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Faster solutions for Black zero lower bound term structure models

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The Black framework offers a theoretically appealing way to model the term structure and gauge the stance of monetary policy when the zero lower bound of interest rates becomes constraining, but it is time consuming to apply using standard numerical methods. I outline a faster Monte Carlo simulation method for Black implementions, illustrate its performance for a one factor model, and then discuss the ready extension to models with multiple factors.

Keywords

Black framework, zero lower bound; shadow short rate; term structure model

JEL Classification

E43, G12, G13

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Faster solutions for Black zero lower bound term structure models

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The Black framework offers a theoretically appealing way to model the term structure and gauge the stance of monetary policy when the zero lower bound of interest rates becomes constraining, but it is time consuming to apply using standard numerical methods. I outline a faster Monte Carlo simulation method for Black implementions, illustrate its performance for a one factor model, and then discuss the ready extension to models with multiple factors.

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1 Introduction

In this article I outline a method for speeding up the Monte Carlo simulations of Gaussian affine term structure models (GATSMs) within the Black (1995, hereafter Black) framework.

The Black framework offers a theoretically appealing method for modeling the term structure when the zero lower bound (ZLB) of interest rates becomes materially constraining, which is the case for many major developed economies at the time of writing. The framework also produces estimated shadow GATSM term structures and associated shadow short rates that can be used to provide a measure of the stance of unconventional monetary policy as they evolve to negative levels; e.g. see Ichiue and Ueno (2006) originally, and more recently Ichiue and Ueno (2013), Krippner (2013a), and Bullard (2013). Furthermore, Claus, Claus, and Krippner (2013) and Wu and Xia (2013) respectively show that U.S. shadow short rates impart quantitative effects similar to pre-ZLB federal funds rates on asset prices and macroeconomic variables.

Unfortunately, the Black framework generally requires numerical methods to implement, and the computing times for such methods grows with the power of the number of factors. Therefore, routine estimations of Black models become challenging with more

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than two factors.[1] For that reason, Krippner (2013b) introduces a very fast approximation to the Black framework based on closed form analytic solutions for GATSM bond and bond option prices. That framework has already been shown to provide results close to the Black framework in practice; e.g. see Krippner (2013a), Christensen and Rudebusch (2013), and Wu and Xia (2013).

The speed and approximation properties of the Krippner (2013b) framework raises the possibility of creating a control variate for faster Monte Carlo estimates of Black bond prices.[2] I outline the Black framework and my proposed method in the following section, and then illustrate its application in section 3. Section 4 briefly concludes and then discusses the ready extension to multi-factor Black models.

2 A control variate for Black bond prices

The Black framework is based on the building block of ZLB short rates defined as $\underline{\mathrm{r}}(t) = \max\{\mathrm{r}(t), 0\}$, where $\mathrm{r}(t)$ is the shadow short rate that can adopt negative values, and $\max\{\mathrm{r}(t), 0\}$ imposes the ZLB. Black bond prices, $\underline{\mathrm{P}}(t,\tau)$ at time t and time to maturity τ, may be evaluated via Monte Carlo (MC) simulation as follows:

$$\widehat{\underline{\mathrm{P}}}(t,\tau) = \frac{1}{J}\sum_{j=1}^{J} \underline{\mathrm{P}}_j(t,\tau) \tag{1}$$

where J is the number individual bond price simulations $\underline{\mathrm{P}}_j(t,\tau)$:

$$\underline{\mathrm{P}}_j(t,\tau) = \exp\left[-\sum_{i=0}^{I-1} \underline{\mathrm{r}}_{t,j,i} \cdot \Delta\tau\right] \tag{2}$$

and $\underline{\mathrm{r}}_{t,j,i}$ is obtained from the Black ZLB mechanism $\underline{\mathrm{r}}_{t,j,i} = \max\{0, \mathrm{r}_{t,j,i}\}$. Each simulated path of the shadow short rate $\mathrm{r}_{t,j}$ is obtained using a discretized risk adjusted (i.e. $\mathbb{Q}$ measure) shadow short rate diffusion process. The process could be any multi-factor GATSM specification, but I use the one factor Vasicek (1977, hereafter Vasicek) model for maximum clarity in this article. Hence:

