedging asset, there is no point in borrowing the riskless asset.

Figure B4: Optimal portfolio allocation without investment restrictions (solid) and with a ban on investing in asset *2* (dashed)

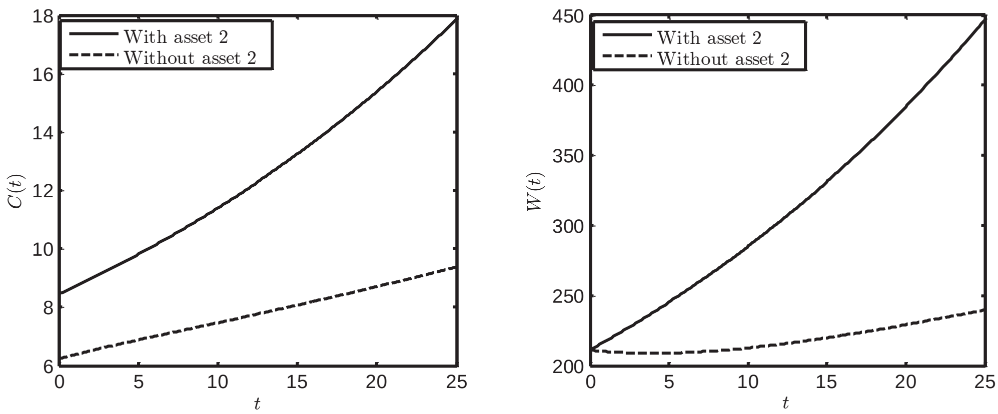
a. Risky assets ***b. Riskless asset***



As stated in proposition 4, being unable to invest in a particular asset will have both a precautionary and a wealth effect on consumption, as illustrated in figure B5. The precautionary effect reduces consumption to build up a buffer stock against unhedged oil price risk. Meanwhile, the wealth effect alters the overall rate of return on total wealth. The net effect depends on the nature of the asset being excluded and the degree of prudence.

Figure B5: Optimal consumption without investment restrictions (solid) and with a ban on investing in asset *2* (dashed)

a. Consumption ***b. Total wealth***

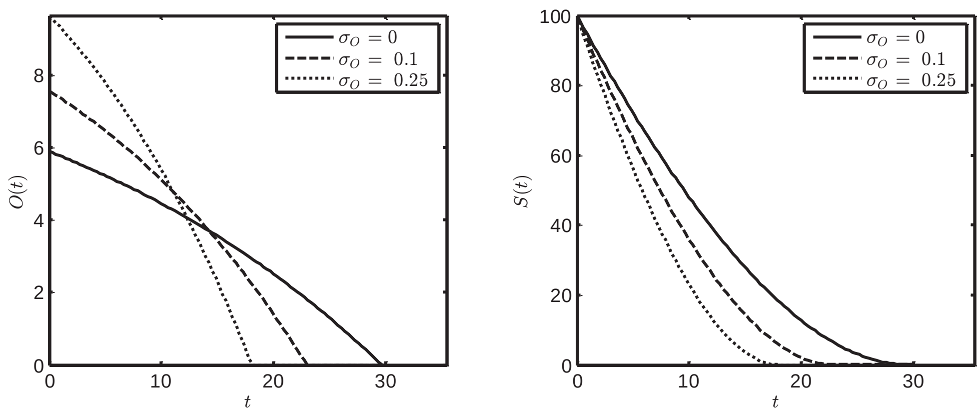


B.3. Endogenous oil extraction

Proposition 5 states that oil should be extracted faster if its price is positively correlated with the financial markets, which is illustrated in figure B6. Initially extracting oil faster allows the rate of production to quickly fall. As the rate of production falls, so too do convex extraction costs. The falling extraction costs increase the rate of return on the oil assets – which provides compensation for the risk of holding oil beneath the ground.

Figure B6: Endogenous oil extraction

a. Oil production ***b. Subsoil reserves***



Appendix C: Data and calibration

The policy experiment on Norway's GPFG is calibrated using monthly equity and bond data and actual Norwegian policy. The properties of financial assets in equation (3) are estimated using maximum likelihood on the FTSE Global All Cap Index, and its ten industry sub-indices, from 31/01/2011 to 31/05/2013. This avoids complications associated with the financial crisis. This gives α_M = 0.06 (per year) and σ_M = 0.15. The risk-free rate is the redemption yield on the US Benchmark 10 year Datastream Government Index, giving r = 0.022 (per year). The initial size of the fund is taken to be US$ 840 billion. The properties of oil prices are also estimated using maximum likelihood on the Brent Crude Oil Current Month FOB price, giving α_O = 0.01, σ_O = 0.22 and ρ_{OM} = 0.52. Oil production is taken from the 2014 BP statistical review giving initial proved reserves as 7.5 billion barrels and $O(0)$ = 0.67 billion barrels per year so that κ = 0.077 for all current reserves to be exhausted. To compare with current policy, we calculate implicit risk preferences by assuming Norway follows a CAPM model that ignores subsoil wealth, so that the weight of equities in the GPFG is $\bar{w} = 0.6$, giving θ = 0.37 from equation (8). The investment restrictions in section 5.2 require an assumption on the volatility of the unspanned component of oil. We begin by assuming $\sigma_h = \sigma_M$, but provide a ±50% sensitivity analysis in figure C1.

Figure C1: Sensitivity to the volatility of the unspanned component of oil