$$\mathrm{r}_{t,j,i} = \mathrm{r}_{t,j,i-1} + \kappa(\theta - \mathrm{r}_{t,j,i-1})\Delta\tau + \sigma\sqrt{\Delta\tau}\varepsilon_{t,j,i} \tag{3}$$

where $\mathrm{r}_{t,j,0} = \mathrm{r}(t)$ is the single state variable, and the parameters are the mean reversion rate κ, the long run level θ, and the volatility (annualized standard deviation) σ. The maturity step is $\Delta\tau = \tau/I$ and the innovations are independent Gaussian draws $\varepsilon_{t,j,i} \sim \mathrm{N}(0,1)$.

The precision of the estimate $\widehat{\underline{\mathrm{P}}}(t,\tau)$ is provided by its associated estimated standard deviation std[·], i.e.:

$$\mathrm{std}\left[\widehat{\underline{\mathrm{P}}}(t,\tau)\right] = \frac{1}{\sqrt{J-1}} \cdot \mathrm{std}\left[\underline{\mathrm{P}}_j(t,\tau)\right] \tag{4}$$

[1] Richard (2013) undertakes a full estimation of a three factor Black model that "requires a long time, literally a month, on large and fast computers to estimate". Once estimated, subsequent implementations would be much faster, but repeated estimations (as required, for example, in simulated real time forecasting exercises or regular model updates) would nevertheless be practically infeasible. Bauer and Rudebusch (2013) instead uses an approximation for their three factor Black model implementation, where a GATSM estimated with pre-ZLB data provides the Black parameters.

[2] James and Webber (2000) chapter 13 contains details on Monte Carlo simulation for interest rate models and the control variate method used in the present article.

Krippner (2013b) introduces an approximation to the Black framework that is based on GATSM bond prices, and GATSM bond call option prices with a strike price of 1. Those prices are used to create closed form analytic solutions for forward bond prices (and forward rates) that embed the ZLB constraint, and therefore enforce that property on the entire term structure. The control variate (CV) I propose uses the Krippner (2013b) concept of combining GATSM bond and bond option prices to approximate Black bond prices. Hence, I begin with the following three expressions:

$$\mathrm{P}_j(t,\tau) = \exp\left[-\sum_{i=0}^{I-1} \mathrm{r}_{t,j,i} \cdot \Delta\tau\right] \tag{5}$$

$$\mathrm{Z}_j(t,\tau) = \sum_{i=0}^{I-1} \mathrm{C}_j(t, i\Delta\tau) \tag{6}$$

$$\mathrm{C}_j(t, i\Delta\tau) = \exp\left[-\sum_{i=0}^{i-2} \mathrm{r}_{t,j,i} \cdot \Delta t\right] \max\left\{\exp\left[-\mathrm{r}_{t,j,i+1} \cdot \Delta\tau\right] - 1, 0\right\} \tag{7}$$

where $\mathrm{r}_{t,j,i}$ are the same shadow short rates previously defined in equation 3, and $\mathrm{C}_j(t, i\Delta\tau)$ are simulated bond call option prices with $\mathrm{Z}_j(t,\tau)$ their cumulative sum.

The population means of $\mathrm{P}_j(t,\tau)$, $\mathrm{Z}_j(t,\tau)$, and $\mathrm{C}_j(t, i\Delta\tau)$, which I respectively denote as $\mathrm{P}(t,\tau)$, $\mathrm{Z}(t,\tau)$, and $\mathrm{C}(t, i\Delta\tau)$, may be obtained using closed form analytic solutions for GATSM bond and bond option prices. Specifically for the Vasicek model, the expressions for $\mathrm{P}(t,\tau)$ and $\mathrm{C}(t, i\Delta\tau)$ are available from Vasicek or textbooks (e.g. James and Webber (2000) p. 186). $\mathrm{Z}(t,\tau)$ is the sum of $\mathrm{C}(t, i\Delta\tau)$, where the latter is evaluated at each point of the maturity grid $i\Delta\tau$. Subtracting the population means from the sample quantities therefore produces the following CV d_j:

$$d_j = \left[\mathrm{P}_j(t,\tau) - \mathrm{Z}_j(t,\tau)\right] - \left[\mathrm{P}(t,\tau) - \mathrm{Z}(t,\tau)\right] \tag{8}$$

which has the required properties of a zero mean and a high correlation to the object being estimated.

The MC/CV estimate of the Black bond price, which I denote as $\widetilde{\underline{\mathrm{P}}}(t,\tau)$, may be obtained from the following OLS regression:

$$\underline{\mathrm{P}}_j(t,\tau) = \alpha + \beta d_j + \varepsilon_j \tag{9}$$

where α is the $\widetilde{\underline{\mathrm{P}}}(t,\tau)$ estimate (as I subsequently illustrate in figure 1), and the standard error of α is $\mathrm{std}\left[\widetilde{\underline{\mathrm{P}}}(t,\tau)\right]$.

3 Empirical application

In this section I estimate Black bond prices using MC simulation and my proposed MC/CV method to illustrate the speed gains from the latter. I use the state variable/parameter set from the Black application of Gorovoi and Linetsky (2004) p. 71 (i.e. $\mathrm{r}(t) = -0.0512/\{\kappa,\theta,\sigma\} = \{0.212, 0.0354, 0.0283\}$ to define the process outlined in equation 3,[3] antithetic draws for ε_t (a standard MC variance reduction technique), $\Delta\tau = 0.01$, and 10,000 replications.

[3] I have also reproduced the Black-Vasicek Gorovoi and Linetsky (2004) table 6.1, p. 68 results using the MC/CV method.

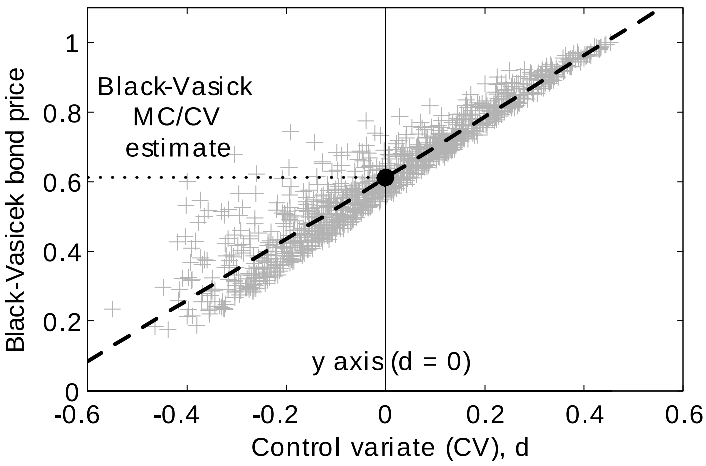

Figure 1: A graphic illustration of the MC/CV data and OLS regression used to obtain the 20-year bond price (i.e. 0.6119 in table 1).

Figure 1 illustrates graphically how the estimate of $\widetilde{\underset{\sim}{P}}(t, \tau)$ for the 20-year Black bond price (i.e. 0.6119 in table 1) is obtained using the MC/CV method. Table 1 also shows the markedly lower estimate of $\mathrm{std}\left[\widetilde{\underset{\sim}{P}}(t, \tau)\right]$ compared to the MC estimate of $\mathrm{std}\left[\widehat{\underset{\sim}{P}}(t, \tau)\right]$, i.e. 0.0005 compared to 0.0019, meaning less simulations will be required to obtain a given precision in the Black bond price estimate. Graphically, that lower relative standard deviation reflects the lower dispersion of $\underset{\sim}{P}_j(t, \tau)$ simulations around the CV regression line compared to the dispersion of $\underset{\sim}{P}_j(t, \tau)$ simulations against a constant (i.e. the dispersion parallel to the x axis).

In practice, interest rates are typically used as observables when estimating term structure models, and a minimum precision is required when generating interest rates from the model to compare to the interest rate data. Hence, I convert the MC and MC/CV bond price results into interest rates using the following standard expressions:

$$\underset{\sim}{R}(t, \tau) = \frac{1}{\tau} \log\left[\underset{\sim}{P}(t, \tau)\right] \tag{10}$$

$$\mathrm{std}\left[\underset{\sim}{R}(t, \tau)\right] = \frac{1}{\tau \cdot \underset{\sim}{P}(t, \tau)} \cdot \mathrm{std}\left[\underset{\sim}{P}(t, \tau)\right] \tag{11}$$

and I also calculate the times it would take to generate a $\mathrm{std}[\underset{\sim}{R}(t, \tau)] = 0.003$ percentage points (pps) for the MC and MC/CV methods given the standard deviations already achieved for the times taken to run the 10,000 simulations.[4] For example, the MC/CV 20-year bond rate estimate is 2.46 pps with a standard deviation of 0.004 pps (to 3 decimal places), and a time of 16.8 seconds would be required to reduce that standard deviation to 0.003 percentage points. Compared to the MC results, the MC/CV results represents a relative time benefit of $0.088 = 7.85/7.15 \times 0.28^2$; i.e. the MC/CV method would produce the same precision in less than $1/10^{\text{th}}$ of the time for the MC method alone. The relative time benefit combines the relative $\mathrm{std}[\cdot]$ benefit of 0.28 (i.e. $\mathrm{std}\left[\widetilde{\underset{\sim}{R}}(t, \tau)\right] / \mathrm{std}\left[\widehat{\underset{\sim}{R}}(t, \tau)\right]$), and the relatively longer time for implementing the MC/CV method (i.e. 7.85/7.15) due to the additional numerical and analytic evaluations required to generate the CV and the OLS regression results.

[4] My choice of 0.003 percentage points is arbitrary for the illustration, but it represents a three standard deviation precision of less than 0.01 percentage points that would be typical in practice.

Table 1: Black-Vasicek simulation results

MC results: time 7.15 seconds for 10,000 simulations					
Maturity τ	1	5	10	20	30
$\widehat{\underline{\mathrm{P}}}(t,\tau)$	0.9998	0.9618	0.8498	0.6112	0.4307
$\mathrm{std}\left[\widehat{\underline{\mathrm{P}}}(t,\tau)\right]$	0.0000	0.0005	0.0013	0.0019	0.0019
$\widehat{\underline{\mathrm{R}}}(t,\tau)$ pps	0.02	0.78	1.63	2.46	2.81
$\mathrm{std}\left[\widehat{\underline{\mathrm{R}}}(t,\tau)\right]$	0.001	0.011	0.015	0.016	0.015
0.003 time*	1	87	172	190	170
MC/CV results: time 7.85 seconds for 10,000 simulations					
Maturity τ	1	5	10	20	30
$\widetilde{\underline{\mathrm{P}}}(t,\tau)$	0.9998	0.9621	0.8506	0.6119	0.4306
$\mathrm{std}\left[\widetilde{\underline{\mathrm{P}}}(t,\tau)\right]$	0.0000	0.0000	0.0001	0.0005	0.0010
$\widetilde{\underline{\mathrm{R}}}(t,\tau)$ pps	0.02	0.77	1.62	2.46	2.81
$\mathrm{std}\left[\widetilde{\underline{\mathrm{R}}}(t,\tau)\right]$	0.000	0.000	0.001	0.004	0.008
0.003 time*	0.0002	0.1	1.8	16.8	51.1
Relative results (i.e. MC/CV results divided by MC results)					
Maturity τ	1	5	10	20	30
std[·] benefit	0.012	0.037	0.097	0.28	0.52
Time benefit	0.0002	0.0015	0.010	0.088	0.30

Note: * estimated time to achieve interest rate std[-] of 0.003 pps.

Table 1 contains the results for a range of maturities from 1 to 30 years. While all results show a relative time benefit from using the MC/CV method, the benefit decreases by maturity. Figure 2 illustrates the latter trend by maturity, and shows a trend for the relative time benefit to worsen as the shadow short rate declines from positive values, to zero, and to negative values.

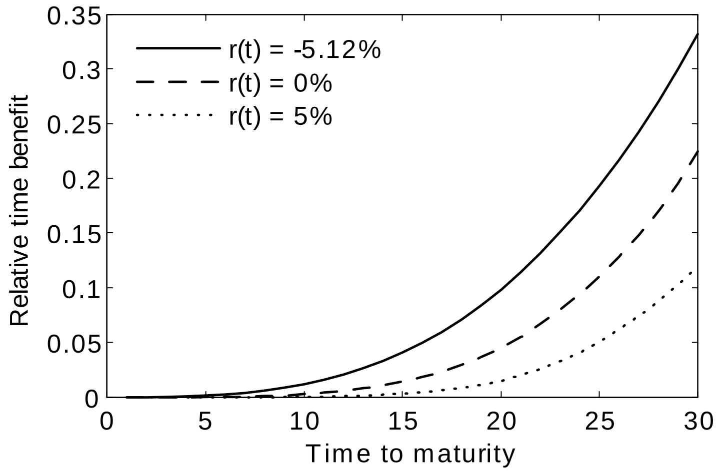

Figure 2: The time to achieve a given interest rate precision for the MC/CV method relative to standard MC simulation.

Both trends are intuitive: when shadow short rates have higher probabilities of being or becoming negative, the differences in ZLB bond price approximations based on shadow short rates will be greater relative to Black solutions obtained from ZLB short rates. In turn, the correlation and therefore effectiveness of the CV will deteriorate.

4 Conclusion and extensions

In this article, I have introduced a CV for the MC simulation of Black bond prices and interest rates, and have shown that it can greatly speed up implementation times. The ultimate extension would be to estimate multi-factor Black models using the MC/CV method, which would in turn produce shadow term structures and associated shadow short rates that can be used to provide a measure of the stance of unconventional monetary policy at the zero lower bound.

From that perspective, the MC/CV method readily extends to multiple factors because MC simulation is well suited for higher dimensional modeling.[5] Importantly also, the number of numerical evaluations required to generate my proposed CV remains invariant to the model specification (i.e. the number of factors and free parameters).

Notwithstanding those points, the full estimation of multi-factor Black models on full spans of yield curve data will remain challenging for the following reasons: (1) the "curse of dimensionality" still applies for MC estimations, even though the CV method facilitates faster implementation times; (2) the robust estimation of ZLB models requires many implementations near the ZLB, due to their highly non-linear nature in that state (see Krippner (2013b) for further discussion); and (3) the speed gains with the CV are relatively less for longer times to maturity.[6] Further investigation of additional speed up methods (e.g. importance sampling and additional CVs in multivariate OLS regressions) may offer further relative time benefits, as would allowing for the heteroskedasticity with my proposed CV.[7]

In the interim, routine Black estimations may now at least be feasible for maturity spans out to 10 years. The MC/CV method will also facilitate the systematic checking of how well the Krippner (2013b) framework approximates the Black framework with respect to different model specifications and the combinations of parameters and state variables within those models.

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[5]The alternative numerical methods of finite difference grids and lattices are typically applied with one or two dimensions.

[6]Items 2 and 3 take the full estimation of the Black-Vasicek model beyond the scope of the present article, given space considerations.

[7]Figure 1 shows apparent heteroskedasticity as a function of the CV, which turns out to be highly significant for all maturities. Hence, more efficient estimates of $\widetilde{\underline{\mathrm{P}}}(t,\tau)$ could readily be obtained.

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